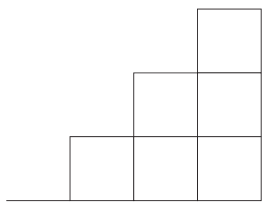
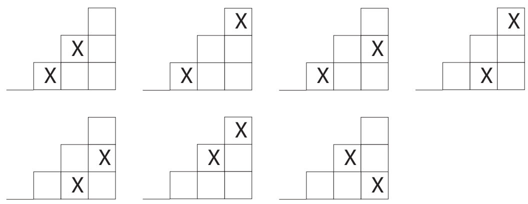
ROOK THEORY & POLY-STIRLING NUMBERS

ABSTRACT. Rook theory is the study of algebraic structures in terms of looking at ways in which to place rooks on a chess board. I propose the study of a new class of generalized Stirling numbers, called *Poly-Stirling numbers*, using this theory.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. We say that $p(x)$ is a polynomial if $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$, where m is a natural number and the number a_k is called the *coefficient* of x^k. For any natural number n, we let $\mathcal{B}_n = F(0, 1, 2, \ldots, n-1)$ denote the n^{th} *staircase board*, which one can think of as a chess board with column heights, from left to right, of $0, 1, 2, \ldots, n-1$. An example of $\mathcal{B}_4$ is shown here.



We wish to place nonattacking rooks on these staircase boards, that is, we will place rooks in $\mathcal{B}_n$ such that no two rooks lie in the same row or column. We will define $r_k(\mathcal{B}_n)$ to be the number of ways of placing k nonattacking rooks in the board $\mathcal{B}_n$, and we will call $r_k(\mathcal{B}_n)$ the k^{th} *rook number of* $\mathcal{B}_n$. As an example, $r_2(\mathcal{B}_4) = 7$, since there are seven ways to place two nonattacking rooks in the board $\mathcal{B}_4$. These seven placements are shown here.



In [13], Goldman, Joichi, and White used this notion of placing rooks on a board to prove that for any natural number n and any number x,

$$x^n = \sum_{k=0}^{n} r_{n-k}(\mathcal{B}_n)(x) \downarrow_k, \tag{0.1}$$

where $(x) \downarrow_k = x(x-1)(x-2)\cdots(x-(k-1))$. This idea of placing rooks on boards to generate mathematical formulas is called *rook theory*, and since this original rook theory paper in which Equation (0.1) appeared, generalizations of this rook theory model have been studied in [3], [4], [5], [6], [7], [12], [15], [16], [19], and [26]. In many of these papers, different types boards are presented (not necessarily the staircase boards), and consequently different types of rook numbers are defined and studied.

Now, in what appears at first to be a total divergence of topics, there exists another class of numbers which have been studied for a very long time, called *Stirling numbers.* These numbers satisfy the recursive relation

$$S_{n+1,k} = S_{n,k-1} + kS_{n,k},$$

where $S_{0,0} = 1$ and $S_{n,k} = 0$ if $n, k < 0$ or $n < k$, and there are many interesting formulas which involve the $S_{n,k}$. Two such formulas are

$$\sum_{k=0}^{n} S_{n,k} \frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k \tag{0.2}$$

and

$$\sum_{k=0}^{n} S_{n,k} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}. \tag{0.3}$$

Many generalizations of these numbers have been studied, most notably in [8], [10], [14], [17], [18], [21], [22], [23], [24], [25], [27], [28], [29], [30], [31], [32], [33], and [34], and in some of these papers one can find the corresponding generalizations of Equations (0.2) and (0.3).

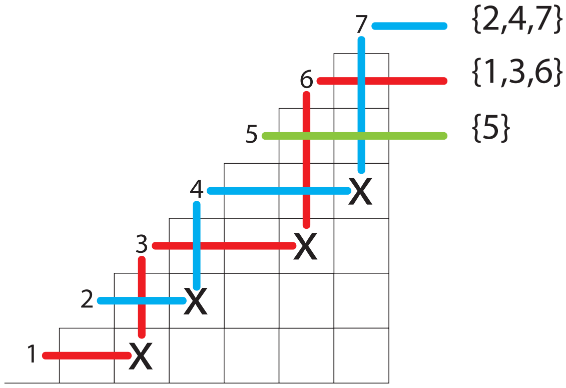
These Stirling numbers are also referred to as the *Stirling set numbers*, because $S_{n,k}$ is equal to the number of ways of partitioning the set $\{1, 2, \ldots, n\}$ into k unordered parts. For example, $S_{4,2} = 7$ since we can partition the set $\{1, 2, 3, 4\}$ into two parts in the following seven ways.

$$\begin{gathered}
\{1\}\{2,3,4\} \\
\{2\}\{1,3,4\} \\
\{3\}\{1,2,4\} \\
\{4\}\{1,2,3\} \\
\{1,2\}\{3,4\} \\
\{1,3\}\{2,4\} \\
\{1,4\}\{2,3\}
\end{gathered}$$

In fact, one may notice that this happens to be the same number of ways of placing two rooks in the board $\mathcal{B}_4$, which is no coincidence. In a paper by Garsia and Remmel [9], it is shown that for all n and k,

$$r_{n-k}(\mathcal{B}_n) = S_{n,k}.$$

In other words, the number of ways of placing $n - k$ rooks in $\mathcal{B}_n$ is always the numbers of ways of partitioning the set $\{1, 2, \cdots, n\}$ into k parts. An example of how one can match off such placements with the correct partition is shown on the next page, where $n = 7$ and $k = 3$. Here, we start at the column labeled "1" and move to the right until we come to a rook. We then go up to the corresponding number and repeat the process. If we move to the right and leave the board, then, we start the process up again at whatever the smallest unused number is. In this way, we turn a placement of $n - k = 4$ rooks in the board $\mathcal{B}_7$ into the partition $\{1, 3, 6\}\{2, 4, 7\}\{5\}$.



In the 20 years since [9] was published, many papers have been written which use the relationship between Stirling numbers and rook boards to study completely different branches of mathematics, such as group and ring theory in [1] and [2]and special functions and hypergeometric series in [11] and [20].

I propose the study of the following generalized Stirling numbers, $S^{p(x)}_{n,k}$, which I call *Poly-Stirling numbers*, defined by the following recursion:

$$S^{p(x)}_{n+1,k} = S^{p(x)}_{n,k-1} + p(k)S^{p(x)}_{n,k}.$$

Here, $p(x)$ is a polynomial with natural number coefficients, $S^{p(x)}_{0,0} = 1$, and $S^{p(x)}_{n,k} = 0$ if $n, k < 0$ or $n < k$. In particular, I would like to investigate the following problems.

Problem I. Are there correlations between special types of rook boards and Poly-Stirling numbers?

Problem II. Are there formulas for the $S^{p(x)}_{n,k}$ which correspond to Equations (0.2) and (0.3)?

Problem III. Can the results from Problems I and II be used to study other problems in mathematics?

As the notion of a Stirling number is one that most undergraduate math majors deal with at some point in their studies, these problems are ideal for undergraduate research. As a junior faculty at Trinity, one of my goals is to involve students from my department in the research that I undertake, and the study of these proposed problems is an excellent venue for such work.

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