Research Experiences for Undergraduates in Mathematics

1 Past REU Summer Programs

(I) Results from Current **REU-NSF Grant**.

Project Director: *Saber N. Elaydi*
Title: Undergraduate Research Experience in Mathematics
Award: DMS-9619837, Award amount: $72,512
Award period: May 15, 1997 - April 30, 2000

In the last three years, application packets were mailed to approximately 200 colleges and universities primarily in Texas, Louisiana, Oklahoma, Arkansas, and New Mexico. A limited number of packets were distributed to selected institutions of high quality outside this area. A web site was also constructed allowing students to directly download the application and reference forms. Participants for each session were selected using the criteria from the proposal: scholastic record, interest in mathematics, and motivation level. In 1997 these students were selected

Noelle Dexheimer	Texas A&M University
Kala Schrager	University of Oregon
John May	University of Oregon
Aaron Heap	Texas Christian University
Phillip Lynch	University of Washington

(i) *Noelle Dexheimer* worked with *Dr. Saphire* on some problems in the area of linkage analysis. Linkage analysis is the study of frequency with which a pair of genes from one grandparent recombine with the same pair of genes from another grandparent in a grandchild.

The linkage fraction. The linkage fraction, $\emptyset$, is defined as the probability that two given genes recombine. The method of maximum likelihood estimate (MLE) is used to estimate $\emptyset$. *Noelle* was able to show that in the case of a phase-known double backcross mating, (1) the bias in the MLE of $\emptyset$ does not change as the sample size increases from an even number to the next odd number; and (2) if the MLE of $\emptyset$ is non-zero for a particular odd sample size, then by increasing the sample size by one, the magnitude of the bias strictly decreases. In addition, she wrote several programs for computing and graphing the bias.

(ii) *Kala Schrager* worked with *Dr. Chapman* in a problem related to minimal systems. A nonempty finite sequence $S = \{g_1, \ldots, g_k\}$ of not necessarily distinct elements of G is called (1) a system of G if $\sum_{i=1}^{k} g_i = 0$, (2) a minimal system of G if S contains no proper subsystems. Let $U(G)$ represent the set of minimal systems of G and set $S_1 \sim S_2$ in G if there exists an automorphism φ of G such that $S_1 = \varphi(S_2)$. Let $N(G)$ denote the number of equivalence classes in $U(G)$ under $\sim$. *Kala* was able to prove the following theorem.

Theorem Let $G = \sum_{i=1}^{k} \mathbb{Z}_2$ be an elementary 2-group. Then $N(G) = t + 1$.

Her work led to the following interesting conjecture.

Conjecture

Let $G = \sum_{i=1}^{t} \mathbb{Z}_3$ be an elementary 3-group. Then $N(G) = t + \sum_{i=2}^{t} 2^i$.

Using Mathematica, *Ms. Schrager* computed $N(G)$ for various finite Abelian groups.

(iii) *John May, Aaron Heap,* and *Phillip Lynch* worked with *Drs. Elaydi* and *Hasfura.* They investigated the dynamical behavior of the family of tent maps $T_w(x) = w(0.5 - |x - 0.5|)$, for $w > 0$. The students determined completely the nature of the sequence $\{w_k\}$, where w_k is the infimum of the values of w for which the corresponding map has points of prime period k. Another startling result is the discovery of the existence of transitive attractors for some of these one-dimensional maps. Most of the results were first conjectured on the basis of vast empirical evidence obtained using Maple.

A joint paper co-authored by *Dr. Hasfura* and *Phillip Lynch* was later submitted for publication in a refereed mathematical journal.

<u>Student activities in the Summer of 1998</u>.

The following students were selected in the summer of 1998:

Michael McQuistan	University of Nebraska
Beck Cantonwine	Hanover College, Indiana
Jason Heller	Elizabethotown College, Pennsylvania
Jeremy Herr	University of Oklahoma
Michael McQuistan	University of Nebraska
Natalie Rooney	University of Texas

(i) *Becky Cantonwine* and *Jason Heller* worked with *Dr. Hasfura.* They considered the problem of characterizing periodic orbits of the family of trapezoidal maps

$$f_{ma}(x) \begin{cases} mx, & 0 \leq x \leq a \\ ma, & a < x \leq 2 - a \\ m(2 - x), & 2 - a < x \leq 2 \end{cases}$$

The students succeeded in proving parts of an interesting combinatorial description of the set of allowed types of periodic orbits.

(ii) *Michael McQuistan* worked under the supervision of *Dr. Bailey* who joined the REU team replacing *Drs. Elaydi* and *Saphire.* He focused on problems in Real Analysis and obtained the following result.

<u>Theorem</u> If $f(y) - f(x) = (y - x)g(x) + (y - x)^2 h(y + 2x)$, where h is continuous, then f is a cubic polynomial.

It is still an open question whether or not this result holds without the continuity of h.

(iii) *Jeremy Herr* and *Natalie Rooney* worked under the supervision of *Dr. Chapman* on problems involving factorization of elements into irreducible in monoids. Let Z be the set of integers and N the set of nonnegative integers. The basis of the project was to study the behavior of factorizations of elements into sums of irreducible elements in additive monoids of the type

$$M(a_1, \ldots, a_n) = \{x_1, \ldots, x_n\} \in N^n : \sum_{j=1}^{n} a_j x_j = 0\},$$

where $a_1, \ldots, a_n \in Z$. The students successfully found the number of nonassociated irreducible factorizations of $x \in M(a_1, \ldots, a_n)$ for all $x \in M$, for the monoids

$$M(1, 1, -1, -1),\ M(1, 1, -2),\ M(1, 1, 1, -2,),\ M(1, 1, 1, \ldots, 1, -2).$$

Moreover, they found a general formula for counting the number of nonassociated factorizations of an element into irreducible elements in an algebraic number ring of class number 2.

A joint paper written by *Herr, Rooney* and *Dr. Chapman* has been submitted for publication in a refereed mathematics journal.

Students activities in the summer of 1999

The following students were selected in the summer of 1999.

Matthew Westerhoff	University of Texas at San Antonio
Nick Neumann	University of Texas A& M University
Bryant Mathews	Howard University
Steven Steincke	University of Arizona
Vic DeLorenzo	Grove City College
Holly Swisher	University of Oregon

(i) *Bryant Mathews* and *Nick Neumann* worked under the supervision of *Dr. Elaydi.* They considered a discrete model of two competing species with planting of the form

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} x_1(e^{p_1 - q_1(x_1+x_2)}) + \alpha \\ x_2 e^{p_2 - q_2(x_1+x_2)} \end{pmatrix}$$

with parameters $p_1, q_1, p_2, q_2 > 0$ and planting coefficients $\alpha \in (0,1)$. The students first investigated the question of persistence of the two species. They showed that, with the possible exception of the case when $(p_1 - \ln(1-\alpha))/q_1 = p_2/q_2$ and $p_1 \leq 2/(1-\alpha) + \ln(1-\alpha)$, one of the species goes extinct. The second question they considered is the existence of a globally attracting 2-cycle. They proved the following result.

