1 Theoretical Applications of Sensitivity Analysis

Sensitivity analysis is a subfield of optimization that began in the 1950s [12, 29, 34, 35] and is concerned with how an optimal solution relies on the data used to construct a specific instance of a model. Other than mathematical interest, the field of sensitivity analysis is important because the results of this area substantiate how practitioners interpret and understand a specific algorithm's solution. As such, sensitivity analysis provides mathematical tools that allow one to query and probe how solution characteristics change when uncertain data is altered. Because modern optimization algorithms and computing power allow us to "... solve far larger problems than we can understand [13]", the field of sensitivity analysis is more important than ever.

Sensitivity analysis is often exclusively perceived as post-solution analysis, and while this is an important part of the field, sensitivity analysis results provide a unique perspective on algorithm design. In particular, implemented interior point algorithms are initiated with an infeasible element, and subsequent iterates rely on parameters that are controlled so that the final iterate is feasible (and optimal). As discussed below, these parameters actually gauge the amounts of change in the right-hand side and cost coefficient vectors. Since most interior point algorithms follow a geometric structure called the central path, understanding and characterizing the convergence of the central path under data perturbation is important. While the analytic properties of the central path have been studied extensively [5, 6, 8, 10, 32, 18, 26, 27, 31, 36, 37, 38, 44], little is known about the convergence of this path under data perturbation.

The research goals of this proposal are listed below.

- Completely characterize convergence of the central path under simultaneous, linear perturbations in the right-hand side and cost coefficient vectors.
- Extend the above analysis to the more difficult case where the linearity of the perturbations is removed.
- Develop an analog of the robust sensitivity analysis based on the linear programming optimal partition for multiple objective linear programming.
- Design and implement algorithms based on these new results.

These goals are chosen for two reasons. First, the questions proposed are challenging, interesting, and will lead to original, publishable quality results. Second, there are many subproblems that are within the grasp of undergraduate students, and as discussed in Section 2, undergraduate students with calculus and linear algebra are able to begin investigating these questions with a semester or two of mentoring. Hence, the project's goals are perfect for incorporating research into the undergraduate curriculum.

1.1 Characterizing Convergence of the Central Path Under Simultaneous Linear Perturbations

Over the last fifteen years, Interior Point methods have "revolutionized" the field of mathematical programming [40]. These algorithms have been intensely studied because of their theoretical diversity and their computational superiority on large problems [24, 25]. There are now several commercial and academic optimization solvers that are based on the theoretical results from the last fifteen years of research into interior point algorithms [42].

For $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, we consider the primal and dual linear programs in standard form,

$$(LP)\min\{c^T x : Ax = b, x \geq 0\} \text{ and } (LD)\max\{b^T y : A^T y + s = c, s \geq 0\},$$

where we assume without loss of generality that $m \leq n$ and that the rank of A is m. We call b the *right-hand side* vector and c the *cost coefficient vector.* The feasible regions for the primal and dual are $\mathcal{P}$ and $\mathcal{D}$, respectively, and the *strict interiors* are $\mathcal{P}^o = \{x \in \mathcal{P} : x > 0\}$ and $\mathcal{D}^o = \{(y, s) \in \mathcal{D} : s > 0\}$. Setting $X = \text{diag}(x_1, x_2, \ldots, x_n)$, we have that the necessary and sufficient optimality conditions for (LP) and (LD) are

$$Ax = b\ ,\ A^T y + s = c\ ,\ Xs = 0\ ,\ x \geq 0\ , \text{and } s \geq 0.$$

The fundamental idea behind an interior point algorithm is to replace the *complementarity constraint* $Xs = 0$ with $Xs = \mu e$, where e is the vector of ones and $\mu > 0$. With this replacement, we have

$$Ax = b\ ,\ A^T y + s = c\ ,\ Xs = \mu e\ ,\ x \geq 0\ , \text{and } s \geq 0,$$

which are the necessary and sufficient conditions for the penalized linear programs

$$\begin{gathered}\min\left\{c^T x - \sum_{i=1}^{n} \ln(x_i) : Ax = b, x \geq 0\right\} \text{ and} \\ \max\left\{b^T y + \sum_{i=1}^{n} \ln(s_i) : A^T y + s = c, s \geq 0\right\}.\end{gathered} \tag{1}$$

The logarithm terms guarantee that the primal objective function is strictly convex and that the dual objective function is strictly concave. Hence, for $\mu > 0$ there is a unique solution, denoted by $x(\mu)$ and $(y(\mu), s(\mu))$. The set $\{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$ is called the *central path.* Similarly, $\{x(\mu) : \mu > 0\}$ is called the *primal central path*, and $\{(y(\mu), s(\mu)) : \mu > 0\}$ is called the *dual central path.* A fundamental result is that both $x(\mu)$ and $(y(\mu), s(\mu))$ converge to an optimal solution as $\mu \downarrow 0$ [9] —i.e. the primal and dual central paths converge to an optimal solution. The iterates of a path following interior point algorithm stay within a neighborhood of the central path and follow this path towards optimality. So, to understand path following interior point algorithms (which are most commonly implemented) one must understand the primal and dual central paths.

The limits of the central path as $\mu \downarrow 0$ are denoted by x^* and (y^*, s^*), and unless there is a unique solution, the characteristics of these limits are different than those of a basic optimal solution. In particular, these solutions are *strictly complementary*, meaning that $(x^*)^T s^* = 0$ and $x^* + s^* > 0$. The strict complementarity of x^* and (y^*, s^*) means that this solution induces the *optimal partition*, which is defined by

$$B = \{i : x_i^* > 0\} \text{ and } N = \{1, 2, \ldots, n\}\backslash B.$$

The sets B and N have simple, but important, interpretations. If an index i is in N, the decision variable x_i is zero in every optimal solution, and if i is in B, x_i is positive in some optimal solution. In the presence of degeneracy, such information is not available from a basic optimal solution, and the only way to acquire this knowledge is to generate every basic optimal solution and investigate the unbounded directions. Techniques for generating all the basic optimal solutions are known [11], but even with anti-cycling rules they are computationally expensive.

