Real Analysis
Fall 2004
Take Home Test 1

Due Monday 10/4 in my mailbox before 5:00 p.m.

1. Use the definition of a limit to show that

   (a) $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

   (b) $\lim_{n\to 3^+} \frac{1}{x-3} = \infty$

2. (a) Let $\{x_n\}$ be a bounded sequence of real numbers and $\{x_{k_n}\}$ be a monotone subsequence. Prove that $\{x_{k_n}\}$ converges to a limit.

   (b) If $\lim_{n\to\infty} x_n = L = \emptyset$, prove that $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{L}$

3. Let $x_{n+1} = \frac{1}{3+x_n}$, $x_1 > 0$. Prove that the sequence $\{x_n\}$ converges and then compute the limit of the sequence.

4. Prove that the function $f(x) = \frac{1}{x^2}$ is continuous for all real numbers $x \neq 0$.

5. Suppose that f is continuous on $[a,b]$, one-to-one (if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$), and $f(a) < f(b)$. Prove that f is monotonically increasing (strictly) on $[a,b]$.

6. Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}^1$ and that $S = S_1 \cap S_2 \cap \cdots \cap S_n$, $S \neq \emptyset$. Let $B_i = \sup S_i$, $b_i = \inf S_i$, $1 \leq i \leq n$. Find a formula relating $\sup S$ and $\inf S$ in terms of the $\{b_i\}$ and $\{B_i\}$.

7. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to -\infty} g(x) = a$. Suppose that for some positive number M, we have $g(x) \neq a$ for $x < -M$. Prove that $\lim_{x\to -\infty} f(g(x)) = \infty$.

8. If $f(x)$ is continuous on $[a,b]$, if $a < c < d < b$, and $M = f(c) + f(d)$, prove there exists a number ξ between a and b such that $M = 2f(\xi)$.

9. Suppose that $f(x)$ and $g(x)$ are functions defined for $x > 0$, $\lim_{x\to 0^+} g(x)$ exists and is finite, and $|f(b) - f(a)| \leq |g(b) - g(a)|$ for all positive real number a and b. Prove that $\lim_{x\to 0^+} f(x)$ exists and is finite.

10. Evaluate $\lim_{n\to\infty} \dfrac{2 + 2^{\frac{1}{2}} + 2^{\frac{1}{3}} + \cdots + 2^{\frac{1}{n}}}{n}$.

(You need not give a proof but you should show some work or justification. Quote a theorem or what have you. Calculator results or graphical analysis are not acceptable.)