Theorem. The system has a 2-cycle if and only if $p_1/q_2 > (p_1 - \ln(1-\alpha)/q_1$. Moreover, if $p_1 = 1.5, q_1 = q_2 = 1$, and $\alpha = 0.5$, then the 2-cycle is asymptotically stable for $1.5 + \ln 2 \leq p_2 \leq 3.411822071$.

Partial results on the global attractivity of the 2-cycle were obtained. *Dr. Elaydi* is still in communication with the students and more progress has been made in recent weeks. These results will be written in a paper to be submitted publication.

(ii) *Steven Steincke* and *Matthew Westerhoff* worked under the supervision of *Dr. Hasfura.* They investigated the dynamics of the following family of maps:

$$f_{ab}(x) = \begin{cases} \left(\frac{1-a}{b}\right) x, & 0 \leq x \leq a \\ \frac{x-1}{1-2b} + \frac{b}{1-2b}, & a < x \leq 1-a \\ \frac{1-a}{b}(x-1) + 1, & 1-a < x \leq 1, \end{cases}$$

where $0 \leq a \leq 1$ and $0 < b < 1/2$. The students showed that (1) if $a \leq 1/2$, then f_{ab} is chaotic on [0,1], (2) if $1/2 < a < 1-a$, then f_{ab} has a global attractor on $([0,a] \subset [1-a,1])$ on which it is chaotic. A remarkable dichotomy concerning the existence of certain periodic orbits was obtained.

Theorem. If $1/4 < b < 1/2$, there exist numbers a_k increasing to (1-b) such that the map f_{ab} has a periodic orbit of length $2(2k+1)$ if and only if $a < a_k$. On the other hand, if $0 < b < 1$. and $0 < a < 1-b$, then the map $f_{a,b}$ has periodic orbits of length six.

Partial results on topological conjugacy between family members were established, for example between $f_{0,1/3}$ and $f_{0,2/5}$. *Steven Steincke* has continued to make more progress on the project after leaving San Antonio and will incorporate his new findings in a final report to be sent to *Dr. Hasfura* shortly.

(iii) During the summer of 1999, *Vic DeLorenzo* and *Holly Swisher* worked under the direction of Professor Scott Chapman at Trinity University's Mathematics REU Program. Their work centered on the asymptotic behavior of irreducible factorizations in block monoids. Let G be an Abelian group written additively and let

$$\mathcal{F}(G) = \{\prod_{g \in G} g^{v_g} \mid v_g \in \mathbb{Z}^+ \cup \{0\}\}$$

be the multiplicative free Abelian monoid with basis G. Given $F \in \mathcal{F}(G)$, we write $F = \prod_{g \in G} g^{v_g(F)}$.

Definition 1. *The block monoid over G is*

$$\mathcal{B}(G) = \{B \in \mathcal{F}(G) \mid \sum_{g \in G} v_g(B)g = 0\}.$$

The elements of $\mathcal{B}(G)$ are called blocks. Note that the empty block acts as identity in $\mathcal{B}(G)$. If G is a finite Abelian group, then $\mathcal{B}(G)$ is an atomic monoid (each nonempty block can be written as a product of irreducible blocks). If $\mathcal{B}(G)$ is a block monoid and x a block in $\mathcal{B}(G)$, then consider the following functions:

$$\eta(x) = \text{ the number of non-associated irreducible factorizations of } x \in R,$$

and for a positive integer d the associated limit

$$\bar{\eta}_d(x) = \lim_{k \to \infty} \frac{\eta(x^k)}{k^d}.$$

Definition 2. *Let H be an atomic monoid and x a nonzero nonunit of R. Set*

$$\sigma(x) = \min\{d \mid d \in \mathbb{N}_0 \text{ and } 0 < \bar{\eta}_d(x) < \infty\},$$

$$\Xi(H) = \sup\{\sigma(x) \mid x \text{ a non-zero non-unit in } H\},$$

and

$$\Psi(H) = \sup\{\sigma(x) \mid x \in \mathcal{I}(H)\}.$$

Theorem 3. *Let n and m be positive integers with $n \leq m$. There exists an atomic monoid H such that $\Psi(H) = n$ and $\Xi(H) = m$.*

Theorem 4. *If m is any positive integer then*

$$\limsup_{n \to \infty} \frac{\Psi(\mathcal{B}(\mathbb{Z}_n))}{n^m} = \infty.$$

Theorem 5. *Let G be a finite Abelian group. The following statements are equivalent.*

1) *G is an elementary p-group for some prime integer p.*
2) *If $x = g_1^{n_1} \cdots g_t^{n_t}$ is an irreducible block in $\mathcal{B}(G)$ with $\sigma(x) = 1$, then $n_1 = n_2 = \cdots = n_t = 1$.*
3) *There exists a prime integer p such that for all irreducible $x \in \mathcal{B}(G)$ with $\sigma(x) = 1$, $\bar{\eta}_1(x) = 1/p$.*
4) *There exists a positive rational q such that for all irreducible $x \in \mathcal{B}(G)$ with $\sigma(x) = 1$, $\bar{\eta}_1(x) = q$.*

Assessment of the program

We have been exceedingly fortunate in the selection of our REU students. They were very bright and self-motivated. The students worked extremely hard and made many mathematical discoveries. All of the supervisors were very impressed by the insight and deep understanding the students had gained in a short period of time. The problems given to the students were by no means trivial and most of them are known open problems in the mathematics literature. As an outcome of the students research, two papers written jointly with supervisors have been submitted for publication in refereed mathematical journals. Two more papers are under preparation and will be submitted for publication shortly.

The student's level of satisfaction with the program has been consistently high over the last three summers. One student commented, "My first time to do research one on one with a professor was a great experience." A second student said, "I thought that the flexibility of the faculty to let the students explore options within the research project was also exemplary." A third student wished that the program had been longer and a fourth student liked the daily tea-hour. Other students responded in the following manner in their evaluations of the program:

> I have really enjoyed my summer research experience at Trinity. I feel that it has definitely given me a better understanding of what mathematics research is like. This will be very helpful in making decisions about graduate school.
>
> The two main strengths of Trinity's REU are the high expectations held for its participants and the small size of the program. The research in which we have participated, though basic enough to be accessible to undergraduates, has also been ambitious enough to make us stretch and enhance our mathematical knowledge.

