**Real Analysis**
**Fall 2004**
**Take Home Final Key**

1. Suppose that $f$ is uniformly continuous on a set $S \subset \mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in $S$. Prove that $\{f(x_n)\}$ is a Cauchy sequence. ($f$ is not assumed to be continuous outside $S$, so you cannot use Theorem 3.2, p. 60).

   *Proof.* Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $S$, there exists $\delta > 0$ such that

   $$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon \quad \text{for all} \quad x, y \in S. \tag{1}$$

   For this $\delta > 0$ there exists $N \in \mathbb{Z}^+$ such that for $n, m > N$, $|x_n - x_m| < \delta$. By (1) it follows that for $n, m > N$, $|f(x_n) - f(x_m)| < \varepsilon$. Hence $\{f(x_n)\}$ is Cauchy. □

2. Let $M > 0$ and let $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$, satisfy the condition $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in D$. Show that $f$ is uniformly continuous.

   *Proof.* Let $\varepsilon > 0$. Put $\delta = \dfrac{\varepsilon}{2M}$. Then if $|x - y| < \delta$,

   $$\begin{aligned} |f(x) - f(y)| &\le M|x - y| \\ &\le M\delta \\ &= M\frac{\varepsilon}{2M} \\ &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

   Hence $f$ is uniformly continuous. □

3. Suppose that for some constant $M$ with $0 < M < 1$, $|a_{n+2} - a_{n+1}| \le M|a_{n+1} - a_n|$, $n = 1, 2, 3, \dots$. Prove that the sequence $\{a_n\}$ is Cauchy.

   *Proof.* We will first show that

   $$|a_{n+2} - a_{n+1}| \le M^n |a_2 - a_1|. \tag{2}$$

   This is true for $n = 1$ as assumed. Suppose it is true for $n = k$. Then

   $$|a_{k+3} - a_{k+2}| \le M|a_{k+2} - a_{k+1}| \le M^k |a_2 - a_1|.$$

   Hence (2) is true for all $n$.

For a fixed $N$ and $n = N + r$, we have

$$\begin{aligned} |a_n - a_N| &\le |a_{N+r} - a_{N+r-1}| + |a_{N+r-1} - a_{N+r-2}| + \cdots + |a_{N+1} - a_N| \\ &\le M^{N+r-2}|a_2 - a_1| + \cdots + M^{N-1}|a_2 - a_1| \\ &= |a_2 - a_1| M^{N-1}[1 + M + \cdots + M^{r-1}] \\ &\le |a_2 - a_1| \frac{M^{N-1}}{1-M}. \end{aligned} \tag{3}$$

Given $\varepsilon > 0$, let $|a_2 - a_1|\dfrac{M^{N-1}}{1-M} < \varepsilon$. Then $M^{N-1} < \dfrac{(1-M)\varepsilon}{|a_2 - a_1|}$. Take logarithms.

$$\begin{aligned} (N-1)\ln M &< \ln \frac{(1-\varepsilon)\varepsilon}{|a_2 - a_1|} \\ N - 1 &> \frac{\ln \frac{(1-M)\varepsilon}{|a_2-a_1|}}{\ln M} \quad \text{since} \quad \ln M < 0 \\ \text{or } N &> \frac{\ln \frac{(1-M)\varepsilon}{|a_2-a_1|}}{\ln M} + 1 \end{aligned}$$

□

4. Suppose that $f$ and $g$ are continuous functions on the closed interal $[a, b]$ such that $f(r) = g(r)$ for every rational number $r \in [a, b]$. Prove that $f(x) = g(x)$ for all $x \in [a, b]$.

