# Towards a theory of periodic difference equations and its application to population dynamics

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**Abstract**

This survey contains the most updated results on the dynamics of periodic difference equations or discrete dynamical systems this time. Our focus will be on stability theory, bifurcation theory, and on the effect of periodic forcing on the welfare of the population (attenuance versus resonance). Moreover, the survey alludes to two more types of dynamical systems, namely, almost periodic difference equations and stochastic difference equations.

**Keys Words: Discrete dynamical systems, Nonautonomous periodic difference equation, Skew-product systems, Cycles, Stability, Bifurcation, Attenuance and resonance**

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# 1 Introduction

In a series of papers, Elaydi and Sacker [13, 14, 15, 32] embarked on a systematic study of periodic difference equations or periodic dynamical systems. The authors also wrote a survey [16] which has not been readily available to researches. The main purpose of this survey is to update, extend, and broaden the above-mentioned survey.

Since the appearance [16], there have many exciting and new results by many authors as reflected by the extensive list of references.

An emphasis is placed here on bifurcation theory of periodic systems, particularly, those obtained by the authors and their collaborators. In fact, some of the results reported here appear for the first time. A more detailed account of bifurcation theory will appear somewhere else.

Two important omissions should be noted. The first is the extension of Sharkovsky's theorem to periodic difference equations [3]. The second is the study of periodic systems with the Allee effect [29]. One reason for not including these topics is our self-imposed limitations on the size of the survey. A second reason is the limitation in the expertise of the writers of this survey. We promise the reader to explore these two topics in a forthcoming work.

In section 2, we motivate the need for introducing skew-product techniques in the study of nonautonomous difference equations. Section 3 develops the basic construction of skew-product dynamical systems.

Subsequently, in section 4 our study is focused on periodic difference equations. This section includes two important results in the theory of periodic systems, namely, lemma 6 and lemma 8. Then in section 5, we tackle the question of stability in both the space $X$ and in the skew-product $X \times Y$. The section ends with the fundamental result in theorem 14, which states that in a connected topological space, the period of a globally asymptotically stable periodic orbit must divide the period of the system.

In section 6 we extend Singer's theorem to periodic systems. And in section 7 we develop a bifurcation theory for $2-$periodic difference equations. In particular, a unimodal map with the Allee effect is thoroughly analyzed. A bifurcation graph of the parameter space of a $2-$periodic system consisting of these maps is developed using the techniques of resultant in Mathematica software.

In section 8 we address the question of whether the solutions of bifurcation equations are independent of the phase shifts.

In section 9, we present an updated account of results pertaining to attenuance and resonance. The question we tackle here is whether periodic forcing has a deleterious effect on the population (attenuance) or it is advantageous to the population (resonance). Section 10 introduces almost periodicity and contains some of the results obtained in [10]. This is followed by section 11 in which the study of stochastic difference equations is conducted.

# 2 Preliminaries

Let $X$ be a topological space and $\mathbb{Z}$ be the set of integers. A discrete dynamical system $(X, \pi)$ is defined as a map $\pi : X \times \mathbb{Z} \to X$ such that $\pi$ is continuous and satisfies the following two properties

1. $\pi(x, 0) = x$ for all $x \in X$,
2. $\pi(\pi(x, s), t) = \pi(x, s + t), s, t \in \mathbb{Z}$ and $x \in X$ (the group property).

We say $(X, \pi)$ is a discrete semidynamical system if $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$, the set of nonnegative integers, and the group property is replaced by the semigroup property.

Notice that $(X, \pi)$ can be generated by a map $f$ defined as $\pi(x, n) = f^n(x)$, where $f^n$ denotes the $n^{th}$ composition of $f$. We observe that the crucial property here is the semigroup property.

A difference equation is called autonomous if it is generated by one map such as

$$x_{n+1} = f(x_n), n \in \mathbb{Z}^+. \tag{1}$$

Notice that for any $x_0 \in X, x_n = f^n(x_0)$. Hence, the orbit $\mathcal{O}(x_0) = \{x_0, x_1, x_2, \ldots\}$ in Eq. (1) is the same as the set $\mathcal{O}(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$ under the map $f$.

A difference equation is called nonautonomous if it is governed by the rule

$$x_{n+1} = F(n, x_n), n \in \mathbb{Z}^+, \tag{2}$$

which may be written in the friendlier form

$$x_{n+1} = f_n(x_n), n \in \mathbb{Z}^+, \tag{3}$$

where $f_n(x) = F(n, x)$. Here the orbit of a point $x_0$ is generated by the composition of the sequence of maps $\{f_n\}$. Explicitly,

$$\begin{aligned} \mathcal{O}(x_0) &= \{x_0, f_0(x_0), f_1(f_0(x_0)), f_2(f_1(f_0(x_0))), ...\} \\ &= \{x_0, x_1, x_2, ...\}. \end{aligned}$$

It should be pointed out here that equation (2) or equation (3) may not generate a discrete semidynamical system as it may not satisfy the semigroup property. The following example illustrates this point.

**Example 1** *Consider the nonautonomous difference equation*

$$x_{n+1} = (-1)^n \left(\frac{n+1}{n+2}\right) x_n, x(0) = x_0. \tag{4}$$

*The solution of Eq. (4) is*

$$x_n = (-1)^{\frac{n(n-1)}{2}} \frac{x_0}{n+1}.$$

*Let $\pi(x_0, n) = x_n$. Then*

$$\begin{aligned} \pi(\pi(x_0, m), n) &= \pi\left((-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{m+1}, n\right) \\ &= (-1)^{\frac{n(n-1)}{2}} (-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{(n+1)(m+1)} \end{aligned}$$

*However,*

$$\pi(x_0, m+n) = (-1)^{\frac{(n+m)(n+m-1)}{2}} \frac{x_0}{m+n+1} \neq \pi(\pi(x_0, m), n).$$

# 3 Skew-product Systems

Consider the nonautonomous difference equation

$$x_{n+1} = F(n, x_n), n \in \mathbb{Z}^+, \tag{5}$$

where $F(n, \cdot) \in C(\mathbb{Z}^+ \times X, X) = C$. The space $C$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z}^+ \times X$. Let $F_t(n, \cdot) = F(t + n, \cdot)$ and $\mathcal{A} = \{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$ be the set of translates of $F$ in $C$. Then $G(n, \cdot) \in \omega(\mathcal{A})$, the omega limit set of $\mathcal{A}$, if for each $n \in \mathbb{Z}^+$,

$$|F_t(n, x) - G(n, x)| \to 0$$

uniformly for $x$ in compact subsets of $X$, as $t \to \infty$ along some subsequence $\{t_{n_i}\}$. The closure of $\mathcal{A}$ in $C$ is called the hull of $F(n, \cdot)$ and is denoted by $Y = cl(\mathcal{A}) = \mathcal{H}(F)$.

On the space $Y$, we define a discrete semidynamical system $\sigma : Y \times \mathbb{Z}^+ \to Y$ by $\sigma(H(n, \cdot), t) = H_t(n, \cdot)$; that is $\sigma$ is the shift map.

For convenience, one may write equation (5) in the form

$$x_{n+1} = f_n(x_n) \tag{6}$$

with $f_n(x_n) = F(n, x_n)$.

Define the composition operator $\Phi$ as follows

$$\Phi_n^i = f_{i+n-1} \circ \ldots \circ f_{i+1} \circ f_i \equiv \Phi_n(F(i, \cdot)),$$

and the reverse composition operator $\widetilde{\Phi}$ as

$$\widetilde{\Phi}_n^i = f_i \circ f_{i+1} \circ \cdots \circ f_{i+n-1}.$$

When $i = 0$, we write $\Phi_n^0$ as $\Phi_n$ and $\widetilde{\Phi}_n^0$ as $\widetilde{\Phi}$.

The skew-product system is now defined as

$$\pi : X \times Y \times \mathbb{Z}^+ \to X \times Y$$

with

$$\pi((x, G), n) = (\Phi_n(G(i, \cdot)), \sigma(G, n)).$$

If $G = f_i$, then $\pi((x, f_i), n) = (\Phi_n^i(x), f_{i+n})$.

The following commuting diagram illustrates the notion of skew-product systems where $\mathcal{P}(a, b) = a$ is the projection map.

$$\begin{tikzcd}
X \times Y \times \mathbb{Z}^+ \arrow[r, "\pi"] \arrow[d, "\mathcal{P}\times id"'] & X \times Y \arrow[d, "\mathcal{P}"] \\
Y \times \mathbb{Z}^+ \arrow[r, "\sigma"'] & Y
\end{tikzcd}$$

For each $G(n, \cdot) \equiv g_n \in Y$, we define the fiber $\mathcal{F}_g$ over $G$ as $\mathcal{F}_g = \mathcal{P}^{-1}(G)$. If $g = f_i$, we write $\mathcal{F}_g$ as $\mathcal{F}_i$.

**Theorem 2** *[16] $\pi$ is a discrete semidynamical system.*

**Example 3 (Example (1) revisited)** *Let us reconsider the nonautonomous difference equation*

$$x_{n+1} = (-1)^n \left(\tfrac{n+1}{n+2}\right) x_n, x(0) = x_0.$$

*Hence, $F(n,x) = (-1)^n \left(\tfrac{n+1}{n+2}\right) x = f_n(x)$. Its hull is given by $G(n,x) = (-1)^n x$, that is, $g_n$ is a periodic sequence given by $g_0 = g_{2n}$, $g_1 = g_{2n+1}$, for all $n \in \mathbb{Z}^+$, in which $g_0(x) = x$, and $g_1(x) = -x$.*

*It is easy to verify that $\pi$ defined as $\pi((x, f_i), n) = (\Phi_n^i(x), f_{i+n})$ is a semidynamical system.*

# 4 Periodicity

In this section our focus will be on $p-$periodic difference equations of the form

$$x_{n+1} = f_n(x_n), \tag{7}$$

where $f_{n+p} = f_n$ for all $n \in \mathbb{Z}^+$.