Making the convention that a set subscript denotes the sub-vector whose components are indexed within the set, we see that the optimal partition is important because it characterizes the solution sets;

$$\mathcal{P}^* = \{x \in \mathcal{P} : x_N = 0\} \text{ and } \mathcal{D}^* = \{(y, s) \in \mathcal{D} : s_B = 0\}. \tag{2}$$

The limits x^* and (y^*, s^*) are the *analytic centers* of these sets, meaning that they are the unique solutions, respectively, to

$$\max\left\{\sum_{i \in B} \ln(x_i) : x \in \mathcal{P}^*, x_B > 0\right\} \text{ and } \max\left\{\sum_{i \in N} \ln(s_i) : (y, s) \in \mathcal{D}^*, s_N > 0\right\}.$$

As such, x^* and (y^*, s^*) are called the *analytic center solutions*. Prior to the middle 1980s, post solution analysis was based on basic optimal solutions, and researchers thought that a basic optimal solution was required to conduct sensitivity analysis. Megiddo initially addressed the problem of obtaining a basic optimal solution from a strictly complementary solution by providing a polynomial time algorithm that starts with a strictly complementary solution and obtains a basic optimal solution [28]. The commercial solver CPLEX uses a variant of this algorithm, called "crossover", to produce a basic optimal solution when their path following interior point algorithm is used (CPLEX's barrier algorithm).

Several researchers have now shown that the sensitivity analysis allowed by the analytic center solution, and the induced optimal partition, is more robust than the classical results based on a basic optimal solution [2, 3, 8, 15, 16, 17, 18, 21, 23, 30, 32]. In fact, there a several situations where the characteristics of the analytic center solution are preferred to the characteristics of a basic optimal solution [14, 22]. This robust analysis is briefly discussed in Section 1.3, where extensions for the multiple objective case are suggested.

Results in the sensitivity analysis literature have had little influence on algorithm design, which is sad because sensitivity analysis provides a unique perspective to view these algorithms. To see this, we note that infeasible interior point algorithms are usually implemented. The reason for the infeasibility lies with the difficulty of obtaining an initial starting point, which must be *strictly feasible* —i.e. the initial iterate (x^0, y^0, s^0) must be in $\mathcal{P}^o \times \mathcal{D}^o$. The feasibility restriction may be relaxed because one can force any $x^0 > 0$ and (y^0, s^0), $s^0 > 0$, to be strictly feasible to an altered set of constraints. This is done by setting $\delta b = Ax^0 - b$, $\delta c = A^T y^0 + s^0 - c$, and considering the perturbed necessary and sufficient conditions,

$$Ax = b + \rho\delta b\ ,\ A^T y + s = c + \tau\delta c\ ,\ Xs = \mu e\ ,\ x > 0\ ,\ \text{and}\ \ s > 0. \tag{3}$$

For any $(\rho, \tau, \mu) > 0$, we denote the unique solution to the perturbed necessary and sufficient conditions by $x(\rho, \tau, \mu)$ and $(y(\rho, \tau, \mu), s(\rho, \tau, \mu))$. With the three arguments of ρ, τ, and

μ, $\{x(\rho,\tau,\mu) : \rho \geq 0, \tau \geq 0, \mu > 0\}$ is technically not a "path", but we abusively call this set the *primal parameterized central path* for ease with geometric explanations. Likewise, $\{(y(\rho,\tau,\mu), s(\rho,\tau,\mu))\}$ is called the *dual parameterized central path*. When $\rho = \tau = 1$, x^0 and (y^0, s^0) are strictly feasible to the linear programs where b is replaced with $b + \rho\delta b$ and c is replaced with $c + \tau\delta c$. The goal is to construct a sequence of iterates, usually with Newton's Method, that rely on (ρ,τ,μ) for which $(\rho,\tau,\mu) \downarrow 0$. This means that we need to understand how elements react to changes in the right-hand side and cost coefficient vector. One of the major concerns in the field of sensitivity analysis is to answer such perturbation questions, and hence the results of this field allow a unique perspective on algorithm design.

The analysis of infeasible interior point algorithms does provide conditions under which any cluster point of a sequence of iterates is an optimal solution (a nice explanation is found in Chapter 5 of [41]). What is lacking is a complete analysis that characterizes when convergence is guaranteed. Some results are provided by Holder, Sturm, and Zhang [20], where sufficient conditions for primal convergence are developed. However, nowhere in the literature are there necessary and sufficient conditions that guarantee the convergence of $x(\rho,\tau,\mu)$ or $(y(\rho,\tau,\mu), s(\rho,\tau,\mu))$. We demonstrate the complex behavior possible with a simple example.