   *Proof.* Suppose that $f(z) \ne g(z)$ for some irrational number $z$ in $[a, b]$. Let $|f(z) - g(z)| = \alpha$. For $\varepsilon = \dfrac{\alpha}{2}$, there exists $\delta > 0$ such that $|f(z) - f(x)| < \dfrac{\varepsilon}{2}$ and $|g(z) - g(x)| < \dfrac{\varepsilon}{2}$ whenever $|z - x| < \delta$ ($\delta = \min(\delta_1, \delta_2)$). Let $r$ be a rational number with $|z - r| < \delta$. Then $f(r) = g(r)$. Moreover,

$$\begin{aligned} \alpha = |f(z) - g(z)| &\le |f(z) - f(r)| + |g(z) - g(r)| \\ &\le \frac{\alpha}{4} + \frac{\alpha}{4} = \frac{\alpha}{2}, \end{aligned}$$

   a contradiction. Hence $f(z) = g(z)$ for all $z \in [a, b]$. □

5. Let $u_{n+1} = \sqrt{u_n + 1}$, $u_1 = 1$.

   (a) Show that $\{u_n\}$ is bounded and monotone.

   *Proof.* Let $f(x) = \sqrt{x+1}$. Then $f'(x) = \dfrac{1}{2\sqrt{x+1}} > 0$ for $x > -1$. Hence $f$ is increasing. Consider the point $x^* = \dfrac{1+\sqrt{5}}{2}$. Then $f(x^*) = x^*$. For $u_1 = 1 < x^*$,

$u_2 = f(u_1) < f(x^*) = x^*$. And by induction, we have $u_n < x^*$. Hence $\{u_n\}$ is bounded above by $x^*$. Since $u_2 = \sqrt{1+1} = \sqrt{2} > u_1$, it follows by the same reasoning that $\{u_n\}$ is monotonically increasing. Hence by the Bolzano-Weierstran Theorem, $\{u_n\}$ must converge. □

(b) Find $\lim_{n\to\infty} u_n$.

*Proof.* Let $\sqrt{x+1} = x$. Then $x^* = \dfrac{1+\sqrt{5}}{2}$ is the limit point as $\dfrac{1-\sqrt{5}}{2}$ is discarded since it is negative. □

6. Let $S$ be the space of all rational numbers, with $d(p,q) = |p-q|$, and $E$ is the set of all rational numbers $p$ such that $2 < p^2 < 3$. Prove that

(i) $E$ is closed and bounded.

(ii) $E$ is not compact.

*Proof.*

$$\begin{aligned} E &= \{p : 2 < p^2 < 3\} \cap S \\ &= \Big[(-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})\Big] \cap S \\ &= (I_1 \cup I_2) \cap S \end{aligned}$$

(a) $E$ is clearly bounded by $-\sqrt{3}$ and $\sqrt{3}$. Let $x \in E'$, then there is a sequence $\{p_n\}$ in $E$ with $d(p_n, x) \to 0$ as $n \to \infty$. Now $\{p_n\}$ is either in $I_1$ or $I_2$, say $I_n$. Hence it either converges in $I$, and hence $x \in I_1$ or it converges to either $-\sqrt{3}$ or $-\sqrt{2}$ which are not in our space. Hence $E$ is closed.

(b) Consider the open cover $\mathcal{F} = \{V_n : i = 1,2,3,\ldots\}$ where $V_n = \{p \in S : 3 - \frac{n}{n+1} < p^2 < 3\}$. This cover has not finite subcover. Hence it is not compact.
Another solution:
Take the sequence $\{p_n\}$, $p_n^2 = 3 - \frac{1}{n}$. Then $\{p_n\}$ has no convergent subsequences.

□

7. Let $E$ be a nonempty subset of a metric space $(S,d)$. Define the distance from $x \in S$ to the set $E$ by $\rho(x) = \underset{y\in E}{glb}\ d(x,y)$.

(a) Prove that $\rho(x) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that $\rho : S \to R$ is uniformly continuous on $S$.

