**Real Analysis**
**Test 3**
**Fall 2004**

1. Suppose that $f$ and $g$ are positive and continuous on $I = \{x : a \le x \le b\}$. Prove that there is a number $\xi \in I$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

*Proof.* Suppose that $f$ and $g$ are positive and continuous on $[a, b]$. By Theorem 5.3(c),

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx$$

where $m = \min\{f(x) : a \le x \le b\}$, $M = \max\{f(x) : a \le x \le b\}$. Since $f$ is continous, there exists $x_m, x_M \in [a, b] \ni m = f(x_m), M = f(x_M)$. Hence

$$f(x_m) \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le f(x_M).$$

By the Intermediate Value Theorem, there exists $\xi \in [a, b] \ni \int_a^b f(x)g(x)\,dx / \int_a^b g(x)\,dx \le f(\xi)$. □

2. A function $f$ defined on an interval $I$ is called a **step-function** if and only if $I$ can be subdivided into a finite number of subintervals $I_1, I_2, \ldots, I_n$ such that $f(x) = c_i$ for all $x$ interior to $I_i$ where the $c_i$, $i = 1, 2, \ldots, n$, are constants. Prove that every step-function is integrable (whatever values $f(x)$ has at the endpoints of the $I_i$) and find a formula for the value of the integral.

*Proof.* Let $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, where $f_i(x) = c_i$ defined on $I_i$. By Theorem 5.3(b)

$$\underline{\int_a^b} f(x)\,dx \ge \underline{\int_a^{x_1}} f_1(x)\,dx + \underline{\int_{x_1}^{x_2}} f_2(x)\,dx + \cdots + \underline{\int_{x_{n-1}}^{b}} f_n(x)\,dx \tag{1}$$

$$\overline{\int_a^b} f(x)\,dx \le \overline{\int_a^{x_1}} f_1(x)\,dx + \overline{\int_{x_1}^{x_2}} f_2(x)\,dx + \cdots + \overline{\int_{x_n}^{b}} f_n(x)\,dx \tag{2}$$

Now on each $I_i$, $f_i(x)$ is continuous and thus is integrable by Corollary 1, p. 106. Hence the terms in the right hand side of (1) and (2) are equal. Thus $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ and consequently $f$ is integrable. Moreover, by the Mean Value Theorem (Theorem 5.6), $\int_{x_{i-1}}^{x_i} f_i(x)\,dx = f(\xi)(x_i - x_{i-1}) = c_i l(I_i)$. Therefore, $\int_a^b f(x)\,dx = \sum_{i=1}^n c_i l(I_i)$. □

3. (a) Give an example of a function $f$ such that $|f|$ is integrable but $f$ is not integrable.

$$\text{Let} \quad f(x) = \begin{cases} 1 & \text{if } x \in Q \\ -1 & \text{if } x \notin Q \end{cases}$$

Then $f(x)$ is not integrable while $|f(x)|$ is.

(b) Give an example of a function $f$ such that $f^2$ is integrable but $f$ is not integrable.
Let $f(x) = \dfrac{1}{x}$ on $[1, \infty)$. Then $f$ is not integrable, however $f^2(x) = \dfrac{1}{x^2}$ is.

(c) Give an example of a metric space in which there exists $r > 0$ such that the closure of the open ball $B(p_0, r)$ is not equal to the closed ball $B\overline{(p_0, r)} = \{p : d(p, p_0) \leq r\}$.
Let $S = (-\infty, 0] \cup [1, \infty)$. Then $\overline{B(0, 1)} = [-1, 0] \cup \{1\}$. However, the closure of $B(0, 1) = [-1, 0]$.

(d) Give an example of two metrics on a set $S$ that are not equivalent.
On $\mathbb{R}^1$, let $d_1(x, y) = |x - y|$, $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.

4. Show that

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

whether or not $b$ is between $a$ and $c$ so long as all three integrals exist.

*Proof.* On page 103 in the book, the author proved that

$$\overline{\int_a^c} f(x)\, dx = \overline{\int_a^b} f(x)\, dx + \overline{\int_b^c} f(x)\, dx. \tag{3}$$

We now show that

$$\underline{\int_a^c} f(x)\, dx = \underline{\int_a^b} f(x)\, dx + \underline{\int_b^c} f(x)\, dx.$$

There exists a subdivision $\Delta_1$ of $[a, b]$ such that

$$\bar{S}(f, \Delta_1) > \underline{\int_a^b} f(x)\, dx - \frac{1}{2}\varepsilon.$$

Similarly, there exists a subdivision $\Delta_2$ of $[b, c]$ such that

$$\bar{S}(f, \Delta_2) > \underline{\int_b^c} f(x)\, dx - \frac{1}{2}\varepsilon.$$