While we are all pleased with the scientific success of the program, we did not succeed in achieving one of our goals. An important aim of Trinity's REU is to attract a sizable number of minority students. In the first summer we had 40% female students; the second summer, 40% female students; and the third summer, 16% female students. We tried very hard to recruit Hispanic students but, unfortunately, we failed. In the next three years we will double our efforts to attract a sizable number of Hispanic students and we hope to succeed in this endeavor.

2 Trinity and The Mathematics Faculty

Trinity University is a leading undergraduate institution. In fact, for the last 10 years Trinity has been ranked as the number one college in the West by US News and World Report. We routinely attract about 30 national merit finalists per year, and the average SAT and ACT scores for entering students vary slightly from 1270 and 27, respectively. Trinity University is located in San Antonio, TX, which is the eighth largest city in The United States. A large part of the city's economic base is provided from tourism, and there are many local attractions for REU participants to enjoy.

Undergraduate research is heavily emphasized within the division of Science, Mathematics, and Engineering. The departments of Chemistry, Computer Science, and Mathematics have each had REU programs for the last several summers. Each of these programs has been a success, with both faculty and students enjoying fruitful research projects. The participants in the Mathematics and Computer Science REU programs have been encouraged, through the use of afternoon tea-hour, to socialize and exchange ideas. Students find that these informal meetings provide a time to discuss a variety of scientific topics.

The department of mathematics is one of the strongest departments at Trinity, and the Principal Investigators of this proposal are adept at working on research projects with undergraduates. In demonstration of the principal investigators' qualifications, we will: 1) discuss how undergraduate research is implemented as part of the mathematics degree at Trinity, 2) highlight the principal investigators prior experiences with undergraduate research, and 3) show a history of past support.

A student receiving an undergraduate mathematics degree from Trinity University must complete a research project. This requirement is incorporated in the majors' seminar, a mandated part of both the junior and senior curriculum. To complete the majors' seminar, students must address a research question, give a short talk about what they have discovered, and hand in a written report about their findings. In an effort to make sure that each student has a successful project, each student works closely with an advising faculty member. The Computer Science department has a similar program, and members of the mathematics faculty have also directed some of these projects.

The faculty's involvement just described is not the full extent to which faculty are engaged in undergraduate research. Various distinctions of the principal investigators are now highlighted. Dr. Saber Elaydi has served as an editor for a mathematical journal on difference equations and has (co)authored over 60 research articles on topics in difference equations, topological dynamics, and differential equations. In addition to mentoring REU students, Dr. Elaydi supervised the research of five Trinity students who were funded by his RUI-NSF grant (1997-2000). Another principal investigator, Dr. Scott Chapman, received a Fulbright scholarship in 1994, has over 30 refereed publications, and has directed numerous senior projects. Dr. Chapman has also co-authored 3 papers with REU participants. These two principal investigators clearly demonstrate a thorough knowledge of the areas they study. Having such a complete understanding of a field is invaluable when choosing problems for undergraduates. Dr. Roberto Hasfura has mentored REU students for the last three years, and has co-authored a paper with one REU student. In 1993, Dr. Allen Holder directed undergraduate and high school research projects supported by NASA at The University of Southern Mississippi. In short, the principal investigators have the experience required to guide intelligent, young minds through their first experiences with mathematical research.

Over the last ten years, the mathematics department has received 3 PEW Foundation grants to support undergraduate research and to develop courses. Over this same time period, 3 NSF grants were awarded to the mathematics department. The first two were awarded in 1989 and 1994 and were used to set up computer labs. These computer labs are equipped with several software packages and are available to the REU students. The third NSF grant has provided funding for the mathematics REU program since 1996.

The mathematics faculty has had great success in attracting female students, with more than 50% of the mathematics majors being female. Upon graduation about 35% of the mathematics majors go to graduate school, another 25% become teachers, and the remaining graduates seek jobs in industry (often in actuarial science and software engineering). Our graduates have a proven record of success.

3 Participant Recruitment

Students are recruited by a nationwide recruitment campaign. An advertisement describing the Trinity REU program will be sent to schools throughout the US and will be posted on several appropriate web pages. A special effort is made to recruit minorities and women. As the following table indicates, a large portion of the regional student population is comprised of Hispanics (about 43%) and females (about 58%).

Institution	Total	Hispanic	Black	Native American	Female
UTSA	17,494	7092	769	86	9,501
St. Mary's University	2,642	1,734	86	12	1,570
Incarnate Word	2,606	1,355	52	8	1,847
Our Lady of the Lake	2,421	1498	174	8	1,877
Trinity University	2,323	205	54	11	1,228

* indicates latest data available.

To encourage minorities and females to apply, we will send announcements to these local schools and invite their mathematics students to attend talks and colloquia at Trinity. Furthermore, we will distribute announcements at local conferences in an attempt to recruit from regional schools with similar diverse student populations. Having the continued involvement of Dr. Roberto Hasfura, a Hispanic, strengthens these goals. Minority and/or female participants should be at least 6 out of 10 selected.

4 Participant Selection

Student applications for the REU program are ranked according to the following criteria:

1. Scholastic Achievement
2. Interest in Mathematics
3. Motivation

The selection of candidates will be done by the five Principal Investigators. Four points are given to scholastic achievement, which is supported by a transcript. The other two criteria are given three points each, and are assessed from short written statements supplied by the student.

5 Student Participation and Schedule

Before students arrive, the students will be distributed among the participating principal investigators. These decisions are based on a statement written by the student indicating their interest in one of the advertised areas.

During the first five days of the summer program, general topic colloquia will be given (Elaydi & Hasfura - discrete dynamical systems, Chapman - monoid theory, Holder - multi-criteria programming, Ponomarenko - combinatorics). The purpose of these lectures is to let all of the REU participants become familiar with the projects being worked on that summer. In the afternoon, the students will meet with their mentors and begin a deeper investigation of their specific topics. The remaining weeks will be comprised of mentor research seminars. The afternoon tea hour is held every afternoon of the project. The following schedule is used for the first week.

