1. Use the definition of a limit to show that 

(a) \( \lim_{n \to \infty} \frac{\sin n}{n} = 0 \)

*Proof.* Let \( \varepsilon > 0 \) be given. Define \( N > \frac{1}{\varepsilon} \), where \( N \) is a positive integer. Then for 

\[ n > N, \quad \left| \frac{\sin n}{n} - 0 \right| < \frac{1}{n} < \frac{1}{N} < \varepsilon. \]

Hence \( \lim_{n \to \infty} \frac{\sin n}{n} = 0 \). \( \square \)

(b) \( \lim_{n \to 3^+} \frac{1}{x - 3} = \infty \)

*Proof.* Let \( A > 0 \) be given. Define \( \delta = \frac{1}{A} \). Then for \( x - 3 < \delta \), \( f(x) = \frac{1}{x - 3} > \frac{1}{\delta} = A. \)

Hence \( \lim_{x \to 3^+} \frac{1}{x - 3} = \infty \). \( \square \)

2. (a) Let \( \{x_n\} \) be a bounded sequence of real numbers and \( \{x_{k_n}\} \) be a monotone subsequence. Prove that \( \{x_{k_n}\} \) converges to a limit.

*Proof.* By Bolzano-Weierstrass Theorem, the bounded subsequence \( \{y_n\} = \{x_{k_n}\} \) has a convergent subsequence \( \{y_{n_l}\} \) which converges to \( x_0 \). Now either \( \{x_{k_n}\} \) is nondecreasing or nonincreasing. Without loss of generality, assume that \( \{y_{n_l}\} \) is monotonically nondecreasing. Given \( \varepsilon > 0 \), there exists \( N \) such that \( n > N \) implies \( x_0 - \varepsilon < y_{n_l} < x_0 + \varepsilon \). In particular, \( x_0 - \varepsilon < y_{l_{N+1}} < x_0 + \varepsilon \). For \( n > l_{N+1} \) we have \( x_0 - \varepsilon < y_{l_{N+1}} < y_n < x_0 + \varepsilon \). Thus \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} x_{k_n} = x_0 \). \( \square \)

(b) If \( \lim_{n \to \infty} x_n = L = \emptyset \), prove that \( \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{L} \)

*Proof.* Let \( \varepsilon_1 = \frac{|L|}{2} \). Then there exists \( N_1 > 0 \) such that for \( n > N_1, \ |x_n - L| < \frac{|L|}{2} \). Consequently, for \( n > N_1 \),

\[ |x_n| = |x_n - L + L| \geq |L| - |x_n - L| > |L| - \frac{|L|}{2} = \frac{|L|}{2}. \]
For any given \( \varepsilon > 0 \) there exists \( N_2 > 0 \) such that for \( n > N_2, |x_n - L| < \frac{2\varepsilon}{L^2} \).

Let \( N = \max\{N_1, N_2\} \). Then for \( n > N \),

\[
\left| \frac{1}{x_n} - \frac{1}{L} \right| = \frac{|x_n - L|}{|x_n||L|} \leq \frac{|x_n - L|}{L^2/2} < \frac{2\varepsilon/L^2}{L^2/2} = \varepsilon.
\]

Thus \( \lim_{n \to \infty} \frac{1}{x_n} = L \).

\[\square\]

3. Let \( x_{n+1} = \frac{1}{3 + x_n}, x_1 > 0 \). Prove that the sequence \( \{x_n\} \) converges and then compute the limit of the sequence.

**First Proof (My Preference).** Since \( x_1 > 0 \), \( x_{n+1} = \frac{1}{3 + x_n} < \frac{1}{3} \) for all \( n \geq 1 \). Moreover, \( x_{n+1} = \frac{1}{3 + x_n} < \frac{1}{3 + \frac{1}{3}} = \frac{3}{10} \) for all \( n \geq 1 \). Hence \( \frac{3}{10} < x_n < \frac{1}{3} \) for all \( n > 1 \).