Let $\Delta$ be the union of $\Delta_1$ and $\Delta_2$ on $[a, c]$. Then

$$\begin{aligned}\underline{\int_a^c} f(x)\,dx &\geq \bar{S}(f, \Delta) \\ &= \bar{S}(f, \Delta_1) + \bar{S}(f, \Delta_2) \\ &> \underline{\int_a^b} f(x)\,dx + \underline{\int_b^c} f(x)\,dx - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ can be made arbitrarily small,

$$\underline{\int_a^c} f(x)\,dx = \underline{\int_a^b} f(x)\,dx + \underline{\int_b^c} f(x)\,dx. \tag{4}$$

If $f$ is integrable, we combine (3) and (4) to get the desired result. □

5. (a) Given the function $f : x \to x^3$ defined on $I = \{x : 0 \leq x \leq 1\}$. Suppost $\Delta$ is a subdivision and $\Delta'$ is a refinement of $\Delta$ which adds one more point. Show that

$$S^+(f, \Delta') < S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') > S_-(f, \Delta).$$

*Proof.* Let $\Delta = \{x_0, x_1, \ldots, x_i, \ldots, x_n\}$, $\Delta' = \{x_0, x_1, \ldots, x_{i-1}, x_i', x_i, \ldots, x_n\}$ where $x_i'$ is the new point. Then

$$\begin{aligned}S^+(f, \Delta') = \sum_{j=1}^{i-1} M_j l(J_j) + \sum_{j=i+1}^{n} M_j l(I_i) \\ + M_i'(x_i' - x_i) + M_i''(x_1 - x_i')\end{aligned}$$

where $M_i' = \max\limits_{x_{i-1} \leq x \leq x_i'} f(x)$, $M_i'' = \max\limits_{x_i' \leq x \leq x_i} f(x)$. Notice that $M_i' \leq M_i$, $M_i'' \leq M_i$. Thus

$$S^+(f, \Delta') \leq \sum_{j=1}^{n} M_j l(J_j) = S^+(f, \Delta').$$

Similarly, $\underline{S}(f, \Delta') > \underline{S}(f, \Delta)$. □

(b) Give an example of the function $f$ defined on $I$ such that

$$S^+(f, \Delta') = S^+(f, \Delta) \quad \text{and} \quad S_-(f, \Delta') = S_-(f, \Delta)$$

for the two subdivisions in Part (a).
Let $f(x) = c$.

(c) If $f$ is a strictly increasing continuous function on $I$ show that $S^+(f, \Delta') < S^+(f, \Delta)$ where $\Delta'$ is any refinement of $\Delta$.

*Proof.* This is similar to (a). □

6. Prove that

$$\lim \left[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right] = \ln 2.$$

(*Hint:* Use the Fundamental Theorem of Calculus)

*Proof.* Let $f(x) = \dfrac{1}{x+1}$ defined on $[0, 1]$. Then

$$\int_0^1 f(x)\, dx = \int_0^1 \frac{1}{x+1}\, dx = \ln(x+1)\Big]_0^1 = \ln 2.$$

(By the Fundamental Theorem of Calculus.)

Now

$$\begin{aligned}\int_0^1 f(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x, \quad x_i = \frac{2}{n}, \quad \Delta x = \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{\frac{i}{n}+1} \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{i+n}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{i+n} = \ln 2.$$

□

7. Suppose that $f$ is continuous on an interval $I = \{x : a \le x \le b\}$ with $f(x) > 0$ on $I$. Let $S = \{(x, y) \in \mathbb{R}^2 : a \le x \le b, 0 \le y \le f(x)\}$ (Euclidean metric).

   (a) Show that $S$ is closed.
   We will show that $\mathbb{R}^2 - S$ is open. Let $(x_0, y_0) \notin S$. Define $l = \inf\{d((x_0, y_0), (x, y)) : (x, y) \in S\}$. If $y_0 < 0$, then $l = |y_0|$; and if $y_0 > 0$ and $x_0 < a$ or $x_0 > b$, then either $l \ge a - x_0$ or $\ge x_b - b$, respectively. The last case, $a \le x \le b$, $y_0 > 0$, $l > 0$ for otherwise $y_0 = f(x_0)$, a contradiction. In each case $B((x_0, y_0), r) \subset \mathbb{R}^2 - S$ for $r = \frac{l}{2}$.

(b) Find $S'$ and $\bar{S}$.

$$S' = S, \quad \bar{S} = S$$

(c) Find $S^{(0)}$ and prove the result.

$$S^0 = \{(x, y) : a < x < b, \quad 0 < y < f(x)\}$$

(d) Find $\partial S$.

$$\begin{aligned}\partial S = &\{(x, 0) : a \le x \le b\} \cup \{(a, y) : 0 \le y \le f(x), \quad a \le x \le b\} \\ &\cup \{(b, y) : 0 \le y \le f(x), \quad a \le x \le b\} \cup \{(x, y) : a \le x \le b, \quad y = f(x)\}\end{aligned}$$

8. Let $A_1, A_2, \ldots, A_n, \ldots$ be sets in a metric space. Define $B = \bigcup A_i$. Show that $\bar{B} \supset \bigcup \bar{A}_i$ and give an example to show that $\bar{B}$ may not equal $\bigcup \bar{A}_i$.

