Real Analysis
Fall 2004
Take Home Test 1
SOLUTIONS

1. Use the definition of a limit to show that

   (a) $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$

   Proof. Let $\varepsilon > 0$ be given. Define $N > \dfrac{1}{\varepsilon}$, where N is a positive integer. Then for $n > N$, $\left|\dfrac{\sin n}{n} - 0\right| \leq \dfrac{1}{n} < \dfrac{1}{N} < \varepsilon$. Hence $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$. □

   (b) $\lim\limits_{n\to 3^+} \dfrac{1}{x-3} = \infty$

   Proof. Let $A > 0$ be given. Define $\delta = \dfrac{1}{A}$. Then for $x-3 < \delta$, $f(x) = \dfrac{1}{x-3} > \dfrac{1}{\delta} = A$. Hence $\lim\limits_{x\to 3^+} \dfrac{1}{x-3} = \infty$. □

2. (a) Let $\{x_n\}$ be a bounded sequence of real numbers and $\{x_{k_n}\}$ be a monotone subsequence. Prove that $\{x_{k_n}\}$ converges to a limit.

   Proof. By Bolzano-Weierstrass Theorem, the bounded subsequence $\{y_n\} = \{x_{k_n}\}$ has a convergent subsequence $\{y_{l_n}\}$ which converges to x_0. Now either $\{x_{k_n}\}$ is nondecreasing or nonincreasing. Without loss of generality, assume that $\{y_n\}$ is monotonically nondecreasing. Given $\varepsilon > 0$, there exists N such that $n > N$ implies $x_0 - \varepsilon < y_{l_n} < x_0 + \varepsilon$. In particular, $x_0 - \varepsilon < y_{l_{N+1}} < x_0 + \varepsilon$. For $n > l_{N+1}$ we have $x_0 - \varepsilon < y_{l_{N+1}} < y_n < x_0 + \varepsilon$. Thus $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_{k_n} = x_0$. □

   (b) If $\lim\limits_{n\to\infty} x_n = L = \emptyset$, prove that $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{L}$

   Proof. Let $\varepsilon_1 = \dfrac{|L|}{2}$. Then there exists $N_1 > 0$ such that for $n > N_1$, $|x_n - L| < \dfrac{|L|}{2}$. Consequently, for $n > N_1$,

$$|x_n| = |x_n - L + L| \geq |L| - |x_n - L| > |L| - \frac{|L|}{2} = \frac{|L|}{2}.$$

For any given $\varepsilon > 0$ there exists $N_2 > 0$ such that for $n > N_2$, $|x_n - L| < \dfrac{2\varepsilon}{L^2}$.
Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\left|\frac{1}{x_n} - \frac{1}{L}\right| = \frac{|x_n - L|}{|x_n||L|} \leq \frac{|x_n - L|}{L^2/2} < \frac{2\varepsilon/L^2}{L^2/2} = \varepsilon.$$

Thus $\lim_{n\to\infty} \frac{1}{x_n} = L$. □

3. Let $x_{n+1} = \dfrac{1}{3 + x_n}$, $x_1 > 0$. Prove that the sequence $\{x_n\}$ converges and then compute the limit of the sequence.

First Proof (My Preference). Since $x_1 > 0$, $x_{n+1} = \dfrac{1}{3 + x_n} < \dfrac{1}{3}$ for all $n \geq 1$. Moreover, $x_{n+1} = \dfrac{1}{3 + x_n} < \dfrac{1}{3 + \frac{1}{3}} = \dfrac{3}{10}$ for all $n \geq 1$. Hence $\dfrac{3}{10} < x_n < \dfrac{1}{3}$ for all $n > 1$.

(b) If $\lim_{n\to\infty} x_n = L$ exists, then $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \dfrac{1}{3 + x_n} = \dfrac{1}{\lim_{n\to\infty}(3 + x_n)} = \dfrac{1}{3 + L}$

$$\frac{1}{3 + L} = L. \tag{1}$$

Thus

$$L^2 + 3L - 1 = 0. \tag{2}$$

$L = \dfrac{-3 + \sqrt{13}}{2}$.