Day	Time	Activity	Place
Monday	8:00 - 12:00 p.m.	Check into dorm	Witt Residence Hall
	12:00 - 1:00 p.m.	Lunch	Coates Center
	1:00 - 2:00 p.m.	Orientation	Math Department
	2:00 - 3:30 p.m.	Project Meetings	
	3:30 p.m.	Free time	Trinity University
Tuesday	9:00 - 10:00 a.m.	Topic Colloquium	MMS 104
	10:00 - 11:00 a.m.	Library Tour	Library
	11:00 - 12:00 p.m.	Athletic Facilities Tour	Athletic Center
	12:00 - 1:30 p.m.	Lunch	Coates Center
	1:30 - 2:30 p.m.	Project Meetings	MMS 104
	2:30 - 3:30 p.m.	Tea hour and conversation	
	3:30 - 5:00 p.m.	Project Meetings	MMS 104
Wednesday	9:00 - 10:30 a.m.	Topic Colloquium	MMS 104
	10:30 - 11:00 a.m.	Refreshments	
	11:00 - 12:00 p.m.	Project Meetings	MMS 104
	12:00 - 1:30 p.m.	Lunch	Coates Center
	1:30 - 2:30 p.m.	Project Meetings	MMS 104
	2:30 - 3:30 p.m.	Tea hour and conversation	
	3:30 - 5:00 p.m.	Project Meetings	MMS 104
Thursday	Same as Wednesday		
Friday	Same as Wednesday		

Trinity University is fortunate to be near several companies that employ applied mathematicians (The Southwest Research Institute, Motorola, Texaco, The San Antonio Health Sciences Center). Guest lectures from applied mathematicians employed by these companies will be scheduled as time permits.

The last part of the program is devoted to completing a written report about the findings of the project. These reports will become part the department's technical report series and will be posted on our web page. Besides being preprints to publishable articles, these reports will be useful to prospective and incoming students. The final days of the REU program are reserved for student presentations.

6 Evaluation

In a constant effort to improve the quality of the REU program, students are asked to evaluate their experience. Quotes from some of the prior evaluations are found in the first section. Student input is an invaluable source of information when contemplating changes in the program.

The principal investigators are always looking for ways to improve the REU program. Some changes are already in place. We have made substantial changes to the schedule of the first week. The first several days are no longer reserved for talks from the principal investigators on the topics at hand. Instead, there are general lectures throughout the mornings of the first week, and in the afternoons the students can start a more detailed investigations of their respective problems. This allows more time for the participants to work on specific topics, while at the same time having an awareness of the other projects. The principal investigators will meet after each summer to discuss how the program can be strengthened.

The departmental news letter will be sent to participants for one year after their involvment. Futhermore, past REU students will be invited to give presentations in the majors' seminar. As in prior years, student-mentor relationships are continued when scholarly papers are being prepared for submission.

7 Research Topics

Below are short descriptions of problems that REU participants will undertake.

Stability and Chaos in Competitive Discrete Models
by
Dr. Saber N. Elaydi

In this project, we study competitive discrete models. We will start with the analysis of the growth dynamics of two discretely reproducing populations in competition. If $x_i(n)$ is the population density of species i at generation n, $i = 1, 2$, then a reasonable model may be given by the nonlinear difference equation [2]

$$\left.\begin{aligned} x_1(n+1) &= x_1(n)g_1(x_1(n)+x_2(n)) \\ x_2(n+1) &= x_2(n)g_2(x_1(n)+x_2(n)) \end{aligned}\right\} \tag{1}$$

where $g_i : [0, \infty) \to [0, \infty)$ is a strictly decreasing smooth function that takes on positive values bigger than 1 and less than 1 and $\lim_{x_i(n)\to\infty} g_i(x_i(n)) = 0$ [7, 6, 5].

The students will be asked to address the following questions.

(i) If $g_i(x_1+x_2) = e^{r_i - k_i(x_1+x_2)}$, $i = 1,\ 2$, find conditions under which species 2 ultimately excludes species 1.

(ii) Find conditions under which species 2 ultimately excludes species 1.

A population may be subject to external agents (farmers, hunters, for example) which influence the growth dynamics. When an agent acts to increase a population, the action is called planting or stoking. A reasonable model for two competitive species with planting for species 1 is given by the difference equation [7]

$$\left.\begin{aligned} x_1(n+1) &= x_1(n)g_1(x_1(n)+x_2(n)) + p(x_1(n)) \\ x_2(n+1) &= x_2(n)g_2(x_1(n)+x_2(n)) \end{aligned}\right\} \tag{2}$$

The students will be asked to address the following.

(i) Show that every orbit in (2) is bounded.

(iii) If $g_i(x_1+x_2) = e^{r_i - k_i(x_1+x_2)}$, $i = 1,\ 2$, find conditions under which species 2 ultimately excludes species 1.

(ii) Find conditions under which species 2 ultimately excludes species 1.

(iv) Constant Planting: If $p(x_1(n)) = \alpha, g_i(x_1 + x_2) = e^{r_i - k_i(x_1+x_2)}, i = 1, 2$, find conditions under which there is a stable coexistence between the two species.

(v) Variable Planting: $p(x_1(n)) = \alpha * x_1(n)$.

- **(a)** Obtain conditions for the positive period 2 orbit to attract all positive solutions except for one dimensional invariant manifold.
- **(b)** Obtain conditions under which both species in System (2) persists [4].
- **(c)** Numerical simulations indicate that the period 2 orbit undergoes period-doubling bifurcation route to chaos [3, 1]. Investigate analytically and graphically the bifurcation diagram, the existence of a Feigenbaum sequence and the Feigenbaum number. Finally, investigate when does chaos occur.
- **(d)** Let $g_i(x_1 + x_2) = e^{r_i - k_i(x_1+x_2)}$, $i = 1,\ 2$. If $k_1 = k_2 = 1, \alpha = 0.5$ and $r_1 = 1.5$, Show that if $1.5 + \ln 2 \le r_2 \le 3.411822071$, then System (2) has a positive period 2 orbit that attracts all positive solutions except for one dimensional invariant manifold.

References

[1] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, Benjamin/Cummings, Menlo Park, CA, 1986.

[2] S.N. Elaydi, An Introduction to Difference Equations, Second Edition, Springer-Verlag, New York, 2nd ed., 1999.

[3] S.N. Elaydi, Discrete Chaos, Chapman Hall/CRC, 1999.

[4] A. Fonda, Uniformly persistent semi-dynamical systems, Proc. Amer. Math. Soc. 104: 111-116 (1988).

[5] J.E. Franke and A.-A. Yakubu, Mutual exclusion versus coexistence for discrete competitive systems, J. Math. Biol. 30:161-168 (1991).

[6] V.L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht (1993).

[7] A. Yakubu, The effects of planting and harvesting on endangered species in discrete competition systems, Math. Biosc. 126 (1995), 1-20.