Example: *Consider the linear program* $\min\{x_3 : 0 \leq x_1, x_2, x_3 \leq 1\}$*, and set* $\delta b = (1,1,1)^T$ *and* $\delta c = (10, 2, 0)^T$*. The perturbed primal problem is*

$$\min\{\tau(10x_1) + \tau(2x_2) + x_3 : 0 \leq x_1, x_2, x_3 \leq 1 + \rho\}.$$

After incorporating x_4*,* x_5*, and* x_6 *as primal slacks,*

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (1,1,1,1,1,1)$$

is strictly feasible for the primal, and

$$((y_1, y_2, y_3), (s_1, s_2, s_3, s_4, s_5, s_6)) = ((-1,-1,-1), (11, 3, 2, 1, 1, 1))$$

is strictly feasible for the dual. The parameterized central path is defined by

$$\begin{aligned}
x_1(\rho,\tau,\mu) &= \frac{1}{2}\left((1+\rho) + \frac{\mu}{5\tau} - \sqrt{(1+\rho)^2 + \left(\frac{\mu}{5\tau}\right)^2}\right), \\
x_2(\rho,\tau,\mu) &= \frac{1}{2}\left((1+\rho) + \frac{\mu}{\tau} - \sqrt{(1+\rho)^2 + \left(\frac{\mu}{\tau}\right)^2}\right), \text{ and} \\
x_3(\rho,\tau,\mu) &= \frac{1}{2}\left((1+\rho) + 2\mu - \sqrt{(1+\rho)^2 + (2\mu)^2}\right).
\end{aligned}$$

Consider the two situations of $\tau = \sqrt{\mu}$ *and* $5\tau = \mu$*. Allowing* $(\rho,\tau,\mu) \downarrow 0$*, we have the two different limits of*

$$x(\rho, \sqrt{\mu}, \mu) \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } x(\rho, \mu/5, \mu) \to \begin{pmatrix} 2-\sqrt{2} \\ 6-\sqrt{26} \\ 0 \\ \sqrt{2}-1 \\ \sqrt{26}-1 \\ 1 \end{pmatrix}.$$

There are two things to notice. First, the solution with $\tau = \sqrt{\mu}$ does **not** *induce the optimal partition, while the solution with $5\tau = \mu$ does induce the optimal partition. Second, combining these two scenarios easily provides a sequence with no limit. For example, let $\rho^k = 1/k$, $\mu^k = 1/k$, and τ^k be either $1/\sqrt{k}$ or $1/(5k)$ depending on whether k is even or odd. Then the sequence $x(\rho^k, \tau^k, \mu^k)$ does not converge. We mention that even though $x(\rho^k, \tau^k, \mu^k)$ does not converge for this particular choice of (ρ^k, τ^k, μ^k), the dual sequence of $(y(\rho^k, \tau^k, \mu^k), s(\rho^k, \tau^k, \mu^k))$ does converge.*

The simplicity of this example allows an explicit representation of the primal parameterized central path, and we see that the convergence of this path is not guaranteed when $(\rho, \tau, \mu) \downarrow 0$. Moreover, even if the primal parameterized central path converges, the optimal solution does not necessarily induce the optimal partition. So, if one wants to conduct the robust sensitivity analysis that the optimal partition allows, extra care must be taken to ensure that an algorithm provides a strictly complementary solution. Unfortunately, results stating exactly what this "extra care" is are currently not available. The first goal of the research is to provide conditions for (ρ, τ, μ) that completely characterize the convergence of $x(\rho, \tau, \mu)$. In fact, we hope to do more than just characterize the convergence of $x(\rho, \tau, \mu)$ by stating the exact limit when convergence occurs. Such results would show how to control the parameters of an infeasible interior point algorithm so that the solution characteristics are desirable to the end user.

We note that the prior example exhibits the possibility that the primal sequence may diverge, while the dual sequence converges. Because of this, the perturbation analysis found in [7] and [31], where linear complementarity problems are considered, does not extend to provide the sought after necessary and sufficient conditions. The problem here is that linear complementarity problems do not readily allow for independent perturbations in b and c, and since we need ρ and τ to change independent of each other, this framework is not sufficient.

The techniques used in [20] to develop sufficient conditions for $x(\rho, \tau, \mu)$ to converge indicate that the linearity of the perturbation of c simplifies the analysis, and we now discuss why this is the case. Let $(B|N)$ be the optimal partition for (LP) and (LD), and set A_B and A_N to be the sub-matrices of A that correspond to the columns indexed by B and N, respectively. Let $z(b, \eta)$, $\eta > 0$, be the unique solution to the penalized linear program

$$\min \left\{ \delta c_B^T z - \eta \sum_{i \in B} \ln(z_i) : A_B z = b, z > 0 \right\}.$$

In other words, $z(b, \eta)$ is the central path for the linear program $\min\{\delta c_B^T z : A_B z = b, z \geq 0\}$. A key observation explained in [20] is that

$$x_B(\rho, \tau, \mu) = z\left(b + \rho \delta b - A_N x_N(\rho, \tau, \mu), \frac{\mu}{\tau} \right). \tag{4}$$

Two results are now used to guarantee the convergence of $x(\rho, \tau, \mu)$. First, $x_N(\rho, \tau, \mu) \downarrow 0$ as $(\rho, \tau, \mu) \downarrow 0$, which means that the convergence of $x_N(\rho, \tau, \mu)$ is independent of the any particular sequence of $(\rho, \tau, \mu) \downarrow 0$. Second, a continuity result in [36] shows that $z(b, \eta)$ is a continuous function of both b and η, so long as $\eta > 0$. Using these two results, the authors of [20] show that $x(\rho, \tau, \mu)$ converges when $(\rho, \tau, \mu) \downarrow 0$ and either $\tau = o(\mu)$ or

$\mu = \nu\tau$, for some constant ν. However, these are not necessary conditions for convergence, as demonstrated in the example where $x(\rho, \tau, \mu)$ converges for $\tau = \sqrt{\mu}$.

The equality in (4) has a geometric interpretation. Call $x(\rho, \tau, \mu)$ the "μ element" of the central path defined by $(b + \rho\delta b, c + \tau\delta c)$, and notice that

$$\{x : A_B x_B = b + \rho\delta b - A_N x_N(\rho, \tau, \mu), x_N = x_N(\rho, \tau, \mu), x_B \geq 0\}$$

is a shifted copy of $\mathcal{P}^*$. Equality (4) shows that $x_B(\rho, \tau, \mu)$ is the "μ/τ element" of a central path defined on a shifted copy of $\mathcal{P}^*$ defined by $(b + \rho\delta b - A_N x_N(\rho, \tau, \mu), \delta c)$. If the perturbation in c was arbitrary and non-linear, there would not be a δc available to make this interpretation. Hence, the linear change in c is required to obtain this geometric perspective.

