A note on the asymptotic stability of linear Volterra difference equations of convolution type

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Abstract

We show that the condition $|a| + \left|\sum_{l=0}^{+\infty} b_l\right| < 1$ is not necessary, though sufficient, for the asymptotic stability of $x_{n+1} = ax_n + \sum_{l=0}^{+\infty} b_{n-l}x_l$. We prove the existence of a class of Volterra difference equations that violate this condition but whose zero solutions are asymptotically stable.

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1 Introduction

Let us consider the linear convolution Volterra Difference Equations (VDEs)

$$x_{n+1} = ax_n + \sum_{l=0}^{n} b_{n-l} x_l, \; n \geq 0, \tag{1.1}$$

where x_0 is given and $a \in I\!R$, and recall the following definitions of stability [8, p.176].

Definition 1 *The zero solution of (1.1) is said to be*

1. stable *if for all $\epsilon > 0$ there exists $\delta(\epsilon) >$ such that $|x_0| < \delta$ implies $|x_n| < \epsilon$ for all $n \geq 0$;*

2. asymptotically stable *if it is stable and there exists μ such that $|x_0| < \mu$ implies $lim_{n\to+\infty} x_n = 0$.*

In the last two decades, many authors investigated the asymptotic stability of (1.1) [8, p.39] and [2, 3, 5, 6, 7, 9, 10, 12] mostly by means of the $Z-$transform or the Liapunov approach [8, chap.6]. The earliest thorough study of (1.1) was carried out by Elaydi in [5]. However, the condition he proved, as he himself writes, although necessary and sufficient, is not "practical" because it requires the localization of the roots of a complex function related to the $Z-$transform of the sequence $\{b_n\}_{n\geq 0}$ of the coefficients of (1.1).

In the same paper the following explicit criterion for the asymptotic stability of (1.1) was provided:

Theorem 1.1 *[5, 8] Suppose that b_n does not change sign for $n \geq 0$ and*

$$|a| + \left| \sum_{l=0}^{+\infty} b_l \right| < 1, \tag{1.2}$$

then the zero solution of (1.1) is asymptotically stable.

This is a nice sufficient condition directly expressed in terms of the coefficients of the VDE considered and, until now, it is still an open question whether or not (1.2) is also necessary for the asymptotic stability of (1.1) [8, p.296].

The purpose of this paper is to answer this question. Namely, we prove that, starting from any sequence $\{\beta_n\}_{n\geq 0}$ satisfying $(-1)^i \Delta\beta_n \leq 0, \quad i = 0, 1, 2,$ we can construct an infinite number of VDEs of the type (1.1) whose solution is asymptotically stable and whose coefficients satisfy $|a| + \left|\sum_{l=0}^{+\infty} b_l\right| \geq 1$. Hence, we can conclude that (1.2) is not necessary.

In the next section we prove our result by effectively constructing a class of asymptotically stable VDEs that volate (1.2).

2 Stability of VDEs

In this section we describe our result on the stability of the zero solution of (1.1). Since our approach in the study of the asymptotic properties of x_n will go through the Liapunov technique for VDEs, we refer to [1] and we report here the main result for the general Volterra difference equation of unbounded order

$$y_{n+1} = F(n, y_0, \dots, y_n), \; n \geq 0, \; y_n \in \mathbb{R}, \tag{2.1}$$

with $F(n, 0, \dots, 0) = 0$.

Theorem 2.1 *Let $\omega_i(r)$, $r \in \mathbb{R}$, $i = 1, 2$, be scalar continuous increasing functions such that $\omega_i(0) = 0$, $i = 1, 2$. If there exists a scalar function $V(n, y_0, \dots, y_n)$, continuous with respect to all the variables y_0, $y_1, \dots, y_n, \dots$, such that*

a) $V(0, 0) = 0$

b) $V(n, y_0, \dots, y_n) \geq \omega_1(|y_n|), \quad n \geq 0$

c) $\Delta V_n = V(n+1, y_0, \dots, y_n, F(n, y_0, \dots, y_n)) - V(n, y_0, \dots, y_n) \leq 0, \quad n \geq 0$

then the solution of (2.1) is stable. If, in addition

d) $\Delta V_n \leq -\omega_2(|y_n|)$

then the solution of (2.1) is asymptotically stable.

The following theorem shows how to obtain a class of VDEs which is asymptotically stable even though (1.2) is not satisfied.

Theorem 2.2 *Assume there exists a sequence $\{b_n\}_{n \in \mathbb{N}}$ that satisfies*

i. $(-1)^k \Delta^k b_n \leq 0$, $k = 0, 1, 2$, *for each* $n = 0, 1, \dots$

ii. $b_1 - 2b_0 - 2 < 0$

and one of the following two conditions holds

iii$_1$. $b_1 + 2b_0 + 1 \geq 0$, $b_1 < 0$

iii$_2$. $b_1 + 4b_0 + 2 > 0$,

then it is always possible to find $a \in \mathbb{R}$ such that $|a| + |\sum_{l=0}^{\infty} b_l| \geq 1$ and the zero solution of $x_{n+1} = a x_n + \sum_{l=0}^{\infty} b_{n-l} x_l$ is asymptotically stable.