Arithmetic Properties of Monoids Determined by Linear Diophantine Equations

by

Scott T. Chapman

Much of the modern research and literature surrounding Semigroup Theory centers on the non-commutative case. The study of the commutative case has recently gained additional attention (for example, see the books of Redei [13] and Rosales and García-Sánchez [14]). An explanation for this increased interest may lie in the diverse role which these semigroups play in Algebraic Geometry, Commutative Algebra, Number Theory and Computational Algebra. One area where these structures play a critical role is in the study of the arithmetic properties of commutative rings ([8], [3] and [1] provide nice summaries of some known results in this area). Problems in this area were first motivated by classical questions from algebraic number theory (see Carlitz [2]) and their solutions have involved techniques from not only Semigroup Theory, but also from Abelian Group Theory, Number Theory, and Combinatorics.

In a recent paper [5], Chapman, Krause and Oeljeklaus, studied the arithmetic properties of the following class of semigroups. For integers $a_1, \ldots, a_n$ set

$$\mathbf{M}(a_1, \ldots, a_n) = \{(r_1, \ldots, r_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} r_j a_j = 0\}.$$

The $\mathbf{M}(a_1, \ldots, a_n)$ are submonoids of $\mathbb{N}^n$ and while they seem to represent fundamental algebraic objects, surprisingly little seems to be known about their algebraic structures (see Stanley [16] for a discussion). The purpose of this proposal is to extend and strengthen the results of [5] to a larger class of semigroups. In particular, if

$$\begin{array}{ccccccccc} f_1 & = & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ f_k & = & a_{k1}x_1 & + & \ldots & + & a_{kn}x_n & = & 0 \end{array} \tag{3}$$

is a system of homogenous linear diophantine equations with integer coefficients, then denote its monoid of nonnegative solutions by $\mathbf{S}(f_1, \ldots, f_k)$. Hence $\mathbf{M}(a_1, \ldots, a_n) = \mathbf{S}(f_1)$ for an appropriately chosen f_1. We elaborate on the results in [5], their importance, and the specific questions they raise with respect to the $\mathbf{S}(f_1, \ldots, f_k)$ in the following paragraphs. The problems developed below are easily stated and accessible to advanced undergraduate mathematics majors who have completed courses in abstract algebra, number theory and linear algebra. Several of these problems involve a nontrivial programming component which is a good way to get a summer undergraduate projects started. Solutions to many of these problems could gain enough outside interest to merit publication in refereed mathematics journals. The current author has co-authored 2 papers with undergraduate students (see [6] and [4]) and has a third preprint with his two summer 1999 REU students in preparation.

Of fundamental importance in exploring the arithmetic of the $\mathbf{S}(f_1, \ldots, f_k)$ is the notion of a divisor theory. Let $\mathbf{S}$ be a commutative, cancellative monoid with quotient group $\mathbf{Q}(\mathbf{S})$. Let $\leqq$ be the divisibility relation (or quasi-ordering) on $\mathbf{S}$ induced by the following: $x \leqq y$ in $\mathbf{Q}(\mathbf{S})$ if and only if $xz = y$ for some $z \in \mathbf{S}$. Let $\mathcal{F}$ be a nonempty family of homomorphism $f \not\equiv 0$ of $\mathbf{Q}(\mathbf{S})$ into $\mathbb{Z}$ such that for each $x \in \mathbf{Q}(\mathbf{S})$ the set $\mathcal{F}^* = \{f \in \mathcal{F} \mid f(x) \neq 0\}$ is finite. An element of $\mathcal{F}$ is called a *state* on $\mathbf{S}$. If the monoid $\mathbf{S}$ is defined by $\mathcal{F}$ (i.e., $\mathbf{S}= \{x \in \mathbf{Q}(\mathbf{S}) \mid f(x) \geq 0 \ \forall \ f \in \mathcal{F}\}$) then $\mathbf{S}$ is a *Krull Monoid* (see [8], [9] and [11]). A state on $\mathbf{S}$ is *essential* if for any $x, y \in \mathbf{Q}(\mathbf{S})$, there exists $z \in \mathbf{Q}(\mathbf{S})$ such that $x \leqq z$, $y \leqq z$ and $f(z) = \max\{f(x), f(y)\}$. Let $\mathbf{I}$ denote the set of essential states f which are *normalized* (i.e., $f(\mathbf{Q}(\mathbf{S})) = \mathbb{Z}$). Consider the map $\varphi : \mathbf{S} \to \mathbb{Z}_+^{(\mathbf{I})}$ defined by $\varphi(x)_{(f)} = f(x)$. This map satisfies the following two properties: 1) for all $x, y \in \mathbf{S}$, $x \leqq y$ if and only if $\varphi(x) \leq \varphi(y)$ (here $\leq$ denotes the pointwise ordering), and 2) each element of $\mathbb{Z}_+^{(\mathbf{I})}$ is the minimum of finitely many elements for the set $\varphi(\mathbf{S})$. As described above, the essential states of Krull monoid yield a *divisor theory* for $\mathbf{S}$ (see [8], [12], and [15]). The factor monoid $\mathbb{Z}_+^{(\mathbf{I})}/\varphi(\mathbf{S}) \cong \mathbb{Z}^{(\mathbf{I})}/\varphi(\mathbf{Q}(\mathbf{S}))$ is a group known as the *divisor class group of* $\mathbf{S}$ and denoted $\mathrm{Cl}(\mathbf{S})$. In a Krull monoid, every non-zero non-unit can be written as a product of irreducible elements (such a monoid in general is called *atomic*). An atomic monoid is factorial if and only if it is a Krull monoid with trivial divisor class group (see [8]). The $\mathbf{S}(f_1, \ldots, f_k)$ (and hence the $\mathbf{M}(a_1, \ldots, a_n)$) possess a divisor theory and are thus atomic Krull monoids.

Much of the work in [5] centers on the following factorization property first alluded to by Carlitz in [2]. An atomic monoid H is called *half–factorial* if whenever $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are irreducible elements of H with $x_1 \ldots x_n = y_1 \ldots y_m$ then $n = m$. Among the major results in [5] are:

1. the divisor class group of $\mathbf{M}(a_1, \ldots, a_n)$ is either $\mathbb{Z}$ or $\mathbb{Z}_n$ for some $n \geq 2$ (Theorem 1.3),
2. a characterization of when $\mathbf{M}(a_1, \ldots, a_n)$ is factorial (Corollary 1.4),
3. a geometric criterion in terms of hyperplanes which determines the half–factoriality of $\mathbf{M}(a_1, \ldots, a_n)$ (Proposition 2.1),

A question which can be traced back to the early part of the 20th century concerns determining the set of minimal solutions to a system of the form (3) (see [7]). These minimal solutions are merely the irreducible elements of $\mathbf{S}(f_1, \ldots, f_k)$. These irreducible elements can be determined in a Krull monoid by using arguments centered about the divisor class group (see [8]). An answer to the following question might lead to an alternate (and possibly improved) algorithm to the one developed in [14] for computing the set of minimal solutions to (3). An answer to Question 1 would also extend the characterization of factorial monoids for the $\mathbf{M}(a_1, \ldots, a_n)$ presented in [5].

