Real Analysis
Test 2
Fall 2004
SOLUTIONS

1. (20 points) Suppose that f and g are increasing on an interval I and that $f(x) > g(x)$ for all $x \in I$. Denote the inverses of f and g by F and G and their domains by J_1 and J_2, respectively. Prove that $F(x) < G(x)$ for each $x \in J_1 \cap J_2$.

   Proof. Let $x_0 \in J_1 \cap J_2$. Then there exist $a_1, a_2 \in I$ such that $f(a_1) = x_0 = g(a_2)$. Equivalently, $F(x_0) = a_1$, $G(x_0) = a_2$. Claim that $a_1 = F(x_0) < G(x_0) = a_2$. If not, then $a_1 \geq a_2$. Now if $a_1 = a_2$, then $x_0 = f(a_1) > g(a_1) = g(a_2) = x_0$, which is absurd. On the other hand, if $a_1 > a_2$, then $x_0 = f(a_1) > f(a_2) > g(a_2) = x_0$, yet another contradiction. This proves our claim that for $x_0 \in J_1 \cap J_2$, $F(x_0) < G(x_0)$. □

2. (a) (10 points) $\lim\limits_{x \to \frac{\pi}{2}} \sec x - \tan x$

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = 0$$

   (b) (10 points) $\lim\limits_{x \to 0^+} x^x$

   Let

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \lim_{x \to 0^+} \ln y &= \ln \lim_{x \to 0^+} y \\ &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \text{(since ln is continuous).} \end{aligned}$$

   Hence

$$\begin{aligned} \ln \lim_{x \to 0^+} y &= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{-x^2}{1} = 0 \\ \lim_{x \to 0} y &= e^0 = 1. \end{aligned}$$

3. (20 points) Suppose that f and g are uniformly continuous on a subset S of $\mathbb{R}$ (not a closed interval). Prove that the function $h = f - g$ is uniformly continuous.

Proof. Since f, g are uniformly continuous on $S \subset \mathbb{R}$, given $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that for $x_1, x_2 \in S$ with

$$|x_1 - x_2| < \delta_1 \Rightarrow |f(x_1) - f(x_2)| < \frac{\varepsilon}{2} \tag{1}$$
$$|x_1 - x_2| < \delta_2 \Rightarrow |g(x_1) - g(x_2)| < \frac{\varepsilon}{2}. \tag{2}$$

Let $\delta = \min(\delta_1, \delta_2)$. Then (1) and (2) hold if δ_1, δ_2 are replaced by S. Now for $|x_1 - x_2| < \delta$, we have

$$\begin{aligned} |h(x_1) - h(x_2)| &= |f(x_1) - g(x_1) - f(x_2) + g(x_2)| \\ &= |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus h is uniformly continuous on S. □

4. Give examples of

(i) (5 points) A bounded sequence that is not Cauchy.

Consider the sequence $\{(-1)^n\}$, $n = 1, 2, 3, \ldots$. It is bounded but not Cauchy.

(ii) (5 points) A nested sequence $\{I_n\}$ of intervals such that $\bigcap_{n=1}^{\infty} = I_n = \emptyset$

Let $I_n = \left(0, \frac{1}{n}\right)$, $n = 1, 2, 3, \ldots$. Then $I_n \supset I_{n+1}$. But $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(iii) (5 points) A continuous but not uniformly continuous function.

$f(x) = \frac{1}{x}$ on $(0, \infty)$ is continuous but not uniformly continuous.

(iv) (5 points) A Cauchy sequence $\{x_n\}$ and a function f such that $\{f(x_n)\}$ is not Cauchy.

Define $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2, & x = 0 \end{cases}$ on $[0, \infty)$. Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ as $n \to \infty$, but $f(x_n) = n \to \infty \neq f(0) = 2$.

5. Prove directly from the definition that the function $f(x) = x^4$ is uniformly continuous on the closed interval $[0, 1]$. Is f uniformly continuous on $\mathbb{R}$?

Proof. (a) We will show that f is continuous at each point $x \in [0, 1]$. Let $x_0 \in [0, 1]$ and $\varepsilon > 0$ be given. Put $\delta = \frac{\varepsilon}{4}$. Then for $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &= |x^4 - x_0^4| \\ &= |x - x_0||x + x_0||x^2 + x_0^2| \\ &< \delta \cdot 2 \cdot 2 = \varepsilon. \end{aligned}$$

Hence f is continuous on x_0. By Theorem 3.13, f is uniformly continuous on $[0, 1]$.

(b) Let $x_n = n$, $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$. However,

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left| n^4 + 4\frac{n^3}{n} + 6\frac{n^2}{n^2} + 4\frac{n}{n^3} + \frac{1}{n^4} - n^4 \right| \\ &= \left| 4n^2 + 6 + \frac{4}{n^2} + \frac{1}{n^4} \right| \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence f is not uniformly continuous on $\mathbb{R}$.

□