There are two areas where the proposed research will be helpful. First, many interior point algorithms have been adapted to the Positive Semidefinite Programming (SDP) statement. The SDP problem statement is

$$\min_{X \in S^n} \{C \cdot X : X \succeq 0, A^i \cdot X = b^i, i = 1, 2, \ldots, m\},$$

where S^n is the space of real symmetric matrices, the $\cdot$ operator is defined by $M \cdot N = \text{trace}(MN)$, and $X \succeq 0$ means X is positive semidefinite. The study of SDP programming has flourished because the problem statement encompasses many classical optimization problems, and as such, provides a unified approach when studying these problems. As in linear programming, the infeasible interior point algorithms for SDP programming perturb C and b^i, $i = 1, 2, \ldots m$, to acquire an initial, strictly feasible element. Unfortunately, there is very little work that studies the sensitivity analysis of SDP (exceptions being [39] and [43]). The groundwork suggested in this proposal will give us insight into how the central path in SDP programming behaves under data perturbations, and the principal investigator will pursue similar result for the SDP problem statement.

Second, having a thorough knowledge of how the perturbed central path converges will have implications in the design of radiotherapy plans. External beam radiotherapy is a cancer treatment where beams of radiation are focused on a tumorous region. The idea is to find beam intensities and angles so that the radiation is deposited within the tumor but not in surrounding tissues. In [19] the principal investigator has shown how to use parameterization techniques to design "optimal" radiotherapy plans, and the plans most favorable to the patient are the analytic center solutions. So, being able to control the perturbation parameters ρ and τ in order to guarantee that the limit point of an algorithm is the analytic center is of great importance.

1.2 Convergence Under Arbitrary, Simultaneous Perturbations

We next consider the situation where the linearity of the perturbations in b and c is alleviated. Instead of having $b + \rho\delta b$ and $c + \tau\delta c$, we have sequences b^k and c^k that converge to b and c, respectively. The notation for x and (y, s) is changed to $x(b^k, c^k, \mu^k)$ and $(y(b^k, c^k, \mu^k), s(b^k, c^k, \mu^k))$, where these primal and dual elements are the unique solutions to

$$Ax = b^k \ , \ A^T y + s = c^k \ , \ Xs = \mu^k e \ , \ x \geq 0 \ , \ s \geq 0. \tag{5}$$

As previously mentioned, the results in [20] rely on linear perturbations, and the situation of arbitrary perturbations appears considerably more difficult. We demonstrate the increased difficulty with the following simple example.

Example: *Consider the very simple linear program*

$$\min\{c_1^k x_1 + c_2^k x_2 : x_1 + x_2 = b^k, x_1 \geq 0, x_2 \geq 0\}.$$

For any sequence with $c_1^k > 0$, $c_2^k > 0$, *and* $\mu^k > 0$, *we have that* $x_i(b^k, c^k, \mu^k)$, $i = 1, 2$, *is*

$$\frac{\mu^k}{c_1^k + c_2^k} + \frac{1}{2}b^k - \sqrt{\left(\frac{\mu^k}{c_1^k + c_2^k} + \frac{1}{2}b^k\right)^2 - \frac{\mu b^k}{c_1^k + c_2^k}}.$$

Let $b^k \to 1$, $\mu^k \to 0$, *and* c^k *be either* $(1 + 1/k)[1, 1]^T$ *or* $(2 + 1/k)[1, 1]^T$, *depending on whether* k *is even or odd. Then, as* $k \to \infty$ *we have that* $x(b^k, c^k, \mu^k) \to 0$. *This shows that* c^k *need not converge for* $x(b^k, c^k, \mu^k)$ *to converge.*

As this example shows, we can not simply look at converging sequence of (b^k, c^k, μ^k) to characterize the convergence of $x(b^k, c^k, \mu^k)$. This is why characterizing convergence under arbitrary perturbations appears substantially more difficult than characterizing the convergence under linear perturbations.

There is currently no literature that addresses the convergence of $x(b^k, c^k, \mu^k)$ and $(y(b^k, c^k, \mu^k), s(b^k, c^k, \mu^k))$ under arbitrary perturbation, which means this is fertile ground to conduct research. Moreover, the iterates of an infeasible interior point algorithm are not guaranteed to adhere to the linear perturbations of b and c found in (3). This is because the iterates of an infeasible interior point algorithm are not calculated with an exact solution to (3) (in most cases only a single Newton iteration is used). So, an understanding of the solutions to the more robust system found in (5) would provide an increased knowledge of how an infeasible interior point algorithm's iterates depend on b and c. The proposed analysis would also provide insights into sensitivity analysis questions. For example, having a complete understanding of how the central path depends on b and c will allow us to state whether or not the optimal partition is "stable" with respect to changes in b and c.

As was the case with linear perturbations, the analysis for arbitrary, nonlinear perturbations leads immediately to similar questions for the SDP problem. The generality of this problems statement will increase the difficulty of the analysis, but the principal investigator is confident that results are possible.