(b) If \( \lim_{n \to \infty} x_n = L \) exists, then \( L = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \frac{1}{3 + x_n} = \frac{1}{\lim_{n \to \infty} (3 + x_n)} = \frac{1}{3 + L} \)

\[
\frac{1}{3 + L} = L. \tag{1}
\]

Thus

\[
L^2 + 3L - 1 = 0. \tag{2}
\]

(a) If \( x_1 > L \), then \( x_2 = \frac{1}{3 + x_1} < \frac{1}{3 + L} = L \) from (1). More generally, if \( x_{2n-1} > L \), then \( x_{2n} = \frac{1}{3 + x_{2n-1}} < \frac{1}{3 + L} = L \). On the other hand, if \( x_1 < L \), then \( x_{2n-1} < L \) and \( x_{2n} > L \). Without loss of generality, assume \( x_1 > L \). Define the sequence \( \{s_n\} \) as \( s_n = |x_n - L| \). Then claim that \( s_{n+1} < s_n \) for all \( n \geq 1 \). If not, then \( \frac{s_{n+1}}{s_n} \geq 1 \).

Assume \( n = 2k \). Then

\[
\frac{s_{2k+1}}{s_{2k}} = \frac{x_{2k+1} - L}{L - x_{2k}} = \frac{\frac{1}{3+x_{2k}} - L}{L - x_{2k}} \geq 1
\]

\[
1 - 3L - Lx_{2k} \geq 3L - 3x_{2k} + Lx_{2k} - x_{2k}^2
\]

\[
(x_{2k}^2 - 2Lx_{2k} + L^2) - L^2 - 6L + 3x_{2k} + 1 \geq 0
\]

\[
(x_{2k} - L)^2 - 1 - 3L + 3x_{2k} + 1 \geq 0 \quad \text{(using (2))}
\]

\[
(x_{2k} - L)[x_{2k} - L + 3] \geq 0.
\]
Since $x_{2k} - L < 0$, $x_{2k} - L + 3 \leq 0$ or $x_{2k} \leq L - 3 < 0$, a contradiction. A similar conclusion is obtained for $n = 2k - 1$.

Since $s_n$ is bounded and monotonically decreasing, $\lim s_n = \alpha$. Thus $\lim s_{2n} = \alpha = \lim s_{2n+1}$. For the case $x_1 > L$, $x_{2n+1} > L$ and $x_{2n} < L$ and consequently

(i) $s_{2n} = L - x_{2n} \to \alpha \Rightarrow x_{2n} \to L - \alpha$ as $n \to \infty$,

(ii) $s_{2n+1} = x_{2n+1} - L \to \alpha \Rightarrow x_{2n+1} \to L + \alpha$ as $n \to \infty$.

Now

$$\lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} \frac{1}{3 + x_{2n}} \Rightarrow L + \alpha = \frac{1}{3 + L - \alpha}$$

$$L^2 + 3L - 1 + 3\alpha + \alpha L - \alpha^2 - \alpha L = 0$$

$$\alpha(3 - \alpha) = 0 \Rightarrow \alpha = 0 \text{ or } \alpha = 3.$$ 

If $\alpha = 3$, $L - \alpha < 0$ and $L + \alpha > \frac{1}{3}$, a contradiction. Hence $\alpha = 0$ and thus $\lim_{n \to \infty} x_n = L$. 

\[\square\]

\textit{Second Proof.} Let $f(x) = \frac{1}{3 + x}$. Then $f'(x) = -\frac{1}{(3 + x)^2}$. Since $x > 0$, $|f'(x)| = \frac{1}{(3 + x)^2} \leq \frac{1}{3}$ for all $x > 0$. Now\(^1\)

$$\frac{|x_2 - L|}{|x_1 - L|} = \frac{|f(x_1) - L|}{|x_1 - L|} = |f'(\xi_1)| \leq \frac{1}{3}$$

by the Mean Value Theorem, where $\xi_1$ is between $x_1$ and $L$. $|x_2 - L| \leq \frac{1}{3}|x_1 - L|$. Similarly,

$$\frac{|x_3 - L|}{|x_2 - L|} = \frac{|f(x_2) - L|}{|x_2 - L|} = |f'(\xi_2)| \leq \frac{1}{3},$$

$\xi_2$ between $x_2$ and $L$. $|x_3 - L| \leq \frac{1}{3}|x_2 - L| \leq \left(\frac{1}{3}\right)^2 |x_1 - L|$. By induction, $|x_n - L| \leq \left(\frac{1}{3}\right)^{n-1} |x_1 - L| \to 0$ as $n \to \infty$. Thus $\lim_{n \to \infty} x_n = L$. \[\square\]