Question 1. *Compute the divisor class group of* $\mathbf{S}(f_1, \ldots, f_k)$. *In particular, can every finitely generated abelian group serve as the divisor class group of some monoid of the form* $\mathbf{S}(f_1, \ldots, f_k)$*?*

This result leads to a central focus of the proposed research.

Question 2. *Find a geometric connection between the monoids* $\mathbf{S}(f_1, \ldots, f_k)$ *and their factorization properties. In particular, does some form of the hyperplane condition of [5, Proposition 2.1] hold for a monoid of the form* $\mathbf{S}(f_1, \ldots, f_k)$*?*

In [10], Krause begins to explore the relationship between geometry and factorization properties of certain monoids which arise through the study of cones in numerical modules. If $\mathbf{S}$ is a monoid and $\{M_i\}_{i \in \mathcal{I}}$ a family of factorial submonoids of $\mathbf{S}$ such that (i) $M_i \neq M_j$ for $i \neq j$, and (ii) if x is a nonunit of $\mathbf{S}$ then there exists a positive integer n such that $x^n \in \bigcup_{i \in \mathcal{I}} M_i$, then $\mathbf{S}$ along with $\{M_i\}_{i \in \mathcal{I}}$ is called a *factorial complex*. We ask the following.

Question 3. *Characterize monoids of the form $\mathbf{S}(f_1, \ldots, f_k)$ which possess factorial complexes. Specifically, if $\mathbf{S}(f_1, \ldots, f_k)$ and $\{M_i\}_{i\in\mathcal{I}}$ form a factorial complex, are all the irreducibles of $\mathbf{S}(f_1, \ldots, f_k)$ contained in $\bigcup_{i\in\mathcal{I}} M_i$?*

A possible tool in the study of factorization in monoids is the *extraction grade.* If $\mathbf{S}$ is a monoid and $x, y \in \mathbf{S}$, then the extraction grade of x and y in $\mathbf{S}$ is defined by

$$\Lambda_{\mathbf{S}}(x, y) = \sup\{\frac{m}{n} | x^m \text{ divides } y^n, m \in \mathbf{Z}_+, n \in \mathbf{N}\}.$$

A monoid $\mathbf{S}$ is called an *extraction monoid* if the extraction grade for each pair of elements $x, y \in \mathbf{S}$ is rational. Recall that the *elasticity* of an atomic monoid $\mathbf{S}$ is defined as

$$\rho(\mathbf{S}) = \sup\{\frac{m}{n} | x_1 \cdots x_m = y_1 \cdots y_n \text{ where each } x_i, y_j \text{ are irreducible in } \mathbf{S}\}.$$

A survey of the known results concerning elasticity in integral domains can be found in [1].

Question 4. *Is there a relationship between the extraction grades of $\mathbf{S}(f_1, \ldots, f_k)$ and the elasticity of $\mathbf{S}(f_1, \ldots, f_k)$? In general, must an extraction monoid have rational elasticity?*

References

[1] D.F. Anderson, Elasticity of factorization in integral domains: a survey, Lecture Notes in Pure and Applied Mathematics, Marcel-Dekker, 189(1997), chapter 1, 1-30.

[2] L. Carlitz, A characterization of algebraic number fields with class number two, *Proc. Amer. Math. Soc.* **11**(1960), 391-392.

[3] S.T. Chapman and A. Geroldinger, Krull domains and monoids, their sets of lengths and associated combinatorial problems, Lecture Notes in Pure and Applied Mathematics, Marcel-Dekker, 189(1997), chapter 3, 73-112.

[4] S.T. Chapman, J. Herr and N. Rooney, A factorization formula for class number two, to appear in *J. Number Theory.*

[5] S.T. Chapman, U. Krause and E. Oeljeklaus, Monoids determined by a homogenous linear diophantine equation and the half-factorial property, to appear in *J. Pure Appl. Algebra.*

[6] S.T. Chapman and W. Thill, On a generalization of a theorem of Zaks and Skula, *Proc. Royal Irish Aca.* **96A**(1996), 79-83.

[7] J.H. Grace and A. Young, *The Algebra of Invariants*, Cambridge University Press, Cambridge, 1903.

[8] F. Halter-Koch, Halbgruppen mit Divisorentheorie, *Expo. Math.* **8**(1990), 27-66.

[9] U. Krause, On monoids of finite real character, *Proc. Amer. Math. Soc.* **105**(1989), 546-554.

[10] U. Krause, Cones in numerical modules, *Arch. Math.* **58**(1992), 70-80.

[11] U. Krause and C. Zahlten, Arithmetic in Krull monoids and the cross number of divisor class groups, *Mittelungen der Mathematische Gesellschaft Hamburg* **12**(1991), 681-696.

[12] G. Lettl, Subsemigroups of finitely generated groups with divisor-theory, *Mh. Math.* **106**(1988), 205-210.

[13] L. Rédei, The Theory of Finitely Generated Commutative Semigroups, Pergamon, 1965.

[14] J.C. Rosales and P.A. García-Sanáchez, Finitely generated commutative monoids, Novascience publishers, New York 1999.

[15] L. Skula, Divisorentheorie einer Halbgruppe, *Math. Z.* **114**(1970), 113-120.

[16] R. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston, 1983.

Dynamical Properties of Continuous Maps of the Interval
Trinity REU Project Proposal

Roberto Hasfura

We propose here that student participants research the dynamical properties of certain families of continuous functions of the interval $[0, 1]$. Many recent results showcasing surprisingly complex behavior of very simple-looking (for example, piecewise linear) maps, as well as others yet to be discovered, are accessible to students equipped with only the most elementary mathematical tools in the undergraduate curriculum. ([3] is an example.) Such research would in fact build on the successful work carried out by students who participated in REUs that took place at Trinity in the summers of 1997, 1998 and 1999. Collectively, those students have considered the dynamical properties (especially insofar as periodic points are concerned) of the families (i) $T_\omega(x) = \omega(1/2 - |x - 1/2|)$, $0 < \omega \leq 2$ (tent maps); (ii) $f_{m,e}(x) = mx$ or me or $m(2-x)$ depending on whether $0 \leq x < e$ or $e \leq x \leq 2-e$ or $2-e < x \leq 2$, where $m > 1$ and $me \in (2-e, 2]$ (trapezoidal maps); and (iii) the piecewise-linear maps of $[0,1]$ with three pieces, the 'middle one' mapping onto $[0,1]$. Some results obtained last summer concerning a subfamily of the family in (iii) suggest some avenues of student research.