(a) If $x_1 > L$, then $x_2 = \dfrac{1}{3 + x_1} < \dfrac{1}{3 + L} = L$ from (1). More generally, if $x_{2n-1} > L$, then $x_{2n} = \dfrac{1}{3 + x_{2n-1}} < \dfrac{1}{3 + L} = L$. On the other hand, if $x_1 < L$, then $x_{2n-1} < L$ and $x_{2n} > L$. Without loss of generality, assume $x_1 > L$. Define the sequence $\{s_n\}$ as $s_n = |x_n - L|$. Then claim that $s_{n+1} < s_n$ for all $n \geq 1$. If not, then $\dfrac{s_{n+1}}{s_n} \geq 1$. Assume $n = 2k$. Then

$$\frac{s_{2k+1}}{s_{2k}} = \frac{x_{2k+1} - L}{L - x_{2k}} = \frac{\frac{1}{3+x_{2k}} - L}{L - x_{2k}} \geq 1$$
$$1 - 3L - Lx_{2k} \geq 3L - 3x_{2k} + Lx_{2k} - x_{2k}^2$$
$$(x_{2k}^2 - 2Lx_{2k} + L^2) - L^2 - 6L + 3x_{2k} + 1 \geq 0$$
$$(x_{2k} - L)^2 - 1 - 3L + 3x_{2k} + 1 \geq 0 \quad \text{(using (2))}$$
$$(x_{2k} - L)[x_{2k} - L + 3] \geq 0.$$

Since $x_{2k} - L < 0$, $x_{2k} - L + 3 \leq 0$ or $x_{2k} \leq L - 3 < 0$, a contradiction.
A similar conclusion is obtained for $n = 2k - 1$.
Since s_n is bounded and monotonically decreasing, $\lim_{n\to\infty} s_n = \alpha$. Thus $\lim_{n\to\infty} s_{2n} = \alpha = \lim_{n\to\infty} s_{2n+1}$. For the case $x_1 > L$, $x_{2n+1} > L$ and $x_{2n} < L$ and consequently

(i) $s_{2n} = L - x_{2n} \to \alpha \Rightarrow x_{2n} \to L - \alpha$ as $n \to \infty$,

(ii) $s_{2n+1} = x_{2n+1} - L \to \alpha \Rightarrow x_{2n+1} \to L + \alpha$ as $n \to \infty$.

Now

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{1}{3 + x_{2n}} \Rightarrow L + \alpha = \frac{1}{3 + L - \alpha}$$
$$L^2 + 3L - 1 + 3\alpha + \alpha L - \alpha^2 - \alpha L = 0$$
$$\alpha(3 - \alpha) = 0 \Rightarrow \alpha = 0 \text{ or } \alpha = 3.$$

If $\alpha = 3$, $L - \alpha < 0$ and $L + \alpha > \frac{1}{3}$, a contradiction. Hence $\alpha = 0$ and thus $\lim_{n\to\infty} x_n = L$.

□

Second Proof. Let $f(x) = \frac{1}{3+x}$. Then $f'(x) = \frac{-1}{(3+x)^2}$. Since $x > 0$, $|f'(x)| = \frac{1}{(3+x)^2} \leq \frac{1}{3}$ for all $x > 0$. Now[1]

$$\frac{|x_2 - L|}{|x_1 - L|} = \frac{|f(x_1) - L|}{|x_1 - L|} = |f'(\xi_1)| \leq \frac{1}{3}$$

by the Mean Value Theorem, where ξ_1 is between x_1 and L. $|x_2 - L| \leq \frac{1}{3}|x_1 - L|$. Similarly,

$$\frac{|x_3 - L|}{|x_2 - L|} = \frac{|f(x_2) - L|}{|x_2 - L|} = |f'(\xi_2)| \leq \frac{1}{3},$$

ξ_2 between x_2 and L. $|x_3 - L| \leq \frac{1}{3}|x_2 - L| \leq \left(\frac{1}{3}\right)^2 |x_1 - L|$. By induction, $|x_n - L| \leq \left(\frac{1}{3}\right)^{n-1} |x_1 - L| \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} x_n = L$. □

4. Prove that the function $f(x) = \frac{1}{x^2}$ is continuous for all real numbers $x \neq 0$.

[1] L is defined as the fixed point of f, i.e., $f(L) = L = \frac{1}{3+L} \Rightarrow L^2 + 3L - 1 = 0 \Rightarrow L = \frac{-3+\sqrt{13}}{2}$