4. Prove that the function $f(x) = \frac{1}{x^2}$ is continuous for all real numbers $x \neq 0$.

\(^1\)\(L\) is defined as the fixed point of $f$, i.e., $f(L) = L = \frac{1}{3 + L} \Rightarrow L^2 + 3L - 1 = 0 \Rightarrow L = \frac{-3 + \sqrt{13}}{2}$.
Method 1. Let $\varepsilon > 0$ be given. Let $x_0 \in \mathbb{R} \setminus \{0\}$. Put $\delta = \frac{|x_0|}{2}$. Then if $|x - x_0| < \delta = \frac{|x_0|}{2}$, then $\frac{|x_0|}{2} < x < \frac{3|x_0|}{2}$. Let $\delta_2 = \frac{\varepsilon}{10|x_0|x_0^2}$. For $\delta = \min(\delta_1, \delta_2)$, if $|x - x_0| < \delta$, then

\[
\frac{1}{x^2} - \frac{1}{x_0^2} = \frac{|x^2 - x_0^2|}{x_0^2 x^2} = \frac{|x - x_0||x + x_0|}{x_0^2 x^2} < \frac{|x - x_0|5|x_0|}{x_0^2 x_0^2} \leq \frac{|x - x_0| \cdot 10}{|x_0|x_0^2} = \varepsilon
\]

\[\square\]

Method 2. Consider the function $g(x) = x^2$. Given $\varepsilon > 0$, let $\delta_1 = \frac{|x_0|}{2}$. Then $\frac{|x_0|}{2} < x < \frac{3|x_0|}{2}$ for all $|x - x_0| < \delta_1$. Let $\delta_2 = \frac{2\varepsilon}{5|x_0|}$. Then for $\delta = \min(\delta_1, \delta_2)$, and $x \in (x_0 - \delta, x_0 + \delta)$, we have

\[
|x^2 - x_0^2| = |x - x_0||x + x_0| < |x - x_0|\frac{5|x_0|}{2} \leq \varepsilon.
\]

Then use a theorem about the reciprocal of a continuous function is continuous. \[\square\]

5. Suppose that $f$ is continuous on $[a, b]$, one-to-one (if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$), and $f(a) < f(b)$. Prove that $f$ is monotonically increasing (strictly) on $[a, b]$.

Proof. Suppose that $f$ is not monotonically increasing on $[a, b]$.

\[\text{Diagram:}
\]
Then there exists $x_1, x_2 \in [a, b]$ with $x_1 < x_2$ but $f(x_1) > f(x_2)$. We have two cases to consider:

Case (i): If $f(a) < f(x_1)$, then $f(a)$ lies between $f(x_1)$ and $f(x_2)$. Hence by the Intermediate Value Theorem, there exists $c$ between $x_1$ and $x_2$ with $f(c) = f(a)$, a contradiction to the assumption that $f$ is one-to-one.

Case (ii): If $f(a) > f(x_1)$, then since $f(a) < f(b)$, the value $f(a)$ lies between $f(x_1)$ and $f(b)$. Again, by the Intermediate Value Theorem, there exists $d$ between $x_1$ and $b$ with $f(d) = f(a)$, a contradiction. Hence $f$ is increasing.  

6. Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}^1$ and that $S = S_1 \cap S_2 \cap \cdots \cap S_n$, $S \neq \emptyset$. Let $B_i = \sup S_i$, $b_i = \inf S_i$, $1 \leq i \leq n$. Find a formula relating $\sup S$ and $\inf S$ in terms of the $\{b_i\}$ and $\{B_i\}$.

**Proof.** $S = \cap_{i=1}^n S_i$. Claim that $\sup S = \min \{B_i : 1 \leq i \leq n \}$. Let $B_r = \min \{B_i : 1 \leq i \leq n \}$. If $x \in S$, then $x \in S_r$, and hence $x \leq B_r$. Thus $\sup S \leq B_r$. On the other hand, $\sup S \geq x \in S_r$ and by the definition of $B_r$, $\sup S \geq B_r$. Hence $\sup S = B_r$.

One may show in a similar fashion that $\inf S = \max \{b_i : 1 \leq i \leq n \}$.  

7. Suppose that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to -\infty} g(x) = a$. Suppose that for some positive number $M$, we have $g(x) \neq a$ for $x < -M$. Prove that $\lim_{x \to -\infty} f(g(x)) = \infty$.

**Proof.** Since $\lim_{x \to a} f(x) = \infty$, for any $A > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x)| > A$. For this $\delta$ there exists $\tilde{M} > 0$ such that $x < -\tilde{M} \Rightarrow |g(x) - a| < \delta$. Let $K = \max \{\tilde{M}, M \}$. Then for $x < -K$, $|g(x) - a| < \delta$ and thus $|f(g(x))| > A$. Hence $\lim_{x \to -\infty} f(g(x)) = \infty$.