The question that we are going to address is this: What are the permissible periods of the periodic orbits of equation (7)?

We begin by defining an $r-$periodic cycle (orbit).

**Definition 4** *An ordered set of points $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ is $r-$periodic in $X$ if*

$$f_{(i+nr) \ mod \ p}(\overline{x}_i) = \overline{x}_{(i+1) \ mod \ r}, n \in \mathbb{Z}^+.$$

*In particular,*

$$f_i(\overline{x}_i) = \overline{x}_{i+1}, \ 0 \leq i \leq r-2,$$

*and*

$$f_t(\overline{x}_{t \ mod \ r}) = \overline{x}_{(t+1) \ mod \ r}, r-1 \leq t \leq p-1.$$

It should be noted that the $r-$periodic cycle $C_r$ in $X$ generates an $s-$periodic cycle on the skew-product $X \times Y$ of the form $\widehat{C}_s = \{(\overline{x}_0, f_0), (\overline{x}_1, f_1), ..., (\overline{x}_{s \ mod \ r}, f_{s \ mod \ p})\}$, where $s = lcm[r, p]$ is the least common multiple of $r$ and $p$.

The r-periodic orbit $C_r$ is called an $r-$**geometric cycle**, and the $s-$periodic orbit $\widehat{C}_r$ is called an $s-$**complete cycle**.

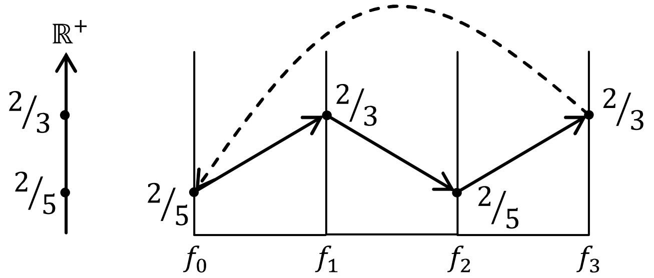


Figure 1: A 2−periodic cycle in a 4−periodic difference equation.

**Example 5** *Consider the nonautonomous periodic Beverton-Holt equation*

$$x_{n+1} = \frac{\mu_n K_n x_n}{K_n + (\mu_n - 1)x_n}, \tag{8}$$

*with* $\mu_n > 1, K_n > 0, K_{n+p} = K_n$, *and* $\mu_{n+p} = \mu_n$, *for all* $n \in \mathbb{Z}^+$.

1. *Assume that* $\mu_n = \mu > 1$ *is constant for all* $n \in \mathbb{Z}^+$. *Then one may appeal to Corollary 6.5 in [14] to show that equation (8) has no nontrivial periodic cycles of period less than p. In fact, equation (8) has a unique globally asymptotically stable cycle of minimal period p.*

2. *Assume that* $\mu_n$ *is periodic. Let* $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$, $K_0 = 1$, $K_1 = \frac{6}{17}$, $K_2 = 2$, *and* $K_3 = \frac{4}{11}$. *This leads to a* 4−*periodic difference equation. There is, however, a* 2−*geometric cycle, namely,* $C_2 = \left\{\frac{2}{5}, \frac{2}{3}\right\}$ *(see Figure 1). This* 2−*periodic cycle in the space* $X$ *generates the following* 4−*complete cycle on the skew-product* $X \times Y$

$$\widehat{C}_4 = \left\{(\tfrac{2}{5}, f_0), (\tfrac{2}{3}, f_1), (\tfrac{2}{5}, f_2), (\tfrac{2}{3}, f_3)\right\},$$

*where* $f_0(x) = \frac{3x}{1+2x}$, $f_1(x) = \frac{24x}{6+51x}$, $f_2(x) = \frac{4x}{2+x}$, *and* $f_3(x) = \frac{5x}{1+11x}$.

We are going to provide a deeper analysis of the preceding example. Let $d = \gcd(r,p)$ be the greatest common divisor of $r$ and $p$, $s = lcm[r,p]$ be the least common multiple of $r$ and $p$, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. The following result is one of two crucial lemmas in this survey.

**Lemma 6** *[14] Let* $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ *be a set of points in a metric space* $X$. *Then the following statements are equivalent.*

1. $C_r$ *is a periodic cycle of minimal period* $r$.

2. *For* $0 \le i \le r-1$, $f_{(i+nd)\ mod\ p}(\overline{x}_i) = \overline{x}_{(i+1)\ mod\ r}$.

3. *For* $0 \le i \le r-1$, *the graphs of the functions*

$$f_i, f_{(i+d)\ mod\ p}, \dots, f_{(i+(m-1)d) mod\ p}$$

*intersect at the* $\ell$ *points*

$$(\overline{x}_i, \overline{x}_{(i+1)\ mod\ r}), (\overline{x}_{(i+d)\ mod\ r}, \overline{x}_{(i+1+d)\ mod\ r}), \dots (\overline{x}_{(i+(\ell-1)d)\ mod\ r}, \overline{x}_{(i+(\ell-1)d+1)\ mod\ r}).$$

**Corollary 7** *[29] Assume that the one-parameter family* $F(\alpha, x)$ *is one to one in* $\alpha$. *Let* $f_n(x_n) = F(\alpha_n, x_n)$. *Then if the* $p-$*periodic difference equation, with minimal period* $p$,

$$x_{n+1} = f_n(x_n) \tag{9}$$

*has a nontrivial periodic cycle of minimal period* $r$, *then* $r = tp$, $t \in \mathbb{Z}^+$.

**Proof.** Suppose that equation (9) has a periodic cycle $C_r = \{\overline{x}_0, \overline{x}_1, \dots, \overline{x}_{r-1}\}$ of period $r < p$, and let $d = \gcd(r, p)$, $s = lcm[r, p]$, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. Then by Lemma 6, the graphs of the maps $f_0, f_d, \dots, f_{(m-1)d}$ must intersect at the points $(\overline{x}_0, \overline{x}_1), (\overline{x}_d, \overline{x}_{d+1}), \dots, (\overline{x}_{(\ell-1)d}, \overline{x}_{(\ell-1)d+1})$.

Since $F(\alpha, x)$ is one to one in $\alpha$, the maps $f_0, f_d, \dots, f_{(m-1)d}$ do not intersect, unless they are all equal. Similarly, one may show that $f_i = f_{i+d} = \dots = f_{i+(m-1)d}$. This shows that equation (9) is of minimal period $d$, a contradiction. Hence $r$ is equal to $p$ or a multiple of $p$. ∎

Applying corollary 7 to the periodic Beverton-Holt equation with $K_{n+p} = K_n, \mu_n = \mu$, for all $n \in \mathbb{Z}^+$, shows that the only possible period of a nontrivial periodic cycle is $p$. However, for the case $\mu_n$ and $K_n$ are both periodic of common period $p$, the situation is murky as was demonstrated by Example 5, case 2.

For the values $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$, $K_0 = 1$, $K_1 = 6/17$, $K_2 = 2$, and $K_3 = 4/11$, we have $f_0(x) = \frac{3x}{1+2x}$, $f_1(x) = \frac{24x}{6+51x}$, $f_2(x) = \frac{4x}{2+x}$, and $f_3(x) = \frac{5x}{1+11x}$. Let $\mathcal{F} = \{f_0, f_1, f_2, f_3\}$. Clearly $x^* = 0$ is a fixed point of the periodic system $\mathcal{F}$. To have a positive fixed point (period 1) or a periodic cycle of period 3, we must have the graphs of $f_0, f_1, f_2, f_3$ intersect at points $(\overline{x}_0, \overline{x}_1), (\overline{x}_1, \overline{x}_2), \dots, (\overline{x}_{\ell-1}, \overline{x}_\ell)$, where $\ell = 1$ or $\ell = 3$. Simple computation shows that this is not possible. Moreover, one may show that the graphs of $f_0$ and $f_2$ intersect at the points $(2/5, 2/3)$ and the graphs of $f_1$ and $f_3$ intersect at the points $(2/3, 2/5)$. Hence $C_2 = \{2/5, 2/3\}$ is a $2-$periodic cycle. Moreover, the equation has the $4-$periodic cycle $\{\frac{238}{361}, \frac{119}{298}, \frac{238}{417}, \frac{238}{607}\}$.

Suppose that the p-periodic difference equation

$$x_{n+1} = f_n(x_n), f_{n+p} = f_n, n \in \mathbb{Z}^+ \tag{10}$$

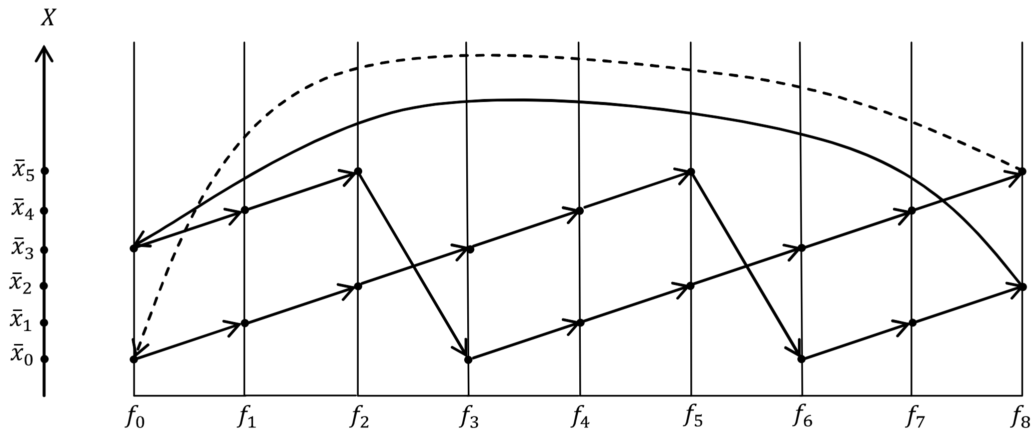


Figure 2: A 6−periodic cycle in a 9−periodic system

has a periodic cycle of minimal period $r$. Then the associated skew-product system $\pi$ has a periodic cycle of period $s = lcm[r, p]$ ($s-$complete cycle). There are $p$ fibers $\mathcal{F}_i = \mathcal{P}^{-1}(f_i)$. Are the $s$ periodic points equally distributed on the fibers? i.e. is the number of periodic points on each fiber equal to $\ell = s/p$?