1.3 Sensitivity Analysis and Multiple Objective Programming

The third objective of this proposal is to investigate how sensitivity analysis results for linear programming can be used in multiple objective linear programming (MOLP). The MOLP problem statement is

$$\text{(MOLP)} \ \min\{Cx : Ax = b, x \geq 0\},$$

where $C \in \mathbb{R}^{q \times n}$. The q linear objectives are represented by the q rows of C and are denoted by c^i, $i = 1, 2, \ldots q$. Unlike linear programming, MOLP allows analysis in either *decision space*, $\mathcal{P} = \{x : Ax = b, x \geq 0\}$, or *objective space*, $\mathcal{O} = \{Cx : Ax = b, x \geq 0\}$, each providing their own unique perspective on algorithm and solution analysis. For the MOLP problem statement, minimization means to find a *pareto* optimal solution –i.e. a feasible element x such that a decrease in one $c^i x$ implies an increase in at least one of the remaining objectives. The set of all pareto optimal solutions is called the *Efficient Frontier*, denoted by $\mathcal{E}$. Practitioners often desire a complete description of the efficient frontier because such a description shows how improving one objective effects the status of the remaining objectives.

Because we are extending the results from the linear programming literature, we briefly describe the robust sensitivity analysis allowed by the linear programming optimal partition. Let $(B|N)$ be the optimal partition for (LP) and (LD). For the time being assume that there is no perturbation in the cost coefficient vector and that the right-hand side vector is parameterized along δb. Set $x^*(\rho)$ to be an optimal solution for the right-hand side $b + \rho\delta b$. A classic result [35] is that the objective function, $z^*(\rho) = c^T x^*(\rho)$, is a piecewise linear, convex function along δb. To see the inadequacies of the basic optimal approach, let $\mathbf{B}$ be an optimal basis for $\rho = 0$. To see how far this particular basis remains optimal along δb one would find the greatest ρ that satisfies $x^*(0) + \rho\mathbf{B}^{-1}\delta b \geq 0$. The problem is that this information is relative to the particular basis $\mathbf{B}$, and the maximum value of ρ may not indicate the end of the current linearity interval for $z^*(\theta)$ —i.e. more bases may be required to continue the parameterization along δb. Unlike an optimal basis, the optimal partition is unique and as the equalities in (2) show, the optimal sets characterize the optimal sets. To find the end point of the current linearity interval of $z^*(\rho)$ one must really solve

$$\max\{\rho : A_B x_B - \rho\delta b = b, x_B \geq 0\}.$$

However, this linear program is easily solved and stated only because of the optimal partition. The maximum value of ρ is the end point of the linearity interval of $z^*(\rho)$, and this value of ρ indicates where the optimal partition must change. In fact, the values of ρ for which $z^*(\rho)$ is not differentiable are precisely the places where the optimal partition must change to accommodate further parameterization. A similar analysis exists for perturbations in the cost coefficient vector, and interested readers are directed to [30] and [33]. We also mention that the sensitivity information from different simplex based solvers can vary greatly (an excellent example is found in [21]). The sensitivity analysis based on the optimal partition is not only robust, but also provides correct and consistent information.

While interior point algorithms have substantially influenced the manner in which we solve and analyze linear programs, the related field of MOLP has been the focus of only a small amount of research. Two exceptions are [1] and [4], with the former containing a development of a novel interior point algorithm that operates in *objective space* (generating an efficient face in polynomial time), and the algorithm in the latter generates iterates in *decision space* (using a linear combination of the search directions for the corresponding single objective linear programs). Outside of these two works there is little known about how interior point algorithms can provide new results for MOLP. In particular, there are two important areas that we will address. First, we will investigate how the analytic center solution may be used to provide information about the efficient frontier, and second we will see how the sensitivity analysis results based on the optimal partition may be extended to the MOLP problem statement.

Recently, Caron, Greenberg, and Holder [8] used a theory of *repelling inequalities* to incorporate the situation of multiple objectives into the convergence analysis of the primal and dual central paths. To understand the proposed research questions, we now give a brief description of their results. Let k_i, $i = 1, 2, \ldots, q$, be an upper bound on the single objectives $c_i x$ Instead of solving (MOLP) directly, the authors consider

$$\max\left\{\mu \sum_{i=1}^{q} \ln(k_i - c_i x) + \sum_{i=1}^{n} \ln(x_i) : Ax = b, x > 0, c_i x < k_i, i = 1, 2, \ldots, k\right\}.$$

As in the penalized linear programs in (1), the objective function is strictly concave, and hence, this problem has a unique solution, which we denote as $x(\mu)$. Similar to the linear programming case, the set $\{x(\mu) : \mu > 0\}$ is called the MOLP central path. One of the main results in [8] is that $x(\mu)$ converges as $\mu \downarrow 0$. Moreover, this limit is the analytic center of the following level set of $\mathcal{P}$,

$$\mathcal{L} = \operatorname{argmax}\left\{\sum_{i=1}^{q} \ln(k_i - c_i x) : x \in \mathcal{P}, c_i x < k_i, i = 1, 2, \ldots, q\right\}.$$

We note that $\mathcal{L}$ is not necessarily contained within the boundary of $\mathcal{P}$. This is in stark contrast to the linear programming case where the limits of the central path are the analytic centers of $\mathcal{P}^*$ and $\mathcal{D}^*$, both of which are lower dimensional faces of their corresponding feasible regions (making the tacit assumption that the objective functions are not constant over the feasible regions). While the results in [8] provide the convergence of the MOLP central path, there is no analysis showing how the limit of this path provides information about the efficient frontier. For example, in linear programming the optimal partition induced by the analytic center solution completely characterizes the optimal set. There is currently no MOLP counterpart to the linear programming optimal partition, and subsequently there is no analogue for the robust sensitivity analysis allowed in linear programming. We will investigate how the linear programming analysis extends to the MOLP problem. This means that we will define a MOLP optimal partition and show how to use this concept to analyze the efficient frontier. The difficulty of such analysis is compounded because $\mathcal{E}$ is generally composed of several faces of $\mathcal{P}$.