For specific examples, let's consider the two-parameter family of continuous maps $f_{a,b}$ of $[0,1]$ given by

$$f_{a,b}(x) = \begin{cases} \frac{1-a}{b}x, & 0 \leq x \leq a \\ \frac{1}{1-2b}(x-1) + \frac{b}{1-2b}, & a < x \leq 1-a \\ \frac{1-a}{b}(x-1)+1, & 1-a < x \leq 1 \end{cases}$$

where $0 \leq a \leq 1$ and $0 < b < 1/2$. A number of dynamical properties of this family were established last summer ([4]) including some that beg for closer examination. The following surprising dichotomy provides one good such example: for $1/4 < b < 1/2$ there exist numbers a_k increasing to $1-b$ such that the map $f_{a,b}$ has a periodic orbit of length $2 \cdot (2k+1)$ if and only if $a < a_k$; however, for $0 < b < 1/4$ and for all $0 < a < 1-b$ the map $f_{a,b}$ always has periodic orbits of length six.

Another important question considered last summer was that of possible topological conjugacies between maps in the family. A student constructed an interesting conjugacy between $f_{0,1/3}$ and $f_{0,2/5}$. The construction, fractal-like in its recursive nature, possibly gives rise to a conjugacy which appears to be almost everywhere non-differentiable. Elucidating this and other related questions (for instance: given b and $a' \neq a''$ with no a_k (as described above) satisfying $a' < a_k < a''$, are the maps $f_{a',b}$ and $f_{a'',b}$ topologically conjugate?) appears possible for motivated junior mathematics majors with no more than a Real Analysis course in their background.

There are other questions that, albeit of slightly different flavor, are of relevance to the dynamical behavior of maps of the interval and that may be susceptible to the efforts of bright undergraduates and of those inclined to computer experimentation. For example, the students will be introduced to the notion of invariant measure and to the standard hierarchy of mixing properties: ergodicity, weak- and strong-mixing, K-ness and Bernoullicity. We will then ask whether a given map of the interval admits an absolutely continuous (with respect to Lebesgue) invariant measure and, if so, what the statistical properties of that measure are—is it Bernoulli? or, more generally, to what extent is it mixing? Here is a simple example due to Ulam and von Neumann (see [2]) of this situation: the map $f(x) = 4x(1-x)$ of the interval $[0,1]$ preserves the measure $\frac{dx}{\pi\sqrt{x(1-x)}}$. With this measure, the map is measurably isomorphic to the one-sided Bernoulli 2-shift, a paradigm of random behavior.

Another situation that, time permitting, the students will be asked to consider is that where a family of interval maps (such as those above) indexed by a subset of the integers is coupled locally. Then we will ask whether we can choose the form of the coupling and its strength to produce certain types of global behavior. Here is a version of that query: Can we couple maps with simple behavior in a way to produce global complicated behavior? Work along these lines has been published by L. A. Bunimovich and co-authors. (See, for example, [1].) Again, these questions provide ground for computer experimentation and their resolution can sometimes be achieved with only a few mathematical tools available in the usual undergraduate curriculum.

References

[1] L.A. Bunimovich and S. Venkatagiri, *Onset of chaos in coupled map lattices via the peak-crossing bifurcation,* Nonlineariy, **9**, 1996.

[2] P. Collet and J.P. Eckmann, *Iterated maps on the interval as dynamical systems*, Birkhauser, Boston, 1980.

[3] R. Hasfura and P. Lynch, *Periodic points of the tent family*, Submitted.

[4] Steven Steincke, *The dynamical properties of a family of piecewise linear maps of the interval*, In preparation.

Multiple Objective Linear Programming and
Interior Point Algorithms

by

Allen Holder

A multiple objective linear program (MOLP) is an optimization problem where one attempts to optimize several linear functions relative to linear constraints. The standard form of an MOLP is

$$\min\{Cx : Ax = b, x \geq 0\},$$

where $C \in \mathbb{R}^{p \times n}$ and $A \in \mathbb{R}^{m \times n}$. Because there are multiple objective functions, to find an optimal solution really means to find a *pareto optimal solution.* This type of solution is a feasible vector, x, such that any other feasible vector that decreases one of the objective functions also increases another objective function. The set of pareto optimal solutions is called the *efficient frontier.* From a practical standpoint, it is useful to know how the objective functions behave on this entire set. Recent investigations into parameterizing the efficient frontier are found in [5].

Although interior point algorithms have revolutionized the field of linear programming (LP) [13], very little has been done with their connection to MOLP, exceptions being [1, 3]. Recently, Caron, Greenberg, and Holder [4] have developed new techniques to prove convergence of the analytic center path, which most interior point algorithms trace to optimality. This path terminates at the *analytic center solution* [11], which is a unique, interior solution. The uniqueness of this LP solution finally allowed those using LP to discuss a common solution that is readily computable. Hence, post solution analysis and interpretation could be based on a solution that is easily referenced. Using the analytic center solution, many mathematicians have worked on (re)inventing the topics of LP post solution analysis, see [6, 7, 8, 9, 10, 12]. The analytic center solution of an LP also induces the *optimal partition* [11], which completely defines the optimal face.

The new results of Caron, Holder, and Greenberg [4] have natural extensions to MOLP and may allow new classes of algorithms for this problem statement. Furthermore, there are several theoretical questions to explore, a few of which are now presented.

Question 1 How does one define a unique, computable pareto optimal solution comparable to the analytic center solution for LP?

Once the above definition is established, we are highly motivated to have a complete understanding of the information this solution contains. The following questions demonstrate the types of information that are sought.

Question 2 Does the analytic center pareto optimal solution induce a partition similar to that of the optimal partition?

Question 3 Is the analytic center pareto optimal solution somehow interior to the efficient frontier? Does it define an interior facet of the efficient frontier?

Meaningful answers to these questions would be extremely interesting. Because there is so little known about the above topics, students will find it a fruitful area to work. Moreover, several modern software routines can be used to gain insight. Such computer work is often tangible and rewarding to beginning researchers.

