Name: ________________________

Math 3360
Fall 2002
Take Home Test I

You can use your textbook or notes. You cannot discuss this test with your classmates. Good luck!

1. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.
   (a) Prove that $n^2 > n+1$ for all integers $n \geq 2$.
   (b) Prove that $n! > n^2$ for all integers $n \geq 4$. [Recall that $n! = n(n-1)\cdots 2\cdot 1$; for example $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.]

2. Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers*. Prove that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. *Hint*: First show $\{ r + \sqrt{2} : r \in \mathbb{Q} \} \subseteq \mathbb{I}$.

3. Let $a, b \in \mathbb{R}$. Show that if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

4. Show that $\sup\{ r \in \mathbb{Q} : r < a \} = a$ for each $a \in \mathbb{R}$.

5. Let x be any real number. Prove that there exists a unique integer r such that $r \leq x < r+1$.