Method 1. Let $\varepsilon > 0$ be given. Let $x_0 \in \mathbb{R}\backslash\{0\}$. Put $\delta = \frac{|x_0|}{2}$. Then if $|x - x_0| < \delta = \frac{|x_0|}{2}$, then $\frac{|x_0|}{2} < x < \frac{3|x_0|}{2}$. Let $\delta_2 = \frac{\varepsilon}{10}|x_0|x_0^2$. For $\delta = \min(\delta_1, \delta_2)$, if $|x - x_0| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| &= \frac{|x_0^2 - x^2|}{x_0^2 x^2} \\ &= \frac{|x - x_0||x + x_0|}{x_0^2 x^2} \\ &< \frac{|x - x_0|\frac{5}{2}|x_0|}{x_0^2 \frac{x_0^2}{4}} \\ &\leq \frac{|x - x_0| \cdot 10}{|x_0|x_0^2} \\ &= \varepsilon\end{aligned}$$

□

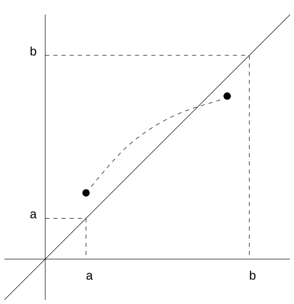
Method 2. Consider the function $g(x) = x^2$. Given $\varepsilon > 0$, let $\delta_1 = \frac{|x_0|}{2}$. Then $\frac{|x_0|}{2} < x < \frac{3|x_0|}{2}$ for all $|x - x_0| < \delta_1$. Let $\delta_2 = \frac{2\varepsilon}{5|x_0|}$. Then for $\delta = \min(\delta_1, \delta_2)$, and $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$\begin{aligned}|x^2 - x_0^2| &= |x - x_0||x + x_0| \\ &< |x - x_0|\frac{5|x_0|}{2} \\ &\leq \varepsilon.\end{aligned}$$

Then use a theorem about the reciprocal of a continuous function is continuous. □

5. Suppose that f is continuous on $[a, b]$, one-to-one (if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$), and $f(a) < f(b)$. Prove that f is monotonically increasing (strictly) on $[a, b]$.

Proof. Suppose that f is not monotonically increasing on $[a, b]$.



Then there exists $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ but $f(x_1) > f(x_2)$. We have two cases to consider:

Case (i): If $f(a) < f(x_1)$, then $f(a)$ lies between $f(x_1)$ and $f(x_2)$. Hence by the Intermediate Value Theorem, there exists c between x_1 and x_2 with $f(c) = f(a)$, a contradiction to the assumption that f is one-to-one.

Case (ii): If $f(a) > f(x_1)$, then since $f(a) < f(b)$, the value $f(a)$ lies between $f(x_1)$ and $f(b)$. Again, by the Intermediate Value Theorem, there exists d between x_1 and b with $f(d) = f(a)$, a contradiction. Hence f is increasing.

□

6. Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}^1$ and that $S = S_1 \cap S_2 \cap \cdots \cap S_n$, $S \neq \emptyset$. Let $B_i = \sup S_i$, $b_i = \inf S_i$, $1 \leq i \leq n$. Find a formula relating $\sup S$ and $\inf S$ in terms of the $\{b_i\}$ and $\{B_i\}$.

Proof. $S = \cap_{i=1}^{n} S_i$. Claim that $\sup S = \min\{B_i, 1 \leq i \leq n\}$. Let $B_r = \min\{B_i : 1 \leq i \leq n\}$. If $x \in S$, then $x \in S_r$, and hence $x \leq B_r$. Thus $\sup S \leq B_r$. On the other hand, $\sup S \geq x \in S_r$ and by the definition of B_r, $\sup S \geq B_r$. Hence $\sup S = B_r$.

One may show in a similar fashion that $\inf S = \max\{b_i | 1 \leq i \leq n\}$. □

7. Suppose that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to -\infty} g(x) = a$. Suppose that for some positive number M, we have $g(x) \neq a$ for $x < -M$. Prove that $\lim_{x \to -\infty} f(g(x)) = \infty$.