Before giving the definitive answer to this question, let us examine the diagram present in Figure 2 in which $p = 9$, and $r = 6$.

There are two points $\left(2 = \frac{lcm[6,9]}{9}\right)$ on each fiber. Since $d = \gcd(6, 9) = 3$, the graphs $f_0$, $f_3$, and $f_6$ intersect at the two points $(\overline{x}_0, \overline{x}_1)$, $(\overline{x}_3, \overline{x}_4)$; the graphs $f_1, f_4$, and $f_7$ intersect at the two points $(\overline{x}_1, \overline{x}_2)$, $(\overline{x}_4, \overline{x}_5)$; and the graphs $f_2, f_5, f_8$ intersect at the points $(\overline{x}_2, \overline{x}_3)$, $(\overline{x}_5, \overline{x}_0)$.

Note that the number of periodic points on each fiber is 2, which is $\ell = \frac{lcm[r,p]}{p}$. The following crucial lemma proves this observation.

**Lemma 8** *[13] Let $s = lcm[r, p]$. Then the orbit of $(\overline{x}_i, f_i)$ in the skew-product system intersect each fiber $\mathcal{F}_j$, $j = 0, 1, \ldots, p-1$, in exactly $\ell = s/p$ points and each of these points is periodic under the skew-product $\pi$ with period $s$.*

**Proof.** Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be a periodic cycle of minimal period $r$. Then the orbit of $(\overline{x}_0, f_0)$ in the skew-product has a minimal period $s = lcm[r, p]$. Now $\mathcal{S} = \mathcal{O}\left((\overline{x}_0, f_0)\right) = \{\pi((\overline{x}_0, f_0), n) : n \in \mathbb{Z}^+\} \subset X \times Y$ is minimal, invariant under $\pi$ and has $s$ distinct points.

For each $i, 0 \leq i \leq p-1$, the maps

$$f_i : \mathcal{S} \cap \mathcal{F}_i \to \mathcal{S} \cap \mathcal{F}_{(i+1)\ mod\ p} \tag{11}$$

are surjective. We now show that it is injective.

Let $N_i$ be cardinality of $\mathcal{S}\cap\mathcal{F}_i$. Then $N_i$ is a non-increasing integer valued function and thus stabilizes at some fixed value from which it follows that $N_i$ is constant. Thus each $\mathcal{S}\cap\mathcal{F}_i$ contains the same number of points, namely $\ell = s/p$. ■

# 5 Stability

We begin this section by stating the basic definitions of stability.

**Definition 9** *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle in the $p-$periodic equation (10) in a metric space $(X, \rho)$ and $s = lcm[r, p]$ be the least common multiple of $p$ and $r$. Then*

1. *$C_r$ is stable if given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\rho(z, \overline{x}_{i \ mod \ r}) < \delta \ \textit{implies} \ \rho(\Phi^i_n(z), \Phi^i_n(\overline{x}_{i \ mod \ r})) < \epsilon$$

   *for all $n \in \mathbb{Z}^+$, and $0 \leq i \leq p-1$. Otherwise, $C_r$ is said unstable.*

2. *$C_r$ is attracting if there exists $\eta > 0$ such that*

$$\rho(z, \overline{x}_{i \ mod \ r}) < \eta \ \textit{implies} \ \lim_{n\to\infty} \Phi^i_{ns}(z) = \overline{x}_{i \ mod \ r}.$$

3. *We say that $C_r$ is asymptotically stable if it is both stable and attracting. If in addition, $\eta = \infty$, $C_r$ is said to be globally asymptotically stable.*

**Lemma 10** *[29] An $r-$periodic cycle $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ in equation (10) is*

1. *asymptotically stable if $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| < 1$,*
2. *unstable if $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| > 1$,*

*where $s = lcm[r, p]$ is the least common multiple of $p$ and $r$.*

Consider the skew-product system $\pi$ on $X \times Y$ with $X$ a metric space with metric $\rho$, $Y = \{f_0, f_1, \ldots, f_{p-1}\}$ equipped with the discrete metric $\widetilde{\rho}$, where

$$\widetilde{\rho}(f_i, f_j) = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}.$$

Define a metric $D$ on $X \times Y$ as

$$D((x, f_i), (y, f_j)) = \rho(x, y) + \widetilde{\rho}(f_i, f_j).$$

Let $\pi^1(x,f) = \pi((x,f),1)$, then $\pi^n(x,f) = \pi((x,f),n)$. Thus $\pi^1 : X\times Y \to X\times Y$ is a continuous map which generates an autonomous system on $X\times Y$. Consequently, the stability definitions of fixed points and periodic cycles follow the standard ones that may be found in [11, 9].

Now we give a definition of stability for a complete periodic cycle in the skew-product system.

**Definition 11** *A complete periodic cycle* $\widehat{C}_s = \{(\overline{x}_0, f_0), ..., (\overline{x}_{s \ mod \ r}, f_{s \ mod \ p})\}$ *is*

1. *stable if given* $\epsilon > 0$*, there exits* $\delta > 0$*, such that*

$$D((z,f_i),(\overline{x}_0,f_0)) < \delta \textit{ implies } D(\pi^n(z,f_i),\pi^n(\overline{x}_0,f_0)) < \epsilon, \ \forall n \in \mathbb{Z}^+.$$

*Otherwise,* $\widehat{C}_s$ *is said unstable.*

2. *attracting if there exists* $\eta > 0$ *such that*

$$D((z,f_i),(\overline{x}_0,f_0)) < \eta \textit{ implies } \lim_{n\to\infty} \pi^{ns}(z,f_i) = (\overline{x}_0,f_0).$$

3. *asymptotically stable if it is both stable and attracting. If in addition,* $\eta = \infty$*,* $\widehat{C}_s$ *is said to be globally asymptotically stable.*

Since $f_{i+ns} = f_i$ for all $n$, it follows from the above convergence that $f_i = f_0$. Hence, stability can occur only on each fiber $X \times \{f_i\}, 0 \leq i \leq p-1$.

It should be noted that one may reformulate lemma 10 in the setting of the skew-product theorem. However, to do so, one needs to develop the notion of derivative in the space $X \times Y$.

**Definition 12** *Let* $g = \pi^p : X \times Y \to X \times Y$ *defined as* $g(x,f_i) = (\Phi_p^i(x), f_i)$*. The generalized derivative of* $g$ *is defined as* $g'(x,f_i) = \frac{d}{dx}\left(\Phi_p^i(x)\right) = \left(\Phi_p^i\right)'(x)$*.*

**Lemma 13** *A complete periodic cycle* $\widetilde{C}_s = \{(\overline{x}_0, f_0), ..., (\overline{x}_{s \ mod \ r}, f_{s \ mod \ p})\}$ *of the skew-product system* $\pi$ *on* $X \times Y$ *is*

1. *asymptotically stable if* $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| < 1$*,*
2. *unstable if* $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| > 1$*,*

*where* $s = lcm[r,p]$ *is the least common multiple of* $p$ *and* $r$*.*

We are now ready to state our main result in this survey.

**Theorem 14** *[13] Assume that $X$ is a connected metric space and each $f_i \in Y$ is a continuous map on $X$, with $f_{i+p} = f_i$. Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be a periodic cycle of minimal period $r$. If $C_r$ is globally asymptotically stable, then $r$ divides p. Moreover, $r = p$ if the sequence $\{f_n\}$ is a one-parameter family of maps $F(\mu_n, x)$ and $F$ is one to one with respect to $\mu$.*

**Proof.** The skew-product system $\pi$ on $X \times Y$ has the periodic orbit

$$\{(\overline{x}_0, f_0), (\overline{x}_1, f_1), \ldots, (\overline{x}_{s \bmod r}, f_{s \bmod p})\}$$

which is globally asymptotically stable. But as we remarked earlier, globally stability can occur only on fibers. By Lemma 8, there are $\ell = s/p$ points on each fiber. If $\ell > 1$, we have a globally asymptotically $\ell$-periodic cycle in the connected metric space $X \times \{f_i\}$ under the map $\pi^p$. This violates Elaydi-Yakubu Theorem [12]. Hence $\ell = 1$ and consequently $r|p$.

Note that by Lemma 6, the graphs of the maps $f_i$, $f_{i+d}$, ,$f_{i+(m-1)d}$, $0 \leq i \leq p-1$, must intersect at $\ell$ points. However, since $\{f_i\}$ is a one parameter family of maps $F(\mu_n, x)$ where $F$ is one to one with respect to the parameter $\mu$, it follows that $f_i = f_{i+d}$, $0 \leq i \leq p-1$. This implies that $d$ is the period of our system and since $p$ is the minimal period of the system, this implies that $d = p$. Hence $r = p$. ■

# 6 An extension of Singer's Theorem

One of the well known work done by Singer is present in his famous paper [33] and currently known by Singer's theorem. It is a useful tool in finding an upper bound for the number of stable cycles in autonomous difference equations. In this section we present the natural extension of this theorem to the periodic nonautonomous difference equations.

Recall that the Schwarzian derivative, $Sf$, of a map $f$ at $x$ is defined as

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Let $f : I \to I$ be a $C^3$ map with a negative Schwarzian derivative for all $x \in I$, defined on the closed interval $I$. If $f$ has $m$ critical points in $I$, then $f$ has at most $m + 2$ attracting period cycles of any given period.