*Proof.* Let $x \in \cup_{i=1}^n \bar{A}_i$. Then $x \in \bar{A}_j$ for some $j$. Then either $x \in A_j$ or $x \in A_j'$. If $x \in A_j$, then $x \in \cup_{j=1}^n A_j = B \subset \bar{B}$. On the other hand if $x \in A_j'$, then there exists a sequence $\{a_n\}$ in $A_j$, $a_n \neq x$, such that $\lim_{n\to\infty} a_n = x$. Since $a_n \in A_j$, $a_n \in B$. Hence $x \in B' \subset \bar{B}$. Thus $x \in \bar{B}$. Hence $\bar{B} \supset \cup_{i=1}^n \bar{A}_i$. □

9. Let $d$ be a metric on a nonempty set $S$. Let $\tilde{d}(x, y) = \min(1, d(x, y))$, where $x, y \in S$.

(a) Show that $\tilde{d}$ is a metric on $S$.

*Proof.* $\bar{d}(x, y) = \min(1, d(x, y))$

i. $\bar{d}(x, y)$ is either equal 1 or $d(x, y)$ and in both cases $\bar{d}(x, y) \ge 0$. Now $\bar{d}(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$.

ii. $\bar{d}(x, y) = \min(1, d(x, y)) = \min(1, d(y, x)) = \bar{d}(y, x)$

iii. $\bar{d}(x, y) = \min(1, d(x, y)) \le \min(1, d(x, z)+d(z, y))$. We must show that $\min(1, d(x, z)+d(z, y)) \le \min(1, d(x, z)) + \min(1, d(z, y))$
Case (a) $d(x, z) + d(z, y) < 1$. Then $d(x, z) < 1$ and $(z, y) < 1$. Furthermore,

$$\begin{aligned}\min(1, d(x, z) + d(z, y)) &= d(x, z) + d(z, y) \\ \min(1, d(x, z)) &= d(x, z) \\ \min(1, d(z, y)) &= d(z, y).\end{aligned}$$

Therefore

$$\min(1, d(x, z) + d(z, y)) = \min(1, d(x, z)) + \min(1, d(z, y)).$$

Case (b) $d(x,z)+d(z,y) \geq 1$. Then

$$\min(1, d(x,z)+d(z,y)) = 1,$$
$$\min(1, d(x,z)) = \begin{cases} 1 & \text{if } d(x,z) > 1 \\ d(x,z) & \text{otherwise} \end{cases}$$
$$\min(1, d(z,y)) = \begin{cases} 1 & \text{if } d(y,z) > 1 \\ d(z,y) & \text{otherwise} \end{cases}.$$

In either case, $\min(1, d(x,z)+d(z,y)) \leq \min(1, d(x,z)) + \min(1, d(z,y))$.

□

(b) Show that $d$ and $\tilde{d}$ are equivalent.

*Proof.* $d$ is not equivalent to $\tilde{d}$. For if it is true, then there are positive constant $c_1$ and $c_2$ such that $c_1 d(x,y) \leq \tilde{d}(x,y) \leq c_2 d(x,y)$ for all $x, y \in S$.

i. If $d(x,y) < 1$, then $\tilde{d}(x,y) = d(x,y)$ and thus

$$c_1 d(x,y) \leq d(x,y) \leq c_2 d(x,y)$$

$$c_1 \leq 1 \leq c_2.$$

ii. If $d(x,y) > 1$, then $\tilde{d}(x,y) = 1$ and thus

$$c_1 d(x,y) \leq 1 \leq c_2 d(x,y)$$

$$c_1 \leq \frac{1}{d(x,y)}.$$

As $d(x,y)$ gets larger, $c_1$ has no positive lower bound but zero. If the set $S$ is bounded, then $d(x,y) \leq M$. Then $c_2 = 1$, $c_1 = \frac{1}{M}$ work and $d$ is equivalent to $\tilde{d}$.

□

10. Let $S$ be a set and $d$ a function from $S \times S$ into $\mathbb{R}^1$ with the properties:

(i) $d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,z) \leq d(x,y) + d(z,y)$ for all $x, y, z \in S$.

Show that $d$ is a metric and hence that $(S,d)$ is a metric space.

*Proof.* From (ii)

$$\begin{aligned} 0 = d(x,x) &\leq d(x,y) + d(x,y) \\ 2d(x,y) &\geq 0. \end{aligned}$$

Hence $d(x,y) \geq 0$ for all $x, y \in S$.

From (ii)

$$\begin{aligned} d(x,z) &\leq d(x,x) + d(z,x) = d(z,x) \\ d(z,x) &\leq d(z,z) + d(x,z) = d(x,z) \end{aligned}$$

Hence $d(x,z) = d(z,x)$ for all $x, z \in S$.

Now $d(x,z) \leq d(x,y) + d(z,y) = d(x,y) + d(y,z)$ from above.

Hence $d$ is a metric on $S$. □