Proof. Let b_n be such a sequence and choose a such that

$$\frac{b_1 - 2b_0 - 2}{2} < a \leq \min\{-(1 + b_0), b_1 + b_0\}. \tag{2.2}$$

With this form for b_n and the values prescribed in (2.2) for a, it is obvious that $|a| + \left|\sum_{l=0}^{+\infty} b_l\right| \geq 1$ and hence (1.2) is not satisfied. Let us consider the Liapunov function

$$\begin{aligned} V(n, x_0, \dots, x_n) = &- (b_n + a\delta_{n,0}) \left(\sum_{j=0}^{n} x_j\right)^2 \\ &+ (b_0 + a + 2)x_n^2 + \sum_{i=0}^{n-1} (b_{n-i} - b_{n-1-i} - a\delta_{n-1,i}) \left(\sum_{j=0}^{n} x_j\right)^2, \end{aligned} \tag{2.3}$$

where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$, and set

$$\begin{aligned} \omega_1(y) =& (b_0 + a + 2)y^2 \\ \omega_2(y) =& - (b_1 - 2(b_0 + a) - 2)y^2. \end{aligned} \tag{2.4}$$

V is continuous with respect to $x_0, \dots, x_n$ and ω_1 and ω_2 are continuous. Notice that from Conditions i and ii, we have $b_1 > -2$. This implies using (2.2) that $b_0 + a + 2 > 0$,and $b_1 - 2(b_0 + a) - 2 < 0$. Hence, $\omega_1(y)$ and $\omega_2(y)$ in (2.4) are positive increasing functions. Now we show that V satisfies hypotheses a)-d) of Theorem 2.1.

Of course, it is clear from (2.3) that $V(0, 0) = 0$ and

$$V(n, x_0, \dots, x_n) \geq \omega_1(|x_n|).$$

In order to prove that V satisfies c), let us consider

$$\begin{aligned} \Delta V_n &= V(n+1, x_0, x_1, \dots, x_{n+1}) - V(n, x_0, x_1, \dots, x_n) \\ &= -(b_{n+1} + a\delta_{n+1,0}) \left(\sum_{j=0}^{n+1} x_j\right)^2 + (b_0 + a + 2)x_{n+1}^2 \\ &\quad + \sum_{i=0}^{n} (b_{n+1-i} - b_{n-i} - a\delta_{n,i}) \left(\sum_{j=i}^{n+1} x_j\right)^2 - V(n, x_0, \dots, x_n) \end{aligned}$$

$$\begin{aligned} = \; & -(b_{n+1} + a\delta_{n+1,0}) \left(\left(\sum_{j=0}^{n} x_j\right)^2 + x_{n+1}^2 + 2x_{n+1} \sum_{j=0}^{n} x_j\right) + (b_0 + a + 2)x_{n+1}^2 \\ & + \sum_{i=0}^{n} (b_{n+1-i} - b_{n-i} - a\delta_{n,i}) \left(\left(\sum_{j=i}^{n} x_j\right)^2 + x_{n+1}^2 + 2x_{n+1} \sum_{j=i}^{n} x_j\right) \\ & - V(n, x_0, x_1, \dots, x_n), \quad n \geq 0. \end{aligned}$$

By manipulating this expression we get:

$$
\begin{aligned}
\Delta V_n = & -(b_{n+1}+a\delta_{n+1,0})\left(\sum_{j=0}^{n} x_j\right)^2-(b_{n+1}+a\delta_{n+1,0})\left(x_{n+1}^2+2x_{n+1}\sum_{j=0}^{n} x_j\right)\\
&+(b_0+a+2)x_{n+1}^2+\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})x_{n+1}^2\\
&+2x_{n+1}\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})\sum_{j=i}^{n}x_j+\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})\left(\sum_{j=i}^{n}x_j\right)^2\\
&+(b_n+a\delta_{n,0})\left(\sum_{j=0}^{n}x_j\right)^2-(b_0+a+2)x_n^2-\sum_{i=0}^{n-1}(b_{n-i}-b_{n-1-i}-a\delta_{n-1,i})\left(\sum_{j=i}^{n}x_j\right)^2\\
= & B_n-(b_{n+1}+b_n-a\delta_{n,0})\left(\sum_{j=0}^{n}x_j\right)^2\\
&+\sum_{i=0}^{n-1}(b_{n+1-i}-2b_{n-i}+b_{n-1-i}+a\delta_{n-1,i})\left(\sum_{j=i}^{n}x_j\right)^2+(b_1-2(b_0+a)-2)x_n^2, \quad n\geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
B_n &= 2x_{n+1}\left(\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})\sum_{j=i}^{n}x_j-(b_{n+1}+a\delta_{n+1,0})\sum_{j=0}^{n}x_j\right)\\
&\quad +x_{n+1}^2\left(-(b_{n+1}+a\delta_{n+1,0})+(b_0+a+2)+\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})\right).
\end{aligned}
$$

