Real Analysis
Test 3
Fall 2004

1. Suppose that f and g are positive and continuous on $I = \{x : a \le x \le b\}$. Prove that there is a number $\xi \in I$ such that
$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

2. A function f defined on an interval I is called a **step-function** if and only if I can be subdivided into a finite number of subintervals $I_1, I_2, \ldots, I_n$ such that $f(x) = c_i$ for all x interior to I_i where the c_i, $i = 1, 2, \ldots, n$, are constants. Prove that every step-function is integrable (whatever values $f(x)$ has at the endpoints of the I_i) and find a formula for the value of the integral.

3. (a) Give an example of a function f such that $|f|$ is integrable but f is not integrable.
   (b) Give an example of a function f such that f^2 is integrable but f is not integrable.
   (c) Give an example of a metric space in which there exists $r > 0$ such that the closure of the open ball $B(p_0, r)$ is not equal to the closed ball $\overline{B(p_0, r)} = \{p : d(p, p_0) \le r\}$.
   (d) Give an example of two metrics on a set S that are not equivalent.

4. Show that
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$
whether or not b is between a and c so long as all three integrals exist.

5. (a) Given the function $f : x \to x^3$ defined on $I = \{x : 0 \le x \le 1\}$. Suppost Δ is a subdivision and Δ' is a refinement of Δ which adds one more point. Show that
$$S^+(f, \Delta') < S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') > S_-(f, \Delta).$$
   (b) Give an example of the function f defined on I such that
$$S^+(f, \Delta') = S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') = S_-(f, \Delta)$$
for the two subdivisions in Part (a).
   (c) If f is a strictly increasing continuous function on I show that $S^+(f, \Delta') < S^+(f, \Delta)$ where Δ' is any refinement of Δ.

6. Prove that
$$\lim \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right] = \ln 2.$$
(*Hint:* Use the Fundamental Theorem of Calculus)

7. Suppose that f is continuous on an interval $I = \{x : a \le x \le b\}$ with $f(x) > 0$ on I. Let $S = \{(x, y) \in \mathbb{R}^2 : a \le x \le b, 0 \le y \le f(x)\}$ (Euclidean metric).

   (a) Show that S is closed.

   (b) Find S' and $\bar{S}$.

   (c) Find $S^{(0)}$ and prove the result.

   (d) Find ∂S.

8. Let $A_1, A_2, \ldots, A_n, \ldots$ be sets in a metric space. Define $B = \bigcup A_i$. Show that $\bar{B} \supset \bigcup \bar{A}_i$ and give an example to show that $\bar{B}$ may not equal $\bigcup \bar{A}_i$.

9. Let d be a metric on a nonempty set S. Let $\tilde{d}(x, y) = \min(1, d(x, y))$, where $x, y \in S$.

   (a) Show that $\tilde{d}$ is a metric on S.

   (b) Show that d and $\tilde{d}$ are equivalent.

10. Let S be a set and d a function from $S \times S$ into $\mathbb{R}^1$ with the properties:

    (i) $d(x, y) = 0$ if and only if $x = y$.

    (ii) $d(x, z) \le d(x, y) + d(z, y)$ for all $x, y, z \in S$.

    Show that d is a metric and hence that (S, d) is a metric space.