Now consider the $p-$periodic system $\mathcal{F} = \{f_0, f_1, f_2, ..., f_{p-1}\}$ of continuous maps defined on a closed interval $I$.

Assume that there are $m_i$ critical points for the map $f_i$, $0 \leq i \leq p-1$. On the fiber $\mathcal{F}_0 = I \times f_0$, there are $m_0$ critical points of $f_0$, at least $m_1$ critical points consisting of all the pre-images under $f_0$ of the $m_1$ critical points of $f_1$,... and at least $m_{p-1}$ critical points that consist of all the pre images, under $\Phi_{p-2}$, of the $m_{p-1}$ critical points of $f_{p-1}$. Since each critical point of $\Phi_p$ is mapped, under compositions of our maps, to one of the original critical points of one of the maps $f_i$, it follows that the number of significant critical points is $\sum_{i=0}^{p-1} m_i$.

By Singer's Theorem, there are at most $\left[\sum_{i=0}^{P-1} m_i + 2\right]$ attracting periodic cycles of any given period. Notice that periodic cycles that appear on fiber $\mathcal{F}_i$ are just phase shifts of periodic cycles that appear on fiber $\mathcal{F}_0$. Hence we conclude that there are at most $\sum_{i=0}^{P-1} m_i + 2$ attracting cycles of any given period (See [2] for details).

So a consequence of this extension, one may show that if the maps are the logistic maps

$$f_i(x) = \mu_i x(1-x), \mu_i > 0,\ 0 \leq i \leq p-1,$$

defined on the interval $[0,1]$, then the p-periodic system $\{f_0, f_1, ..., f_{p-1}\}$ has at most $p-$attracting cycles of any given period $r$. Notice that each map $f_i$ has one critical point, $x = 1/2$, and the boundary points 0 and 1 are attracted only to 0.

# 7 Bifurcation

The study of various notions of bifurcation in the setting of discrete nonautonomous systems is still at its infancy stage. The main contribution in this area are the papers by Henson [24], Al-Sharawi and Angelos [2], Oliveira and D'Aniello [30], and recently Luís, Elaydi and Oliveira [29].

The main objective in this section is to give the pertinent definitions, notions, terminology and results done in [29]. Though our focus here will be on $2-$periodic systems, the ideas presented can be easily extended to the general periodic case.

Throughout this section we assume that the maps $f_0$ and $f_1$ arise from a one-parameter family of maps such that $f_1 = f_{\alpha_1}$ and $f_0 = f_{\alpha_0}$ with $\alpha_0 = q\alpha_1$ for some real number $q > 0$. Thus one may write, without loss of generality, our system as $\mathcal{F} = \{f_0, f_1\}$.

Moreover, we assume that the one-parameter family of maps is one-to-one with respect to the parameter. Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of $\mathcal{F}$. Then by corollary 7 the latter assumption implies that $r = 2m$, $m \geq 1$.

With $\Phi_2 = f_1 \circ f_0$, one may write the orbit of $\overline{x}_0$ as (see figure 3)

$$\begin{aligned}\mathcal{O}(\overline{x}_0) &= \left\{\overline{x}_0, f_0(\overline{x}_0), \Phi_2(\overline{x}_0), f_0 \circ \Phi_2(\overline{x}_0), \Phi_4(\overline{x}_0), ..., \Phi_{2(m-1)}(\overline{x}_0), f_0 \circ \Phi_{2(m-1)}(\overline{x}_0)\right\} \\ &= \{\overline{x}_0, \Phi_1(\overline{x}_0), \Phi_2(\overline{x}_0), ..., \Phi_{2m-1}(\overline{x}_0)\} \qquad (12)\end{aligned}$$

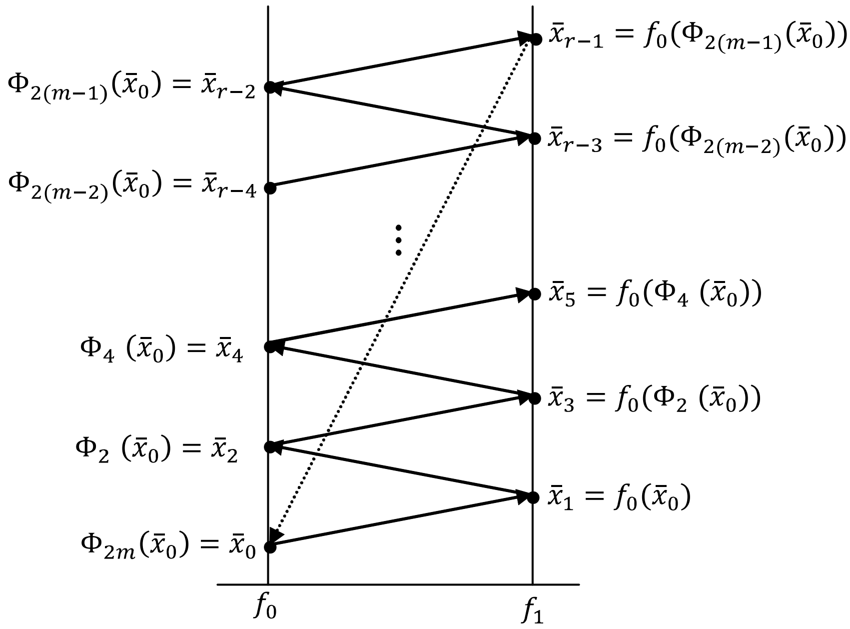


Figure 3: Sequence of the periodic points $\{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ in the $2-$periodic system $\mathcal{F} = \{f_0, f_1\}$ illustrated in the fibers, where $\Phi_2 = f_1 \circ f_0$ and $r = 2m, m \geq 1$.

Equivalently, one may write the sequence of points given in (12) as

$$\begin{aligned} \mathcal{O}(\overline{x}_1) &= \Big\{ f_1 \circ \widetilde{\Phi}_{2(m-1)}(\overline{x}_1), \overline{x}_1, f_1(\overline{x}_1), \widetilde{\Phi}_2(\overline{x}_1), f_1 \circ \widetilde{\Phi}_2(\overline{x}_1), ..., \widetilde{\Phi}_{2(m-1)}(\overline{x}_1) \Big\} \\ &= \{\widetilde{\Phi}_{2m-1}(\overline{x}_1), \overline{x}_1, \widetilde{\Phi}_1(\overline{x}_1), \widetilde{\Phi}_{2m}(\overline{x}_1), ..., \widetilde{\Phi}_{2m-2}(\overline{x}_1)\} \end{aligned} \tag{13}$$

where $\widetilde{\Phi} = f_0 \circ f_1$. Hence the order of the composition is irrelevant to the dynamics of the system.

The dynamics of $\mathcal{F}$ depends very much on the parameter as the qualitative structure of the dynamical system changes as the parameter changes. These qualitative changes in the dynamics of the system are called bifurcations and the parameter values at which they occur are called bifurcation points. For autonomous systems or single maps the bifurcation analysis may be found in Elaydi [11].

In a one-dimensional systems generated by a one-parameter family of maps $f_\alpha$, a bifurcation at a fixed point $x^*$ occurs when $\frac{\partial f}{\partial x}(\alpha^*, x^*) = 1$ or $-1$ at a bifurcation point $\alpha^*$. The former case leads to a saddle-node bifurcation, while the latter case leads to a period-doubling bifurcation.

Now we are going to extend this analysis to $2-$periodic difference equations or $\mathcal{F} = \{f_0, f_1\}$. To simplify the notation we write $\Phi_2(\alpha, x)$ instead of $\Phi_2(x)$ and $\widetilde{\Phi}_2(\alpha, x)$ instead of $\widetilde{\Phi}_2(x)$. Then $\Phi_{2m}(\overline{x}_{2i}) = \overline{x}_{(2i)modr}$ and $\widetilde{\Phi}_{2m}(\overline{x}_{2i+1}) = \overline{x}_{(2i+1)modr}$, $1 \leq i \leq m$. In general, we have $\Phi_{2nm}(\overline{x}_{2i}) = \overline{x}_{(2i)modr}$ and $\widetilde{\Phi}_{2nm}(\overline{x}_{2i+1}) = \overline{x}_{(2i+1)modr}$, $n \geq 1$.

Assuming $\frac{\partial \Phi_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ at a bifurcation point $\overline{\alpha}$, by the chain rule, we have

$$\frac{\partial \Phi_2}{\partial x}(\overline{\alpha}, \overline{x}_{2m-2}) \frac{\partial \Phi_2}{\partial x}(\overline{\alpha}, \overline{x}_{2m-4}) ... \frac{\partial \Phi_2}{\partial x}(\overline{\alpha}, \overline{x}_2) \frac{\partial \Phi_2}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$$

or

$$f_1'(\overline{x}_{2m-1}) f_0'(\overline{x}_{2m-2}) f_1'(\overline{x}_{2m-3}) f_0'(\overline{x}_{2m-4}) ... f_1'(\overline{x}_3) f_0'(\overline{x}_2) f_1'(\overline{x}_1) f_0'(\overline{x}_0) = 1 \tag{14}$$

Applying $f_0$ on both sides of the identity $\Phi_{2m}(\overline{\alpha}, \overline{x}_0) = \overline{x}_0$, yields $\widetilde{\Phi}_{2m}(\overline{\alpha}, \overline{x}_1) = \overline{x}_1$. Differentiating both sides of this equation yields

$$\frac{\partial \widetilde{\Phi}_2}{\partial x}(\overline{\alpha}, \overline{x}_{2m-1}) \frac{\partial \widetilde{\Phi}_2}{\partial x}(\overline{\alpha}, \overline{x}_{2m-3}) ... \frac{\partial \widetilde{\Phi}_2}{\partial x}(\overline{\alpha}, \overline{x}_3) \frac{\partial \widetilde{\Phi}_2}{\partial x}(\overline{\alpha}, \overline{x}_1) = 1$$

or equivalently

$$f_0'(\overline{x}_0) f_1'(\overline{x}_{2m-1}) f_0'(\overline{x}_{2m-2}) f_1'(\overline{x}_{2m-3}) ... f_0'(\overline{x}_4) f_1'(\overline{x}_3) f_0'(\overline{x}_2) f_1'(\overline{x}_1) = 1. \tag{15}$$

Hence Eq. (14) is equivalent to Eq. (15). More generally the following relation holds

$$\frac{\partial \Phi_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_{2j}) = \frac{\partial \widetilde{\Phi}_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_{2j-1}), j \in \{1, 2, ..., m\}. \tag{16}$$

Now we are ready to write the two main results of this section.