By interchanging the order of summation in $\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i}-a\delta_{n,i})\sum_{j=i}^{n}x_j$, it is easy to prove that

$$
\begin{aligned}
B_n &= 2x_{n+1}\left(\sum_{j=0}^{n}x_j\left(\sum_{i=0}^{j}(b_{n+1-i}-b_{n-i})-\sum_{i=0}^{j}a\delta_{n,i}-b_{n+1}\right)\right)\\
&\quad +x_{n+1}^2\left(-(b_{n+1}+b_0+a+2)+\sum_{i=0}^{n}(b_{n+1-i}-b_{n-i})-a\right),
\end{aligned}
$$

and by using the fact that $\sum_{i=0}^{j}(b_{n+1-i}-b_{n-i})=b_{n+1}-b_{n-j}$, it comes out that $B_n=0$. For the other quantities involved in the expression of ΔV_n, we have that

$$
b_{n+1}+b_n-a\delta_{n,0}=\begin{cases} b_1+b_0-a & \text{if } n=0\\ b_{n+1}+b_n & \text{if } n>0\end{cases} \tag{2.5}
$$

and

$$
b_{n+1-i}-2b_{n-i}+b_{n-1-i}-a\delta_{n-1,i}=\begin{cases} b_2-2b_1+b_0+a & \text{if } i=n-1\\ b_{n+1-i}-2b_{n-i}+b_{n-1-i} & \text{otherwise.}\end{cases} \tag{2.6}
$$

As a consequence of the hypotheses on b_n and of our choice (2.2) for a, both (2.5) and (2.6) are less than or equal to zero. Therefore,

$$\Delta V_n \leq -\omega_2(|x_n|),\ n = 0, 1, \dots ,$$

where $\omega_2(|\cdot|)$ is given in (2.4).

In conclusion, V satisfies all the hypotheses of Theorem 2.1, and consequently, the zero solution of (1.1) is asymptotically stable. $\square$

Notice that the result in theorem (2.2) is valid subject to the existence of a sequence b_n which satisfies Conditions i, ii, and either iii$_1$ or iii$_2$. The following theorem provides an algorithm on how to construct such a sequence.

Theorem 2.3 *There exists an infinite number of sequences $\{b_n\}_{n\in\mathbb{N}}$ such that*

i. $(-1)^k\Delta^k b_n \leq 0,\ k = 0, 1, 2,\ n \geq 0$

ii. $b_1 - 2b_0 - 2 < 0$

iii$_2$. $b_1 + 4b_0 + 2 > 0,$

hold.

Proof. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence whose terms satisfy $(-1)^k\Delta^k\beta_n \leq 0$, for $k = 0, 1, 2$ and $n \geq 0$ (for instance $\beta_n = -\dfrac{1}{2^n}$). Set $b_n = -\dfrac{\beta_n}{\theta\beta_0}$ where $\theta > 2 + \dfrac{\beta_1}{\beta_0}$. From the assumptions on β_n, it immediately follows that $b_n \leq 0$, for $n \geq 0$, $b_n - b_{n-1} \geq 0$ and $b_{n+1} - 2b_n + b_{n-1} \leq 0$, for all $n \geq 1$ and then i. holds. Now consider

$$b_1 - 2b_0 - 2 = \frac{-\beta_1 - 2(\theta - 1)\beta_0}{\theta\beta_0}$$

and

$$b_1 + 4b_0 + 2 = \frac{-\beta_1 - 2(2 - \theta)\beta_0}{\theta\beta_0}.$$

Since β_0 and β_1 are negative and $\theta > 2 + \dfrac{\beta_1}{\beta_0}$, the first expression above is negative, while the second is positive. Hence, ii. and iii$_2$. hold. The result stated in the theorem is then proved. $\square$

Example 2.4 *It is easy to check that the sequence $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = -\dfrac{1}{3}\dfrac{1}{2^n}$ satisfies the hypotheses of Theorem 2.2. In this case, any a belonging to $\left(-\frac{3}{4}, -\frac{2}{3}\right)$ leads to the asymptotic stability of (1.1).*

3 Concluding remarks

In this note we propose a constructive analysis on the asymptotic stability of a class of VDEs of convolution type. In particular we show the non necessity of condition (1.2) for the asymptotic stability of (1.1) by proving the existence of infinite sequences $\{b_n\}_{n\in\mathbb{N}}$ which, for certain values of a, lead to the asymptotic stability of (1.1) and that, all the same, don't fulfill (1.2). The problem of finding "easily verifiable" necessary and sufficient condition for the asymptotic stability of the zero solution of (1.1) remain therefore an open question!

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