8. If $f(x)$ is continuous on $[a, b]$, if $a < c < d < b$, and $M = f(c) + f(d)$, prove there exists a number $\xi$ between $a$ and $b$ such that $M = 2f(\xi)$.

**Proof.** Let $M = f(c) + f(d)$. If $f(c) \leq f(d)$, then $\frac{M}{2} = \frac{f(c) + f(d)}{2} \leq \frac{f(d) + f(d)}{2} = f(d)$, and $\frac{M}{2} = \frac{f(c) + f(d)}{2} \geq \frac{f(c) + f(c)}{2} = f(c)$. By the Intermediate Value Theorem (since $f(c) \leq \frac{M}{2} \leq f(d)$), there exists $\xi$ between $c$ and $d$ (hence between $a$ and $b$) such that $f(\xi) = \frac{M}{2}$. Thus $M = 2f(\xi)$. 


9. Suppose that \( f(x) \) and \( g(x) \) are functions defined for \( x > 0 \), \( \lim_{x \to 0^+} g(x) \) exists and is finite, and \( |f(b) - f(a)| \leq |g(b) - g(a)| \) for all positive real number \( a \) and \( b \). Prove that \( \lim_{x \to 0^+} f(x) \) exists and is finite.

**Proof.** Suppose that \( \lim_{x \to 0^+} g(x) = L \) and \( |f(b) - f(a)| \leq |g(b) - g(a)| \). If \( \lim_{x \to 0^+} f(x) \) does not exist, there exists \( \varepsilon > 0 \) and two sequences \( x_n > 0, z_n > 0 \) such that \( \lim_{n \to \infty} x_n = 0, \lim_{n \to \infty} z_n = 0 \) and for all \( n \) \(|f(x_n) - f(z_n)| \geq \varepsilon \). For this \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( 0 < x < \delta \Rightarrow |g(x) - L| < \frac{\varepsilon}{2} \). Now for \( 0 < \tilde{x}_1, \tilde{x}_2 < \delta \),

\[
|g(\tilde{x}_1) - g(\tilde{x}_2)| = |g(\tilde{x}_1) - L - g(\tilde{x}_2) + L| \\
\leq |g(\tilde{x}_1) - L| + |g(\tilde{x}_2) - L| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]

For this \( \delta \) there exists \( N \) such that \( n > N, 0 < x_n, z_n < \delta \). Hence for \( n > N \) \(|f(x_n) - f(z_n)| \leq |g(x_n) - g(z_n)| < \varepsilon \), a contradiction. Thus \( \lim_{x \to 0^+} f(x) \) exists. \( \square \)

10. Evaluate \( \lim_{n \to \infty} \frac{2 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n}}{n} \).

(You need not give a proof but you should show some work or justification. Quote a theorem or what have you. Calculator results or graphical analysis are not acceptable.)

**Proof.** Let \( s_n = \frac{2 + \frac{1}{2} + \cdots + \frac{1}{n}}{n} \). Then \( 1 < \frac{n2^{\frac{1}{n}}}{n} < s_n < \frac{n \cdot 2^{\frac{1}{n}}}{n} = 2 \). Notice \( \lim_{n \to \infty} 2^{\frac{1}{n}} = 1 \). Thus \( s_n \) is bounded. Moreover, we claim that \( s_{n+1} < s_n \). Suppose the contrary, that is \( s_{n+1} \geq s_n \). Then

\[
\left(2 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) \geq \frac{(2 + \frac{1}{2} + \cdots + \frac{1}{n})}{n+1} \geq \frac{(2 + \frac{1}{2} + \cdots + \frac{1}{n})}{n}.
\]

Hence

\[
n(2 + \frac{1}{2} + \cdots + \frac{1}{n}) + n2^{\frac{1}{n+1}} \geq n(2 + \frac{1}{2} + \cdots + \frac{1}{n}) + 2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n+1}}
\]

or

\[
n2^{\frac{1}{n+1}} \geq 2 + 2^{\frac{1}{2}} + \cdots + 2^{\frac{1}{n+1}} \geq n2^{\frac{1}{n}}, \quad n > 1
\]

which is a contradiction.

Hence \( \{s_n\} \) is a monotone bounded sequence and thus is converges by the Bolzano-Weierstrass Theorem. Moreover, \( \lim_{n \to \infty} s_n = 1 \). \( \square \)