**Theorem 15 (Saddle-node Bifurcation for $2-$periodic systems [29])** *Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a periodic $r-$cycle of $\mathcal{F}$. Suppose that both $\frac{\partial^2 \Phi_2}{\partial x^2}$ and $\frac{\partial^2 \Phi_2}{\partial^2}$ exist and are continuous in a neighborhood of a periodic orbit such that $\frac{\partial \Phi_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ for the periodic point $\overline{x}_0$. Assume also that*

$$A = \frac{\partial \Phi_{2m}}{\partial \alpha}(\overline{\alpha}, \overline{x}_0) \neq 0 \text{ and } B = \frac{\partial^2 \Phi_{2m}}{\partial x^2}(\overline{\alpha}, \overline{x}_0) \neq 0.$$

*Then there exists an interval $J$ around the periodic orbit and a $C^2-$map $\alpha = h(x)$, where $h : J \to \mathbb{R}$ such that $h(\overline{x}_0) = \overline{\alpha}$, and $\Phi_{2m}(x, h(x)) = x$. Moreover, if $AB < 0$, the periodic points exists for $\alpha > \overline{\alpha}$, and, if $AB > 0$, the periodic points exists for $\alpha < \overline{\alpha}$.*

When $\frac{\partial \Phi_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ but $\frac{\partial \Phi_{2m}}{\partial \alpha}(\overline{\alpha}, \overline{x}) = 0$, two types of bifurcations appear. The first is called transcritical bifurcation which occurs when $\frac{\partial^2 \Phi_{2m}}{\partial x^2}(\overline{\alpha}, \overline{x}_0) \neq 0$ and the second is called pitchfork bifurcation which appears when $\frac{\partial^2 \Phi_{2m}}{\partial x^2}(\overline{\alpha}, \overline{x}_0) = 0$. For more details about this two types of bifurcation see table 2.1 in [11, pp. 90], and [30]. In the former work the author presents many cases for autonomous maps while in the latter article the authors study the pitchfork bifurcation for nonautonomous $2-$periodic systems in which the maps have negative Schwarzian derivative.

The next result gives the conditions for the period-doubling bifurcation.

**Theorem 16 (Period-Doubling Bifurcation for $2-$periodic systems [29])** *Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a periodic $r-$cycle of $\mathcal{F}$. Assume that both $\frac{\partial^2 \Phi_2}{\partial x^2}$ and $\frac{\partial \Phi_2}{\partial \alpha}$ exist and are continuous in a neighborhood of a periodic orbit, $\frac{\partial \Phi_{2m}}{\partial x}(\overline{\alpha}, \overline{x}_0) = -1$ for the periodic point $\overline{x}_0$ and $\frac{\partial^2 \Phi_{4m}}{\partial \alpha \partial x}(\overline{\alpha}, \overline{x}_0) \neq 0$. Then, there exists an interval $J$ around the periodic orbit and a function $h : J \to \mathbb{R}$ such that $\Phi_{2m}(x, h(x)) \neq x$ but $\Phi_{4m}(x, h(x)) = x$.*

Now we are going to apply these two results with an interesting example from [29]. First we need the following definition.

**Definition 17** *A unimodal map is said to have the Allee[1] effect if it has three fixed points $x_1^* = 0$, $x_2^* = A$, and $x_3^* = K$, with $0 < A < K$, in which $x_1^*$ is asymptotically stable, $x_2^*$ is unstable, and $x_3^*$ may be stable or unstable.*

**Remark 18** *Note that if $\mathcal{F}$ is a periodic set formed by unimodal Allee maps, neither the zero fixed point nor the threshold point can contribute to bifurcation, since the former is always asymptotically stable and the latter is always unstable. Hence bifurcation may only occur at the carrying capacity of $\mathcal{F}$.*

**Example 19** *[29] Consider the $2-$periodic system $\mathcal{W} = \{f_0, f_1\}$, where*

$$f_i(x) = a_i x^2 (1 - x),\ i = 0, 1$$

[1]The Allee effect is a phenomenon in population dynamics attributed to the American biologist Warder Clayde Allee 1885-1955 [1]. Allee proposed that the per capita birth rate declines at low density or population sizes. In the languages of dynamical systems or difference equations, a map representing the Allee effect must have tree fixed points, an asymptotically stable zero fixed point, a small unstable fixed point, called the threshold point, and a bigger positive fixed point, called the carrying capacity, that is asymptotically stable at least for smaller values of the parameters.

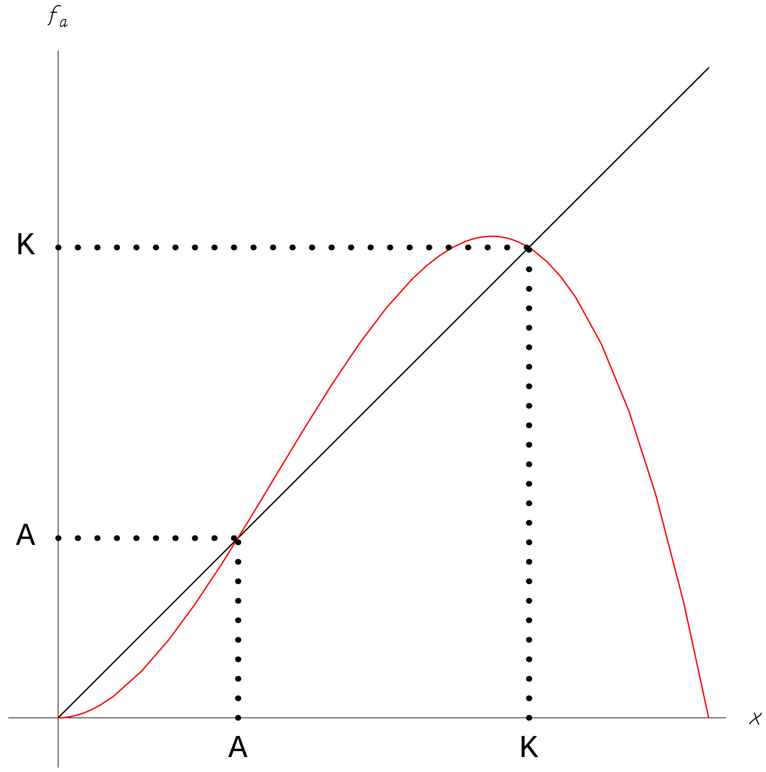


Figure 4: A unimodal Allee map with three fixed points 0, $A$ and $k$.

*in which $x \in [0,1]$ and $a_i > 0$, $i = 0, 1$. For an individual map $f_i$, if $a_i < 4$ we have a globally asymptotically stable zero fixed point and no other fixed point. At $a_i = 4$ an unstable fixed point is born after which $f_i$ becomes a unimodal map with an Allee effect (see Figure 4). Henceforth, we will assume that $a_0, a_1 > 4$.*

*Since 0 is the only fixed point under the system $\mathcal{W}$, we focus our attention on $2-$periodic cycles $\{\overline{x}_0, \overline{x}_1\}$ with $f_0(\overline{x}_0) = \overline{x}_1$, and $f_1(\overline{x}_1) = \overline{x}_0$.*

*A Saddle-node bifurcation occurs when $\frac{\partial}{\partial t}\left(\Phi_2(t)\right)\Big|_{t=\overline{x}_0} = \Phi_2'(\overline{x}_0) = 1$, and a period-doubling bifurcation occurs when $\frac{\partial}{\partial t}\left(\Phi_2(t)\right)\Big|_{t=\overline{x}_0} = \Phi_2'(\overline{x}_0) = -1$.*

*For the saddle-node bifurcation we then solve the equations*

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(f_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = 1 \end{cases} \tag{17}$$

*and for the period-doubling bifurcations we solve the equations*

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(f_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = -1 \end{cases} \tag{18}$$

*Using the command "resultant"*[2] *in Mathematica or Maple Software, we eliminate the variable* $\overline{x}_0$ *in equations (17) and (18). Eq. (17) yields*

$$16777216 + 16384a_0a_1 - 576000a_0^2a_1 + 84375a_0^3a_1 - 576000a_0a_1^2 + 914a_0^2a_1^2 - 350a_0^3a_1^2 + 84375a_0a_1^3 - 350a_0^2a_1^3 + 19827a_0^3a_1^3 - 2916a_0^4a_1^3 - 2916a_0^3a_1^4 + 432a_0^4a_1^4 = 0$$

*while Eq. (18) yields*

$$100000000 - 120000a_0a_1 - 2998800a_0^2a_1 + 453789a_0^3a_1 - 2998800a_0a_1^2 - 4598a_0^2a_1^2 + 2702a_0^3a_1^2 + 453789a_0a_1^3 + 2702a_0^2a_1^3 + 89765a_0^3a_1^3 - 13500a_0^4a_1^3 - 13500a_0^3a_1^4 + 2000a_0^4a_1^4 = 0$$

*For each one of these last two equations we invoke the implicit function theorem to plot, in the* $(a_0, a_1)-$*plane, the bifurcation curves (see figure 5). The black curves are the solution of the former equation at which saddle-node bifurcation occurs, while the gray curves are the solution of the latter equation at which period-doubling bifurcations occurs. The black cusp is the curve of pitchfork bifurcation. In the regions identified by letters one can conclude the following.*