2 Student Involvement

The research problems just discussed require a basic knowledge of analysis, linear algebra, and optimization. All mathematics students at Trinity University are required to have course work in linear algebra and real analysis, and hence, would only require sufficient exposure to optimization to start working on these problems. This is one of the reasons that these research topics are suggested.

The principal investigator has formed a small group of second year students, called the APplied Optmization Group (APOG), as a pilot program (a web page describing this group is found at `http://www.trinity.edu/aholder/research/apog/`). The goals for APOG are:

- To expose students to research level questions.
- To show students that mathematical research is an enjoyable challenge.

- To encourage and foster an interest in pursuing advanced degrees in mathematics.

The group has met for two hours a week over the semester, and the students now have a clear understanding of the mathematical questions. They are beginning to implement some of the robust sensitivity analysis algorithms based on the optimal partition, but because the mathematics department does not have either the computing power or the optimization software required to develop academic level applications, the students are using their own computers (together with the free optimization solver PCx). Such a situation is not pleasant, and we are requesting funds for one Unix based workstation and several optimization packages (at least CPLEX, GAMS, and MATLAB). Having a common machine for the students to work on is extremely valuable as it (1) provides a uniform platform for development, (2) provides nightly backups, and (3) allows for easy version control. The small group means that one machine, with only a few terminals, will suffice. The equipment will be located in the mathematics computer lab so that the students will have access in the evenings and weekends. This last semester, the Division of Mathematics and Sciences was granted its own system administrator, so the technical support for the new equipment is already in place.

After a year of participation, the students in APOG will have developed some mathematical maturity and will have had the topics reinforced by implementing algorithms. In the subsequent 3^{rd} and 4^{th} years the students will be asked to work on original problems and implement new algorithms that rely on their work. The students in APOG will also be asked to present their work at the local MAA meetings, and depending on the quality of the work, may be asked to present material at research oriented conferences. The mathematics and computer science work of the students will be submitted for publication in undergraduate journals such as *Rose-Hulman Undergraduate Mathematics Journal*, *The Morehead Electronic Journal of Applications in Mathematics*, and *The Furman University Electronic Journal of Undergraduate Mathematics*. When appropriate, student work will be submitted to refereed, research quality journals.

APOG currently meets from 4:00 - 6:00 on Wednesdays, with students bringing a brown bag dinner. All three of the students in the group are sophomores, and this appears to be the time in their education to introduce them to the research topics. They are completing calculus and linear algebra and enjoy seeing how these topics can be applied to solve real world problems. During their junior year they will take both an algebra and an analysis course, and will acquire the skills to confidently pursue the goals of the project. This pilot program has substantiated the following:

- Studying advanced topics in sensitivity analysis is within the reach of undergraduates after 4-8 months of exposure to optimization.

- Implementing analysis algorithms provides a lively way to introduce mathematical concepts.

- Students find the projects interesting and are willing to undertake the research efforts in addition to their normal course work.

One of the reasons the principal investigator is requesting three years of support is to firmly establish the APplied Optimization Group. The goal is to build a group of nine to twelve students with varying scientific interests where mathematical research questions may be openly discussed. The group will be built in the following way. By October of each academic year, students in their 2^{nd} year will be invited to join APOG, with the invitations being made after a brief interview with the principal investigator. The purpose of the interview is to make sure that the student is completing calculus and linear algebra, to gauge interest in the questions being worked on, explain the possibility of summer support, and to establish expected time commitments. The program will be promoted by advertising on the departments web page and by distributing flyers in mathematics and related courses. Any 2^{nd} year student is welcome to apply, and in fact, a diverse collections of backgrounds is preferred (currently APOG has 1 math major, 1 computer science major, and 1 biology major). A special effort will be made to attract women by distributing flyers within female student groups like the Society of Women Engineers (the mathematics department has an exceptional track record of attracting bright women students into the mathematical sciences, with 20 out of 31 of our majors over the last 4 graduating classes being women). Three or four invitations will be made each year, so that over the next three years the group will have nine to twelve students (3,4 sophomores; 3,4 juniors; 3,4 seniors). The small size and mixed maturity levels will create a pleasant research environment where students can encourage and instruct each other. The meeting will remain during the dinner hour as there are no conflicts with other scheduled activities. The principal investigator is asking for funds to cover refreshments for these meetings.

In March of each academic year the students participating in APOG can apply for summer support to continue their work. The primary goal of supporting a student over the summer is to produce publishable quality material. Applications for summer support will be comprised of an unofficial transcript and a short written statement that indicates the problem they are currently working on and what they expect to accomplish over the summer. While the principal investigator will know the projects being worked on, a written report is a great way to measure how well a student can accurately state a problem, and how well they can communicate a proposed solution method. As such, this instrument will be used to measure the students mathematical maturity. The transcript is included so that grade point average and course experience are available when the principal investigator is deciding on who should get support. The requirements for summer support are

- an overall grade point average of at least 3.0,
- a mathematics grade point average of at least 3.5, and
- at least one math course in each of the prior semesters.

The awards will be announced by April 1 of each year, and the successful application materials will be forwarded to the NSF in subsequent reviews.

3 Conclusion and Assessment

The three tiered research projects of this proposal are interesting mathematically, important to practitioners, and approachable by undergraduate students. For these reasons we believe that this research project is perfect for the RUI program. The research into MOLP is the most open-ended, and the work here could lead to many years of productive research. All the research topics discussed have natural analogues in the popular realm of Positive Semidefinite Programming, the topics discussed here are fundamental to advances in the SDP literature.

