1. Suppose that $f$ and $g$ are positive and continuous on $I = \{x : a \leq x \leq b\}$. Prove that there is a number $\xi \in I$ such that
\[
\int_a^b f(x)g(x) \, dx = f(\xi) \int_a^b g(x) \, dx.
\]

2. A function $f$ defined on an interval $I$ is called a step-function if and only if $I$ can be subdivided into a finite number of subintervals $I_1, I_2, \ldots, I_n$ such that $f(x) = c_i$ for all $x$ interior to $I_i$ where the $c_i, i = 1, 2, \ldots, n$, are constants. Prove that every step-function is integrable (whatever values $f(x)$ has at the endpoints of the $I_i$) and find a formula for the value of the integral.

3. (a) Give an example of a function $f$ such that $|f|$ is integrable but $f$ is not integrable.
(b) Give an example of a function $f$ such that $f^2$ is integrable but $f$ is not integrable.
(c) Give an example of a metric space in which there exists $r > 0$ such that the closure of the open ball $B(p_0, r)$ is not equal to the closed ball $B(p_0, r) = \{p : d(p, p_0) \leq r\}$.
(d) Give an example of two metrics on a set $S$ that are not equivalent.

4. Show that
\[
\int_a^c f(x) \, dx = \int_a^a f(x) \, dx + \int_a^c f(x) \, dx
\]
whether or not $b$ is between $a$ and $c$ so long as all three integrals exist.

5. (a) Given the function $f : x \to x^3$ defined on $I = \{x : 0 \leq x \leq 1\}$. Suppose $\Delta$ is a subdivision and $\Delta'$ is a refinement of $\Delta$ which adds one more point. Show that
\[
S^+(f, \Delta') < S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') > S_-(f, \Delta).
\]
(b) Give an example of the function $f$ defined on $I$ such that
\[
S^+(f, \Delta') = S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') = S_-(f, \Delta)
\]
for the two subdivisions in Part (a).
(c) If $f$ is a strictly increasing continuous function on $I$ show that $S^+(f, \Delta') < S^+(f, \Delta)$ where $\Delta'$ is any refinement of $\Delta$. 
6. Prove that
\[ \lim_{n \to \infty} \left[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right] = \ln 2. \]

*(Hint: Use the Fundamental Theorem of Calculus)*

7. Suppose that \( f \) is continuous on an interval \( I = \{ x : a \leq x \leq b \} \) with \( f(x) > 0 \) on \( I \). Let \( S = \{ (x, y) \in \mathbb{R}^2 : a \leq x \leq b, 0 \leq y \leq f(x) \} \) (Euclidean metric).

(a) Show that \( S \) is closed.
(b) Find \( S' \) and \( \bar{S} \).
(c) Find \( S^{(0)} \) and prove the result.
(d) Find \( \partial S \).

8. Let \( A_1, A_2, \ldots, A_n, \ldots \) be sets in a metric space. Define \( B = \bigcup A_i \). Show that \( B = \bigcup \bar{A}_i \) and give an example to show that \( B \) may not equal \( \bigcup \bar{A}_i \).

9. Let \( d \) be a metric on a nonempty set \( S \). Let \( \tilde{d}(x, y) = \min(1, d(x, y)) \), where \( x, y \in S \).

(a) Show that \( \tilde{d} \) is a metric on \( S \).
(b) Show that \( d \) and \( \tilde{d} \) are equivalent.

10. Let \( S \) be a set and \( d \) a function from \( S \times S \) into \( \mathbb{R}^1 \) with the properties:

(i) \( d(x, y) = 0 \) if and only if \( x = y \).
(ii) \( d(x, z) \leq d(x, y) + d(z, y) \) for all \( x, y, z \in S \).

Show that \( d \) is a metric and hence that \( (S, d) \) is a metric space.