Proof. Since $\lim_{x \to a} f(x) = \infty$, for any $A > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x)| > A$. For this δ there exists $\widetilde{M} > 0$ such that $x < -\widetilde{M} \Rightarrow |g(x) - a| < \delta$. Let $K = \max\{\widetilde{M}, M\}$. Then for $x < -K$, $|g(x) - a| < \delta$ and thus $|f(g(x))| > A$. Hence $\lim_{x \to -\infty} f(g(x)) = \infty$. □

8. If $f(x)$ is continuous on $[a,b]$, if $a < c < d < b$, and $M = f(c) + f(d)$, prove there exists a number ξ between a and b such that $M = 2f(\xi)$.

Proof. Let $M = f(c) + f(d)$. If $f(c) \leq f(d)$, then $\frac{M}{2} = \frac{f(c)+f(d)}{2} \leq \frac{f(d)+f(d)}{2} = f(d)$, and $\frac{M}{2} = \frac{f(c)+f(d)}{2} \geq \frac{f(c)+f(c)}{2} = f(c)$. By the Intermediate Value Theorem (since $f(c) \leq \frac{M}{2} \leq f(d)$), there exists ξ between c and d (hence between a and b) such that $f(\xi) = \frac{M}{2}$. Thus $M = 2f(\xi)$. □

9. Suppose that $f(x)$ and $g(x)$ are functions defined for $x > 0$, $\lim_{x\to 0^+} g(x)$ exists and is finite, and $|f(b) - f(a)| \le |g(b) - g(a)|$ for all positive real number a and b. Prove that $\lim_{x\to 0^+} f(x)$ exists and is finite.

Proof. Suppose that $\lim_{x\to 0^+} g(x) = L$ and $|f(b) - f(a)| \le |g(b) - g(a)|$. If $\lim_{x\to 0^+} f(x)$ does not exist, there exists $\varepsilon > 0$ and two sequences $x_n > 0$, $z_n > 0$ such that $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} z_n = 0$ and for all n $|f(x_n) - f(z_n)| \ge \varepsilon$. For this $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < x < \delta \Rightarrow |g(x) - L| < \frac{\varepsilon}{2}$. Now for $0 < \widehat{x_1}, \widehat{x_2} < \delta$,

$$\begin{aligned} |g(\widehat{x_1}) - g(\widehat{x_2})| &= |g(\widehat{x_1}) - L - g(\widehat{x_2}) + L| \\ &\le |g(\widehat{x_1}) - L| + |g(\widehat{x_2}) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

For this δ there exists N such that $n > N$, $0 < x_n, z_n < \delta$. Hence for $n > N$ $|f(x_n) - f(z_n)| \le |g(x_n) - g(z_n)| < \varepsilon$, a contradiction. Thus $\lim_{x\to 0+} f(x)$ exists. □

10. Evaluate $\lim_{n\to\infty} \dfrac{2 + 2^{\frac{1}{2}} + 2^{\frac{1}{3}} + \cdots + 2^{\frac{1}{n}}}{n}$.
(You need not give a proof but you should show some work or justification. Quote a theorem or what have you. Calculator results or graphical analysis are not acceptable.)

Proof. Let $s_n = \dfrac{2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}}}{n}$. Then $1 < \dfrac{n2^{\frac{1}{n}}}{n} < s_n < \dfrac{n \cdot 2}{n} = 2$. Notice $\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$. Thus s_n is bounded. Moreover, we claim that $s_{n+1} < s_n$. Suppose the contrary, that is $s_{n+1} \ge s_n$. Then

$$\frac{(2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}} + 2^{\frac{1}{n+1}})}{n+1} \ge \frac{(2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}})}{n}.$$

Hence

$$n(2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}}) + n2^{\frac{1}{n+1}} \ge n(2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}}) + 2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}}$$

or

$$n2^{\frac{1}{n+1}} \ge 2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n}} \ge n2^{\frac{1}{n}}, \quad n > 1$$

which is a contradiction.

Hence $\{s_n\}$ is a monotone bounded sequence and thus is converges by the Bolzano-Weierstrass Theorem. Moveover, $\lim_{n\to\infty} s_n = 1$. □