Real Analysis
Fall 2004
Take Home Final

Due Tuesday, December 14, 2004 by 5:00 p.m.

1. Suppose that f is uniformly continuous on a set $S \subset \mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in S. Prove that $\{f(x_n)\}$ is a Cauchy sequence. (f is not <u>assumed</u> to be continuous outside S, so you cannot use Theorem 3.2, p. 60).

2. Let $M > 0$ and let $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$, satisfy the condition $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in D$. Show that f is uniformly continuous.

3. Suppose that for some constant M with $0 < M < 1$, $|a_{n+2} - a_{n+1}| \leq M|a_{n+1} - a_n|$, $n = 1, 2, 3, \ldots$. Prove that the sequence $\{a_n\}$ is Cauchy.

4. Suppose that f and g are continuous functions on the closed interal $[a, b]$ such that $f(r) = g(r)$ for every rational number $r \in [a, b]$. Prove that $f(x) = g(x)$ for all $x \in [a, b]$.

5. Let $u_{n+1} = \sqrt{u_n + 1}$, $u_1 = 1$.

   (a) Show that $\{u_n\}$ is bounded and monotone.

   (b) Find $\lim_{n\to\infty} u_n$.

6. Let S be the space of all rational numbers, with $d(p, q) = |p - q|$, and E is the set of all rational numbers p such that $2 < p^2 < 3$. Prove that

   (i) E is closed and bounded.

   (ii) E is not compact.

7. Let E be a nonempty subset of a metric space (S, d). Define the distance from $x \in S$ to the set E by $\rho(x) = \underset{y\in E}{glb}\, d(x, y)$.

   (a) Prove that $\rho(x) = 0$ if and only if $x \in \bar{E}$.

   (b) Prove that $\rho : S \to R$ is uniformly continuous on S.

8. Suppose that f is continuous on an open interval I containing x_0, suppose that f' is defined on I except possibly at x_0, and suppose that $\lim_{x\to x_0} f'(x) = L$. Prove that $f'(x_0) = L$.

9. Let f and g be continuous functions on $[a, b]$, g is positive and monotonically decreasing and $g'(x)$ exists on $[a, b]$. Prove that there exists a point $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\, d(x) = g(a) \int_a^{\xi} f(x)\, dx.$$

10. Suppose that f is continuous at $x = a$ such that $|f(a)| < 1$. Prove that there exists an open interval $I = (a - \delta, a + \delta)$, $\delta > 0$, such that for all $x \in I$, $|f(x)| \leq M < 1$, for some fixed constant M.