The overriding objective of the proposed research is to produce publishable quality material, and articles supported by this grant will be submitted to journals such as *SIAM Journal on Optimization*, *Mathematical Programming*, *Mathematics of Operations Research*, and *Journal of Optimization Theory and Applications*. The principal investigator would consider this proposal a success provided

- at least five journal articles are published, two of which must include undergraduates,
- APOG is firmly established with 50% of the participants being female, and
- at least 2 of the 3 sophomores currently in APOG decide to attend a top tier graduate school in 2003.

The Impact Statement addresses how the fulfillment of the proposed projects will enhance the research and instructional capabilities of the mathematics department.

Impact Statement

1 An Introduction to Trinity

Located in San Antonio, TX, Trinity University is a private, comprehensive university with a strong liberal arts heritage. The primary focus at Trinity is to provide a top quality education for its 2500 undergraduates. Our students choose from B.A. and B.S. degrees in 40 majors and (pre)professional programs. There are three programs that offer Masters degrees (the highest degree awarded by Trinity): Education, Health Care, and Administration and Accounting. Students are required to live on campus their first 3 years, and they take 5 courses a semester.

In the late 1970s a substantial effort was made to increase the academic quality of the students. Most of the graduate programs were eliminated to strengthen the undergraduate education (a masters degree in Mathematics was cut). The recruiting success is evident in the fact that the average entering SAT score in 1981 was 1095, but has been over 1200 since 1985. Trinity routinely attracts 30 national merit finalist, and for the last 9 years Trinity University has been ranked by *US News & World Report* as the number one Western Regional University.

2 The Mathematics Department

During the 1980s the mathematics department saw a tremendous increase in institutional support. Prior to the fall of 1982 the department consisted of 6 full time faculty who taught four, 3 hour courses. However, from 1985 - 1987 the department grew to 11 faculty, including an endowed chair, and the teaching load decreased to three, 3 hour courses. The departmental budget increased from $4,000 in 1982 to more than $20,000 in 1989. These efforts were made because the University recognized that to become a top notch undergraduate institution meant that the faculty needed to have a balance of teaching and research. Faculty are more energetic and passionate about their fields when they enjoy the challenges of original research. Students are rewarded with educators that truly enjoy showing students the challenges that mathematics can present. The mathematic's faculty responded extremely well to the increased freedom presented them in the late 1980s, and since 1992, the faculty have published 45 papers in refereed journals, have 7 papers in submission, and have been awarded 7 grants (1 NSF-REU grant, 2 NSF-RUI grants, 3 PEW Foundation grants, and 1 NSF-SLiBS grant). Over the last 13 years two faculty have moved into management, and the endowed chair was removed after a retirement in 1987. The faculty currently consists of 8 full time faculty (2 full professors, 1 associate professor, and 5 assistant professors) with one visiting assistant professor.

Graduate courses have not been offered by the mathematics department since its Masters degree program was eliminated. The faculty teach three, 3 hour courses per semester, with two of these courses being service courses for science and engineering majors (almost exclusively calculus). So, each faculty gets to teach one upper divisional course a semester. Because of the emphasis on undergraduate instruction, the faculty are not encouraged to buy themselves out of courses for research, but rather are asked to find ways to incorporate students into their research. The Division of Mathematics and Sciences supports this endevour by providing the junior faculty with membership in the National Council of Undergraduate Research (NCUR). The principal investigator has found this a useful resource, especially because students enjoy speaking at the NCUR conference (the proceedings of this conference are also a good outlet for student projects).

The mathematics department has a strong commitment to undergraduate research, and all mathematics majors are required to complete 4 semesters of a seminar class. Juniors propose and work on problems published in collegiate mathematics journals such as *The American Mathematics Monthly* and *The Fibonoccii Quarterly.* Seniors work closely with a faculty member on a more substantial problem, and in their last semester must prepare a written report and an oral presentation. The senior project is graded by a committee of three faculty: the project advisor, the person conducting the seminar, and one other faculty selected by the student. Examples demonstrating the quality of the work completed by our seniors is found at `http://www.trinity.edu/departments/mathematics/studtechreport.html`. The majors seminar has been a turning point in some of our students educational path. As an example, the principal investigator advised Son Quach's senior project last year, and while completing his work decided that research was enjoyable. He also met Dr. David Morton who is in The University of Texas' Operations Research program, where Son is now in graduate school. The majors seminar is also beneficial to faculty because it allows us an opportunity to present, think about, and be challenged by research topics. Because of the large amount of time faculty spend on education, recognizing the benefits of working with bright undergraduates is imperative to a successful research program.

The department's commitment to undergraduate research is demonstrated in two other manners. First, the department conducted undergraduate research projects in the summers of 1998, 1999, and 2000 through an NSF-REU program. A REU grant for the next three summers is currently submitted. Second, the mathematics faculty advises the Trinity Mathematics Modeling Group (TM^{2}G). The students use weekly meetings to prepare for competitions in the COMAP modeling contest and the Statistical Analysis competition sponsored by Tennessee Technical University. For the last 6 years, TM^{2}G has had at least one team awarded a meritorious score in the COMAP competition. The principal investigator has been a TM^{2}G advisor for the last two years, and has found this to be a great place to meet some of Trinity's brightest students.

Mathematics majors at Trinity have been successful in graduate school. As stated in *Baccalaureate Origins of Doctoral Recipients* (1998), 13 mathematics Ph.Ds were granted to students completing a bachelors degree at Trinity University from 1920 to 1995, and 2 mathematics Ph.Ds were granted from 1986 to 1995. This ranks Trinity University 32nd and 31st, respectively, out of 253 masters degree granting institutions. The REU experiences of students studying at Trinity have also been successful, with several students attending top tier graduate schools. In short, the faculty of the mathematics department have the experience to guide young, talented minds through challenging research questions.