*Proof.* (a) Let $p(x) = 0$. Then $\operatorname*{glb}_{y \in E} d(x, y) = 0$. Hence there is a sequence $\{y_n\}$ in $E$ with $d(x, y_n) \to 0$ as $n \to \infty$. Thus $x \in \overline{E}$. For the converse, let $x \in \overline{E}$. If $x \in E$, then as $d(x, x) = 0$, $p(x) = 0$. If $x \in E \backslash E$, then there exists $y_n \in E$, $y_n \to x$ as $n \to \infty$ or $d(y_n, x) \to 0$ as $n \to \infty$. Hence $p(x) = 0$.

(b) $p : S \to \mathbb{R}$. For $x_1, x_2 \in S$

$$\begin{aligned} |p(x_1) - p(x_2)| &= \left| \operatorname*{glb}_{y \in E} d(x_1, y) - \operatorname*{glb}_{y \in E} d(x_2, y) \right| \\ &\leq \operatorname*{glb}_{y \in E} |d(x_1, y) - d(x_2, y)| \end{aligned} \tag{4}$$

But $d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$ or $d(x_1, y) - d(x_2, y) \leq d(x_1, x_2)$. Similarly $d(x_2, y) - d(x_1, y) \leq d(x_1, x_2)$. Thus $|d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2)$. Hence $|p(x_1) - p(x_2)| \leq d(x_1, x_2)$. Given $\varepsilon > 0$, let $\delta = \varepsilon$. If $d(x_1, x_2) < \delta$, then $|p(x_1) - p(x_2)| \leq d(x_1, x_2) < \delta = \varepsilon$.

□

8. Suppose that $f$ is continuous on an open interval $I$ containing $x_0$, suppose that $f'$ is defined on $I$ except possibly at $x_0$, and suppose that $\lim_{x \to x_0} f'(x) = L$. Prove that $f'(x_0) = L$.

*Proof.* $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ if the limits exists. Since $f$ is continuous, $\lim_{x \to x_0} f(x) = f(x_0)$. Now

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \frac{0}{0} \\ \mathscr{L} \\ &= \lim_{x \to x_0} \frac{f'(x)}{1} = L. \end{aligned}$$

□

9. Let $f$ and $g$ be continuous functions on $[a, b]$, $g$ is positive and monotonically decreasing and $g'(x)$ exists on $[a, b]$. Prove that there exists a point $\xi \in [a, b]$ such that

$$\int_a^b f(x) g(x)\, d(x) = g(a) \int_a^\xi f(x)\, dx.$$

*Proof.* Let $h(x) \int_a^x f(t)\, dt$. Since $g$ is positive, either

$$0 \leq \int_a^b f(x) g(x)\, dx \leq g(a) \int_a^b f(t)\, dt$$

or

$$g(a)\int_a^b f(t)\,dt \le \int_a^b f(x)g(x)\,dx \le 0.$$

In either case, $\dfrac{\int_a^b f(x)g(x)\,dx}{g(a)}$ is between $h(a) = 0$ and $h(b)$. By the Intermediate Value Theorem, there exists $\xi$ between $a$ and $b$ such that

$$\frac{\int_a^b f(x)g(x)\,dx}{g(a)} = h(\xi) = \int_a^\xi f(t)\,dt.$$

□

10. Suppose that $f$ is continuous at $x = a$ such that $|f(a)| < 1$. Prove that there exists an open interval $I = (a-\delta, a+\delta)$, $\delta > 0$, such that for all $x \in I$, $|f(x)| \le M < 1$, for some fixed constant $M$.

*Proof.* Let $L = |f(a)| < 1$. If such an $M$ does not exist for any $\delta$, there exists a sequence $\{x_n\}$ that converges to $a$ with $|f(x_n)| > \dfrac{L+1}{2}$. Since $f$ is continuous, $|f(x_n)| \to |f(a)|$ as $n \to \infty$. But this is not possible as $||f(x_n)| - |f(a)|| > \dfrac{1-L}{2}$ for all $n$. □