References

[1] S. S. Abhyankar, T. L. Morin, and T. Trafalis *Efficient faces of polytopes: Interior point algorithms, parameterization of algebraic varieties and multiple objective optimization*, in Mathematical Developments Arising From Linear Programming, editors J. Lagarias and M. Todd, American Mathematical Society, pp. 319-341, 1990.

[2] I. Adler and R. Monteiro, *A geometric view of parametric linear programming*, Algorithmica **8**, pp. 161-176, 1992.

[3] A. Arbel, *A multiobjective interior primal-dual linear programming algorithm* Computers & Operations Research **21**, pp. 433-445, 1994.

[4] R.J. Caron, H.J. Greenberg, and A.G. Holder *Analytic Centers and Repelling Inequalities*, Technical Report CCM#142, Center for Computational Mathematics, University of Colorado at Denver, Denver, CO, USA, 1999.

[5] I. Das and J. Dennis, *Normal-Boundary Intersection: A new method for generating Pareto Optimal point in Multicriteria Optimization Problems* SIAM Journal on Optimization **8**, pp. 631-657, 1998.

[6] H. Greenberg, *The use of the optimal partition in a linear programming solution for postoptimal analysis*, Operations Research Letters **15**, pp. 179-185, 1994.

[7] H. Greenberg, *Rim Sensitivity Analysis from an Interior Solution*, SIAM Journal on Optimization **10**, pp. 427-442, 2000.

[8] A. Holder, J. Sturm, and S. Zhang, *The analytic central path, sensitivity analysis, and parametric programming*, Technical Report CCM#118, Center for Computational Mathematics, University of Colorado at Denver, Denver, CO, USA, 1997.

[9] B. Jansen, C. Roos, and T. Terlaky, *An interior point approach to post optimal and parametric analysis in linear programming*, Technical Report #92-21, Delft University of Technology, Delft, Netherlands, 1992.

[10] B. Jansen, J.J. de Jong, C. Roos, and T. Terlaky, *Sensitivity analysis in linear programming: Just be careful!*, European Journal of Opterations Research **101**, pp. 15-28, 1997.

[11] L. McLinden, *An analogue of Moreau's proximation theorem, with applications to the nonlinear complementary problem*, Pacific Journal of Mathematics" **88**, pp. 101-161, 1980.

[12] R. Monteiro and S. Mehrotra, *A General Parametric Analysis Approach and its Implication to Sensitivity Analysis in Interior Point Methods*, Mathematical Programming **72**, pp. 65-82, 1996.

[13] M. Wright, *The Interior-Point Revolution in Constrained Optimization*, High-Preformance Algorithms and Software in Nonlinear Optimization, R. Deleone, A. Murli, P. Pardalos, and G. Toraldo eds., pp. 359-381, Kluwer Academic Publishers, 1998.

Jump System Analysis

by

Vadim Ponomarenko

Jump systems are an exciting new combinatorial theory, with diverse applications throughout combinatorics. First described in 1995, jump systems simultaneously generalize (see [1]) two generalizations of matroids: delta-matroids and integral polymatroids. They naturally model bidirected graphs, and degree sequences of graphs[2]. They are also the lattice points in bisubmodular polyhedra[1].

Jump systems have an appealingly simple definition that lends itself well to undergraduate investigation. Fix $n > 0$, and let J be a subset of $\mathbf{Z}^n$. Fix the L^1 metric, where for $x, y \in \mathbf{Z}^n$, $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$. Then J is a jump system provided it satisfies the following axiom:

(JS) $\forall x, y \in J, \forall z_1$ with $x \xrightarrow{y} z_1$, then either:

1. $z_1 \in J$, or
2. $\exists z_2 \in J$ with $x \xrightarrow{y} z_1 \xrightarrow{y} z_2$

where $x \xrightarrow{y} z$ means $d(x,z) = 1$ and $d(x,y) > d(z,y)$.

If $x \xrightarrow{y} z$, we say that z is a *step* from x toward y. **(JS)** is then equivalent to the notion that for every step z from x toward y (with both x and y in the jump system), either that step z is in the jump system, or there is a second step from z toward y that is.

Students first will consider low-dimensional jump systems. While one-dimensional jump systems are easily understood (they have no gaps of size larger than one), taxonomy of two-dimensional jump systems remains open.

Currently there is only a limited collection of jump system results (See [1], [2],[3]). Two that are almost obvious are that the Cartesian product of jump systems is a jump system, and that the intersection of a jump system with a box is a jump system. Some less obvious and easily stated results follow:

1. For two jump systems on $\mathbf{Z}^n$, their sum $J_1 + J_2 = \{x_1 + x_2 : x_i \in J_i\}$ is a jump system.
2. A *face* of a jump system – those elements of a jump system maximizing a linear objective function – is a jump system.
3. The natural projection of a jump system on $\mathbf{Z}^n$ onto $\mathbf{Z}^{<n}$ is a jump system.

Students will be asked to consider these and similar, slightly less easily stated, questions. They will be expected to construct their own arguments for proof; some of the existing proofs are cumbersome and might be improved upon. Due to the young nature of the field, there is also great potential for discovery of other properties of similar flavor.

Next, students will be asked to consider the primary open question of jump system theory: the intersection problem. The intersection problem is whether two given (abstractly defined) jump systems have a point in common. There are several partial solutions ([2],[3]), but the main question remains open. Students will be asked to consider special cases, such as:

1. Jump systems J where for all $(x_1, x_2, \ldots, x_n) \in J$, $\sum_{i=1}^{n} x_i$ is zero.
2. Jump systems J where for all $(x_1, x_2, \ldots, x_n) \in J$, $\sum_{i=1}^{n} x_i$ is constant.
3. Jump systems J where for all $(x_1, x_2, \ldots, x_n) \in J$, $\sum_{i=1}^{n} x_i$ is even (or odd).
4. Jump systems J where for all $x \in J$, $x \geq \overline{0}$.

Students will also be asked to consider algorithmic and computational aspects of solving the intersection problem. If there is no good solution, students will consider whether there is at least a (somewhat) efficient search that improves upon naively checking each element of one jump system against each element of the other.

References

[1] André Bouchet and William Cunningham, *Delta-Matroids, Jump Systems, and Bisubmodular Polyhedra*, Siam J. Disc. Math., 8 (1995) 17-32.

[2] László Lovász, *The membership problem in jump systems*, Technical Report #1101, The Department of Computer Science, Yale University, USA (1995).

[3] Vadim Ponomarenko, *Some Results on Jump Systems and Rota's Conjecture*, Ph.D. Dissertation, Department of Mathematics, University of Wisconsin- Madison (1999).