3 The Impact of the Proposed Research

This RUI grant will effect the research and pedagogical environments of the Department of Mathematics in several ways. As already mentioned, all mathematics majors must complete a senior project, and faculty that are engaged in research always seem to have the best ideas for these projects. Constantly working on original research means that one is continuously asking and answering research questions at all levels, so it is not difficult to propose a suitable question for a senior research project. The level of difficulty of these projects needs to be tailored to the specific student; projects that are too easy or too hard do not provide a rewarding experience for the student. Hence, it is important to have the command of a field with research questions at all levels. Such a command is simply not possible unless the faculty member is active in research. The research of this proposal will generate many interesting questions that will be approachable by our students, and the principal investigator looks forward to working with some of our seniors on related questions.

Many students prefer upper divisional courses that show how their mathematical knowledge can be applied to solve real world problems. This is not to say that students want to learn mathematics only because it is a convenient vehicle for applications, but rather that they respect their mathematics training better because it is useful in an applied context. The field of optimization (and the larger field of Operations Research) is a superb area to deliver both serious mathematical content and important real world applications. While our modeling course does contain some optimization modeling, we currently do not offer a full class in optimization. By advising interesting senior projects and teaching the modeling class, the principal investigator has started to generate interest in an optimization class, and it appears as though there is enough interest to offer a special topics class in mathematical optimization. However, having an optimization class in the permanent rotation has many advantages. First, one of the goals of this course is to make students aware of graduate opportunities in Operations Research (OR). Most students have never heard of this field and are not aware that this applied science relies on the mathematical fields of statistics and optimization. With the proposed optimization course and the two statistics classes currently offered, students are capable of a having a good foundation for graduate work in OR. As already mentioned, one of the principal investigator's students has been accepted in the OR program at the University of Texas. OR programs usually prefer mathematics majors over other majors because they often handle the mathematical rigor of graduate courses much better. Because OR programs are usually located within engineering and business departments, undergraduate students studying mathematics often never see OR programs. However, the principal investigator strongly believes that many students will consider OR as a field of graduate study once they are aware of such possibilities. Second, the principal investigator would benefit from teaching an upper divisional course in optimization. As is usually the case, instructors are considerably more energetic when they are teaching a field that is dear to them. Also, as time permits new research topics can be discussed within the class, and hence students would see questions for which there is no known answer. Third, an optimization class would contain a computing component, something that our alumni have told us that we, as a mathematics department, need to include more often. The equipment and optimization software requested for the APplied Optimization Group (APOG) will make the computing portion of an optimization course possible. As one can see, the benefits of having an optimization class are substantial, and the principal investigator looks forward to developing and teaching this course.

As already mentioned within the project proposal, the principal investigator is coordinating the APplied Optmization Group. Other than providing real research opportunities for undergraduates, one of the main purposes of this activity is to prepare students for graduate school. Students participating in APOG will know how to learn mathematics on their own and will not shy away from difficult problems, both valuable skills when making the transition from course work to research. These students will also graduate with advanced computing skills, a quality that can not be over-valued in today's economy. Finally, students in APOG will travel to conferences and give presentations. These experiences will provide them with confidence in their work, allow them to meet other student researchers, and let them meet faculty at institutions with graduate programs. This last point will be reinforced by bringing in two quality researchers per year. The duration of these visits will vary, but will usually be about a week. During their stay, the visiting researcher will work on projects with the principal investigator and the APOG group. The visiting researcher will be asked to give two talks during their stay. First, they will give a talk in our departmental colloquium so that the entire faculty can experience the expertise the individual brings to campus. Second, they will talk in our majors seminar where they will discuss graduate opportunities at their home institution. To present our students with a wide view of the possibilities available to them, these visiting researchers will be invited from a variety of graduate schools. At the end of 3 years of participation in APOG, students will graduate with skills that are not taught in normal courses, will understand the efforts and rewards of research, and will have made contacts in several graduate programs. The goal of the principal investigator is to have at least 60% of the students participating in APOG placed in quality graduate programs.

The new workstation suggested for APOG will be of great value to the mathematics department. Last year there were two instances where the limited computing resources of the department hindered faculty research. First, Dr. Sabar Elaydi was not able to establish a result that required a numerical solution to a specific system of equations. Second, the principal investigator developed a two dimensional planning system for radio-surgeries, and the capabilities of the software were greatly restricted because of the lack of technological resources. Further still, the new machine will provide students in numerical analysis, mathematical modeling, differential equations, optimization, and statistics far greater abilities than currently possible. The lack of programming experience among our majors is extremely evident in the questionnaires returned by our past graduates (93% of them felt that their computing skills were low, and suggested that we require more programming experience). The mathematics department is redesigning the numerical analysis sequence so that the first course has a large C/C++ computing component. All students in this course will be given accounts on the new workstation and will use the gnu compilers throughout the course. In general, accounts are available to any student that has a need for the increased computing power and/or the optimization software. The mathematics faculty will automatically be given accounts.

The fulfillment of the goals presented in this proposal will have substantial and permanent influence on the principal investigator, the Department of Mathematics, the mathematic's majors, and the students participating in APOG. The principal investigator will have established a sound and sustainable research program, while providing high quality instruction at a predominately undergraduate institution. The research objectives are important, interesting, and will lead to several research publications. The Department of Mathematics benefits by having an energetic researcher who will continue to bring in world class researchers for talks and consultation. Acquiring funding through grants encourages the remaining faculty to write and pursue granting opportunities. The equipment and software supplied by this grant will give the mathematicians at Trinity a new tool for research and pedagogy, and our students will graduate with an increased knowledge of computing. The three years that students spend in APOG will give them the special skills required to be successful in graduate school. In conclusion, the fulfillment of this proposal will greatly enhance all aspects of an already strong department here at Trinity University.

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