- *If* $a_0, a_1 \in A$ *then the fixed point* $x^* = 0$ *is globally asymptotically stable.*
- *If* $a_0, a_1 \in B \backslash D$ *then there are two* $2-$*periodic cycles, one attracting and one unstable.*
- *If* $a_0, a_1 \in D$ *then there are two attracting* $2-$*periodic cycles (from the pitchfork bifurcation) and two unstable* $2-$*periodic cycles.*
- *If* $a_0, a_1 \in (C_1 \cup C_2) \backslash (D_1 \cup D_2)$ *then there is an attracting* $4-$*periodic cycle (from the period doubling bifurcation) and two unstable* $2-$*periodic cycles.*
- *If* $a_0, a_1 \in D_1 \cup D_2$ *then there are two attracting* $4-$*periodic cycles (from pitchfork bifurcation) and two unstable* $2-$*periodic cycles.*
- *If* $a_0, a_1 \in E$ *then there are two attracting* $8-$*periodic cycles (from period doubling bifurcation), two attracting* $4-$*periodic cycles (from pitchfork bifurcation), and four unstable* $2-$*periodic cycles.*

It should be noted here that the bifurcation curves for the system $\mathcal{W}$ in figure 5 are incomplete. If we want to draw more bifurcation curves in the space of the parameters we must do the same for $4-$periodic cycles, $8-$periodic cycles, and so

[2] The command "resultant" is a powerful tool that helps us in finding the implicit solutions for a polynomial equations with low degree. We are not aware of similar techniques that work for nonpolynomial equation such the Ricker map $R_p(x) = xe^{p-x}$, $p, x > 0$.

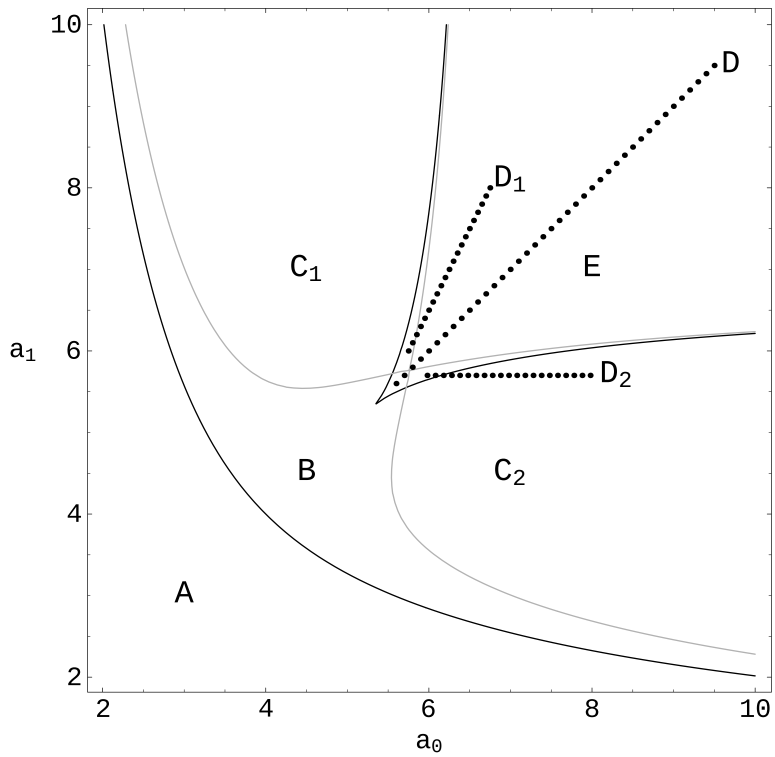


Figure 5: Bifurcations curves for the 2−periodic nonautonomous difference equation with Allee effects $x_{n+1} = a_n x_n^2(1 - x_n)$, in the $(a_0, a_1)$−plane, where $a_{n+2} = a_n$ and $x_{n+2} = x_n$.

on. Finding the implicit solutions of these two new equations involve horrendous computations. The command "resultant" does not produce answers after certain values of the degree of the polynomial. So, for the system $\mathcal{W}$, unfortunately we are unable to draw these curves for the 4−periodic cycle. However, it should be noted that AlSharawi and Angelos [2] have used the command "resultant" to investigate the bifurcations of the periodically forced logistic map, and they were able to draw these curves for the 4−periodic cycles of the 2−periodic system. Moreover, these authors drew the bifurcation surfaces for the 3−periodic cycle of the 3−periodic system in the three dimensional space of the parameters.

Finally, we should mention that Grinfeld et al [20] have used the command "resultant" much earlier to study the bifurcation of 2−periodic logistic systems.

# 8 A note on bifurcation equations

In [5] the authors study the symmetry of degenerate bifurcation equations of periodic orbits in a nonautonomous system with respect to the order of the composition. They proved that the cyclic permutation in the order of the composition do not affect the solutions of the bifurcations in the parameter space.

In order to see this last observation, let $f_0, f_1, \ldots, f_{p-1}$ be a collection of maps

$$\begin{array}{rccc} f_j: & I_j \times \mathbb{R}^K & \longrightarrow & \mathbb{R} \\ & (x, \lambda) & \longrightarrow & f_j(x, \lambda) \end{array}$$

where $\lambda$ is a parameter vector, the fiber $\mathcal{F}_j = I_j \times \{f_j\}$ and $f_j \in \mathcal{C}^m\left(I_j, \mathbb{R}^K\right)$, $j = 0, 1, \ldots, p-1$.

We are concerned with the bifurcations that can occur, in particular with the bifurcations with higher degeneracy conditions on the derivatives of the iteration variable $x$ and not on the degeneracy conditions on the parameters.
These below are the bifurcation equations with the most degenerate conditions that appear with $j$ fixed , $0 \leq j \leq p-1$

$$\begin{aligned} \Phi_{kp}^{j}(x) &= x, \\ \frac{d\Phi_{kp}^{j}}{dx}(x) &= 1, \\ \frac{d^2\Phi_{kp}^{j}}{dx^2}(x) &= 0, \\ \frac{d^3\Phi_{kp}^{j}}{dx^3}(x) &= 0, \\ &\vdots \\ \frac{d^m\Phi_{kp}^{j}}{dx^m}(x) &= 0. \end{aligned} \tag{19}$$

These equations have different solution in terms of $x$, depending on the $j$ we choose.

A natural question arises:

**Do the solutions in the parameter space depend on the particular choice of $\Phi_p^j$?**

This question was posed in [30, 4] and, was positively solved for $p = 2$, and degeneracy conditions of order $m = 2, 3$, that is, for pitchfork and swallowtail, respectively.

We now present the following lemma that is useful for solving general problems of the symmetry of the bifurcation equations.

**Lemma 20** *Let $m \geq 1$ and let $\varphi$ and $\psi$ be real maps satisfying the conditions:*

1. *there exists a such that $\psi(a) = a$ and $\psi$ is a Lipschitz homeomorphism in some open interval $I \ni a$.*
2. *$\varphi$ is a Lipschitz homeomorphism with Lipschitz constant $L$ in a open neighborhood $I_\varphi$ of a point a such that $\psi(a) = a$ and $\psi$ is a Lipschitz homeomorphism in some open interval $I \ni a.a$ such that $\varphi(a) = b$. Let $\varphi^{-1}$ be its inverse in another open neighborhood of $b$, $\varphi^{-1}$ is also Lipschitz continuous with constant $M$.*
3. $$\lim_{x \to a} \frac{|\psi(x) - x|}{|x - a|^m} = 0.$$

*Then the conjugate $\widetilde{\psi}$ of $\psi$ by the homeomorphism $\varphi$*

$$\widetilde{\psi} = \varphi \psi \varphi^{-1}$$

*satisfies*

$$\lim_{y \to b} \frac{\left|\widetilde{\psi}(y) - y\right|}{|y - b|^m} = 0. \tag{20}$$

Using lemma 20 one can prove the following theorem.

**Theorem 21** *[5] Let $f_0, f_1 \ldots, f_{p-1}$ be p functions with a sufficient number of derivatives satisfying the conditions:*

1. *There exists $\overline{x}_0$, $\overline{x}_1$, $\ldots$, $\overline{x}_{p-1}$, fixed points of $\Phi_p$, $\Phi_p^1$, $\ldots$, $\Phi_p^{p-1}$, respectively, that is*
$$\Phi_p^j(\overline{x}_j) = \overline{x}_j$$
2. *The first bifurcation condition holds*
$$\left.\frac{d\Phi_p(x)}{dx}\right|_{x=\overline{x}_0} = \prod_{j=0}^{p-1} \frac{df_j}{dx}(\overline{x}_j) = 1.$$

*3. Higher degeneracy conditions hold for $\Phi_p$*

$$m \geq 2: \quad \frac{d^n \Phi_p}{dx^n}(x)\bigg|_{x=\overline{x}_0} = 0, \qquad 2 \leq n \leq m.$$

*Then the composition operator $\Phi_p^j$ ,$0 \leq j \leq p-1$ satisfies*

$$\frac{d^n \Phi_p^j}{dx^n}(x)\bigg|_{x=\overline{x}_j} = 0, \text{ for } 2 \leq n \leq m.$$

In the case of periodic systems with period two the result is a particular case of the previous theorem.

**Corollary 22** *[5] Let $f_0$ and $f_1$ be maps with a sufficient number of derivatives satisfying the conditions:*

1. $(f_1 \circ f_0)(\overline{x}_0) = \overline{x}_0$ *and* $(f_0 \circ f_1)(\overline{x}_1) = \overline{x}_1$.
2. $\frac{d(f_1 \circ f_0)}{dx}(x)\Big|_{x=\overline{x}_0} = f_1'(\overline{x}_1) f_0'(\overline{x}_0) = 1.$
3. *Fixed* $m \geq 2$: $\frac{d^n(f_1 \circ f_0)}{dx^n}(x)\Big|_{x=\overline{x}_0} = 0$ *for* $2 \leq n \leq m$.

*Then the reverse composition $f_0 \circ f_1$ satisfies*

$$\frac{d^n(f_0 \circ f_1)}{dx^n}(x)\bigg|_{x=\overline{x}_1} = 0, \text{ for } 2 \leq n \leq m.$$

**Example 23** *[5] We will prove directly that the second and third derivatives of alternating maps are both zero, regardless of the order of composition. We do this directly using the higher order chain rule, or Faà di Bruno formula [25] fist proved in [28].*

*Let $f_0$ and $f_1$ be $\mathbf{C}^3$ functions satisfying the conditions:*

1. $(f_0 \circ f_0)(\overline{x}_0) = \overline{x}_0$ *and* $(f_0 \circ f_1)(\overline{x}_1) = \overline{x}_1$ *which is* $f_0(\overline{x}_0) = \overline{x}_1$ *and* $f_1(\overline{x}_1) = \overline{x}_0$.
2. $\frac{d(f_1 \circ f_0)}{dx}(x)\Big|_{x=\overline{x}_0} = f_1'(\overline{x}_1) f_0'(\overline{x}_0) = 1.$
3. $\frac{d^m(f_1 \circ f_0)}{dx^m}(x)\Big|_{x=\overline{x}_0} = 0$ *for* $m = 2, 3$.

*Let us recall the formula of Faà di Bruno for the derivatives of the composition*

$$\frac{d^m\left(f_1\circ f_0\right)}{dx^m}(x)=m!\sum_{n=1}^{m}f_1^{(n)}\left(f_0\left(x\right)\right)\prod_{j=1}^{m}\frac{1}{b_j!}\left(\frac{f_0^{(j)}\left(x\right)}{j!}\right)^{b_j}, \tag{21}$$

*where the sum is over all different solutions $b_j$ in nonnegative integers of the equation*

$$\sum_{j=1}^{m}jb_j=m,\ \text{ and } n:=\sum_{j=1}^{m}b_j.$$

*To avoid to overload this example with indexes we use the notation*

$$\frac{d^m(f_1\circ f_0)}{dx^m}(x)\bigg|_{x=\overline{x}_0}=\left(f_1f_0\right)_m, \qquad \frac{d^m(f_0\circ f_1)}{dx^m}(x)\bigg|_{x=\overline{x}_1}=\left(f_0f_1\right)_m.$$

*With this notation the Faà di Bruno Formula computed at the conditions of the problem is*

$$\left(f_1f_0\right)_m=m!\sum_{n=1}^{m}f_1^{(n)}(\overline{x}_1)\prod_{j=1}^{m}\frac{1}{b_j!}\left(\frac{f_0^{(j)}(\overline{x}_0)}{j!}\right)^{b_j} \tag{22}$$

*and*

$$\left(f_0f_1\right)_m=m!\sum_{n=1}^{m}f_0^{(n)}(\overline{x}_0)\prod_{j=1}^{m}\frac{1}{b_j!}\left(\frac{f_1^{(j)}(\overline{x}_1)}{j!}\right)^{b_j} \tag{23}$$

*Condition 2 in this notation is now*

$$f_0'(\overline{x}_0)f_1'(\overline{x}_1)=1. \tag{24}$$

*Let us consider the first cases. Let $m=2$, we will use the formula 21, so we have to solve the equation*

$$b_1+2b_2=2,$$

*for all possible values of the vector $(b_1,b_2)$ with nonnegative integers. The only solutions are $b_1=0$, $b_2=1$ which gives $n=1$ and $b_1=2$, $b_2=0$ with $n=2$, so we have*

$$\left(f_1f_0\right)_2=2!\left(f_1'(\overline{x}_1)\frac{1}{0!}\left(\frac{f_0'(\overline{x}_0)}{1!}\right)^0\frac{1}{1!}\left(\frac{f_0''(\overline{x}_0)}{2!}\right)^1+f_1''(\overline{x}_1)\frac{1}{2!}\left(\frac{f_0'(\overline{x}_0)}{1!}\right)^2\frac{1}{0!}\left(\frac{f_0''(\overline{x}_0)}{2!}\right)^0\right)=0 \tag{25}$$

$$=f_1'(\overline{x}_1)f_0''(\overline{x}_0)+f_1''(\overline{x}_1)(f_0'(\overline{x}_0))^2=f_1'(\overline{x}_1)f_0''(\overline{x}_0)+\frac{f_1''(\overline{x}_1)}{f_1'(\overline{x}_1)}=0$$

*and*

$$(fg)_2 = 2!\left(f_0'(\overline{x}_0)\frac{1}{0!}\left(\frac{f_1'(\overline{x}_1)}{1!}\right)^0\frac{1}{1!}\left(\frac{f_1''(\overline{x}_1)}{2!}\right)^1 + f_0''(\overline{x}_0)\frac{1}{2!}\left(\frac{f_1'(\overline{x}_1)}{1!}\right)^2\frac{1}{0!}\left(\frac{f_1''(\overline{x}_1)}{2!}\right)^0\right) \tag{26}$$

$$= f_0'(\overline{x}_0)f_1''(\overline{x}_1) + f_0''(\overline{x}_0)(f_1'(\overline{x}_1))^2 = \frac{f_1''(\overline{x}_1)}{f_1'(\overline{x}_1)} + f_0''(\overline{x}_0)(f_1'(\overline{x}_1))^2.$$

*Using Cramer's rule we solve the system with equations (24) and (25) for* $f_1''(\overline{x}_1)$, *we get*

$$f_1''(\overline{x}_1) = \frac{1}{(f_0'(\overline{x}_0))^3}\begin{vmatrix} 0 & f_0''(\overline{x}_0) \\ 1 & f_0'(\overline{x}_0) \end{vmatrix} = -\frac{f_0''(\overline{x}_0)}{(f_0'(\overline{x}_0))^3},$$

*substituting* $f_1'(\overline{x}_1)$ *and* $f_1''(\overline{x}_1)$ *in Eq. (26) we get*

$$(f_0f_1)_2 = -f_0'(\overline{x}_0)\frac{f_0''(\overline{x}_0)}{(f_0'(\overline{x}_0))^3} + f_0''(\overline{x}_0)\frac{1}{(f_0'(\overline{x}_0))^2} = 0.$$

*Now we consider the case* $m = 3$

$$(f_1f_0)_3 = f_1'(\overline{x}_1)f_0'''(\overline{x}_0) + 3f_1''(\overline{x}_1)f_0'(\overline{x}_0)f_0''(\overline{x}_0) + f_1'''(\overline{x}_1)(f_0'(\overline{x}_0))^3 \tag{27}$$

$$= f_1'(\overline{x}_1)f_0'''(\overline{x}_0) + \frac{3f_1''(\overline{x}_1)f_0''(\overline{x}_0)}{f_1'(\overline{x}_1)} + \frac{f_1'''(\overline{x}_1)}{(f_1'(\overline{x}_1))^3} = 0.$$

*We use Cramer's rule to solve the system consisting of equations (24), (25) and (27) for* $f_1'''(\overline{x}_1)$

$$\begin{aligned} f_1'''(\overline{x}_1) &= \frac{1}{(f_0'(\overline{x}_0))^6}\begin{vmatrix} 0 & 3f_0'(\overline{x}_0)f_0''(\overline{x}_0) & f_0'''(\overline{x}_0) \\ 0 & (f_0'(\overline{x}_0))^2 & f_0''(\overline{x}_0) \\ 1 & 0 & f_0'(\overline{x}_0) \end{vmatrix} \\ &= -\frac{f_0'''(\overline{x}_0)}{(f_0'(\overline{x}_0))^4} - \frac{3\left(-\frac{f_0''(\overline{x}_0)}{(f_0'(\overline{x}_0))^3}\right)f_0''(\overline{x}_0)}{(f_0'(\overline{x}_0))^2} \\ &= \frac{3(f_0''(\overline{x}_0))^2}{(f_0'(\overline{x}_0))^5} - \frac{f_0'''(\overline{x}_0)}{(f_0'(\overline{x}_0))^4}. \end{aligned}$$

*In the case of the reverse order composition the third derivative (substituting* $f_1'(\overline{x}_1)$, $f_1''(\overline{x}_1)$ *and* $f_1'''(\overline{x}_1)$ *by the solutions obtained previously) is given by*

$$(f_0f_1)_3 = f_0'(\overline{x}_0)f_1'''(\overline{x}_1) + 3f_0''(\overline{x}_0)f_1'(\overline{x}_1)f_1''(\overline{x}_1) + f_0'''(\overline{x}_0)(f_1'(\overline{x}_1))^3 \tag{28}$$

$$= f_0'(\overline{x}_0)\left(\frac{3(f_0''(\overline{x}_0))^2}{(f_0'(\overline{x}_0))^5} - \frac{f_0'''(\overline{x}_0)}{(f_0'(\overline{x}_0))^4}\right) + 3f_0''(\overline{x}_0)\frac{1}{f_0'(\overline{x}_0)}\left(-\frac{f_0''(\overline{x}_0)}{(f_0'(\overline{x}_0))^3}\right) + \frac{f_0'''(\overline{x}_0)}{(f_0'(\overline{x}_0))^3}$$

$$= 0.$$

Finally we end this section presenting the extension of theorem 21 to the periodic case that answers the question posed in the beginning of this section.

**Theorem 24** *[5] Let $f_0$, $f_1$, ..., $f_{p-1}$ be maps with a sufficient number of derivatives satisfying the conditions:*

1. *There are periodic orbits with period $k$ for the compositions ($kp$ for the iterates)*

$$\begin{aligned}\Phi_{kp}\left(\overline{x}_0\right) &= \overline{x}_0,\\ \Phi_{kp}^{1}\left(\overline{x}_1\right) &= \overline{x}_1\\ &\vdots\\ \Phi_{kp}^{p-1}\left(\overline{x}_{p-1}\right) &= \overline{x}_{p-1}\end{aligned}$$

2. *The first bifurcation condition holds*

$$\left.\frac{d\Phi_{kp}}{dx}(x)\right|_{x=\overline{x}_0} = 1.$$

3. *Higher degeneracy conditions hold.*
   *Fixed $m \geq 2$:*

$$\left.\frac{d^n\Phi_{kp}}{dx^n}(x)\right|_{x=\overline{x}_0} = 0, \qquad 2 \leq n \leq m.$$

*Then $\Phi_{kp}^{j}$, with $j = 1, \ldots, p-1$, satisfies*

$$\left.\frac{d^n\Phi_{kp}^{j}}{dx^n}(x)\right|_{x=\overline{x}_j} = 0, \text{ for } 2 \leq n \leq m.$$

Now we give an example for a $2-$periodic system where the maps do not arise from a family of maps, one is unimodal and the other is bimodal.

**Example 25** *[30]*
*Let us now consider the maps*

$$\begin{array}{rccc} f_0 : & [-1,1]\times[1,4] & \longrightarrow & \mathbb{R} \\ & (x,\lambda_0) & \longrightarrow & \lambda_0 x^3 + (1-\lambda_0)x \end{array}$$

*and*

$$\begin{array}{rccc} f_1 : & [-1,1]\times[0,2] & \longrightarrow & \mathbb{R} \\ & (x,\lambda_0,\lambda_1) & \longrightarrow & -\lambda_1 x^2 - 1 + \lambda_1 \end{array}.$$

*The composition operator $\Phi_2$ is now defined $\Phi : [-1,1] \times [1,4] \times [0,2] \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} \Phi_2(x, \lambda_0, \lambda_1) &= f_1(f_0(x, \lambda_0), \lambda_1) \\ &= -\lambda_1(\lambda_0 x^3 + (1-\lambda_0)x)^2 - 1 + \lambda_1 \end{aligned}$$

*We consider the pitchfork bifurcation problem. In this case we have $m = 2$. The bifurcation equations are*

$$\begin{aligned} \Phi_{2k}(x, \lambda_0, \lambda_1) &= x, \\ \frac{d\Phi_{2k}}{dx}(x, \lambda_0, \lambda_1) &= 1, \\ \frac{d^2\Phi_{2k}}{dx^2}(x, \lambda_0, \lambda_1) &= 0, \end{aligned} \tag{29}$$

*where we assume that there are no more degeneracy conditions. This problem has two solutions, respectively*

$$\begin{aligned} \overline{x}_0 &\cong -0.247674 \\ (\lambda_0, \lambda_1) &\cong (2.85032, 0.90883) \end{aligned}$$

*and*

$$\begin{aligned} \overline{y}_0 &\cong 0.620345 \\ (\lambda_0, \lambda_1) &\cong (2.20004, 1.70216). \end{aligned}$$

*Hence there are two pitchfork bifurcation points.*

**Example 26** *[30] We can also study $\widetilde{\Phi}_2 = f_0 \circ f_1$, with the same families of example 25, the composition appearing now in the reverse order. It is possible to show, with much more cumbersome computations if treated directly, that this problem has two pitchfork bifurcation points. As in the previous example, exactly at the same values of the parameters*

$$\begin{aligned} \overline{x}_0 &\cong 0.414971 \\ (\lambda_0, \lambda_1) &\cong (2.85032, 0.90883) \end{aligned}$$

*and*

$$\begin{aligned} \overline{y}_1 &\cong -0.219234 \\ (\lambda_0, \lambda_1, x) &\cong (2.20004, 1.70216)\,. \end{aligned}$$

# 9 Attenuance and resonance

## 9.1 The Beverton-Holt equation

In [8] Cushing and Henson conjectured that a nonautonomous $p-$periodic Beverton-Holt equation with periodically varying carrying capacity must be attenuant. This means that if $C_p = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{p-1}\}$ is its $p-$periodic cycle, and $K_i$, $0 \le i \le p-1$ are the carrying capacities, then

$$\frac{1}{p}\sum_{i=0}^{p-1} \overline{x}_i < \frac{1}{p}\sum_{i=0}^{p-1} K_i. \tag{30}$$

Since the periodic cycle $C_p$ is globally asymptotically stable on $(0, \infty)$, it follows that for any initial population density $x_0$, the time average of the population density $x_n$ is eventually less than the average of the carrying capacities, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_i < \frac{1}{p}\sum_{i=0}^{p-1} K_i. \tag{31}$$

Eq. (31) gives a justification for the use of the word "attenuance" to describe the phenomenon in which a periodically fluctuation carrying capacity of the Beverton-Holt equation has a deleterious effect on the population. This conjecture was first proved by Elaydi and Sacker in [14, 13, 13] and independently by Kocic [26] and Kon [27]. The following theorem summarizes our findings.

**Theorem 27** *[14] Consider the $p-$periodic Beverton-Holt equation*

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, n \in \mathbb{Z}^+, \tag{32}$$

*where $\mu > 1$, $K_{n+p} = K_n$, and $K_n > 0$. Then Eq. (32) has a globally asymptotically stable $p-$periodic cycle. Moreover, Eq. (32) is attenuant.*

Kocic [26], however gave the most elegant proof for the presence of attenuance. Utilizing effectively the Jensen's inequality, he was able to give the following more general result.

**Theorem 28** *[26] Assume that $\mu > 1$ and $\{K_n\}$ is a bounded sequence of positive numbers*

$$0 < \alpha < K_n < \beta < \infty.$$

*Then for every positive solution $\{x_n\}$ of Eq. (32) we have*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \overline{x}_i \le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} K_i. \tag{33}$$

## 9.2 Neither attenuance nor resonance

By a simple trick, Sacker [31] showed that neither attenuance nor resonance occurs when periodically forcing the Ricker maps

$$R(x) = xe^{p-x}.$$

So consider the $k-$periodic system

$$x_{n+1} = x_n e^{p_n - x_n}, p_{n+k} = p_n, n \in \mathbb{Z}^+. \tag{34}$$

If $0 < p_n < 2$, Eq. (34) has a globally asymptotically stable $k-$periodic cycle [31].

Let $C_k = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{k-1}\}$ be this unique $k-$periodic cycle. Then

$$\begin{aligned}\overline{x}_0 &= \overline{x}_k = \overline{x}_{k-1} e^{p_{k-1} - \overline{x}_{k-1}} \\ &= \overline{x}_{k-2} e^{p_{k-2} - \overline{x}_{k-2}} e^{p_{k-1} - \overline{x}_{k-1}},\end{aligned}$$

and by iteration we get

$$\overline{x}_0 = \overline{x}_0 e^{\sum_{i=0}^{k-1} p_i - \sum_{i=0}^{k-1} \overline{x}_i}.$$

Hence

$$\tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} p_i = \tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} \overline{x}_i,$$

i.e., neither attenuance nor resonance.

## 9.3 An extension: monotone maps

Using an extension to monotone maps, Kon [27] considered a $p-$periodic difference equation of the form

$$x_{n+1} = g\left(x_n / K_n\right) x_n, n \in \mathbb{Z}^+, \tag{35}$$

where $K_{n+p} = K_n$, $K_n > 0$, $x_0 \in [0, \infty)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which satisfies the following properties

- $g(1) = 1$,
- $g(x) > 1$ for all $x \in (0, 1)$,
- $g(x) < 1$ for all $x \in (1, \infty)$.

**Theorem 29** *[27] Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a positive $r-$periodic cycle of Eq. (35) such that $K_i \neq K_{i+1}$ for some $0 \leq i \leq p-1$. Assume that $zg(z)$ is strictly concave on an interval $(a, b)$, $0 < a < b$ containing all points $\frac{\overline{x}_i}{K_i} \in (a, b)$, $1 \leq i \leq rp$. Then the cycle $C_r$ is attenuant.*

This theorem provides an alternative proof of the attenuance of the periodic Bevertob-Holt equation (32).

Consider the equation [27]

$$x_{n+1} = \left(\frac{x_n}{K_n}\right)^{a-1}, 0 < a < 1, \tag{36}$$

where $K_{n+p} = K_n$, $n \in \mathbb{Z}^+$, and $K_i \neq K_{i+1}$ for some $i \in \mathbb{Z}^+$. The maps belong to the class $\mathcal{K}$ and satisfy the assumption of the preceding theorem. Consequently, Eq. (36) has a globally asymptotically stable $p-$periodic cycle that is attenuant.

## 9.4 The loss of attenuance: resonance.

Consider the periodic Beverton-Holt equation (32) in which both parameters $\mu_n$ and $K_n$ are periodic of common period $p$. This equation may be attenuant or resonant. In fact, when $p = 2$, Elaydi and Sacker [14] showed that

$$\overline{x} = \overline{K} + \sigma\frac{K_0 - K_1}{2} - \Delta\frac{(\mu_0 - 1)(\mu_1 - 1)}{2(\mu_0\mu_1 - 1)}(K_0 - K_1)^2 \tag{37}$$

where

$$\overline{x} = \frac{\overline{x}_0 + \overline{x}_1}{2} \text{ and } \overline{K} = \frac{K_0 + K_1}{2},$$

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0\mu_1 - 1}, 0 \leq |\sigma| < 1,$$

and

$$\Delta = \frac{\mu_0(\mu_1^2 - 1)K_0 + \mu_1(\mu_0^2 - 1)K_1}{\mu_0(\mu_1 - 1)^2K_0^2 + (\mu_0 - 1)(\mu_1 - 1)(\mu_0\mu_1 + 1)K_0K_1 + \mu_1(\mu_0 - 1)^2K_1^2} > 0.$$

It follows that attenuance is present if either $(\mu_1 - \mu_0)(K_0 - K_1) < 0$ (out of phase) or the algebraic sum of the last two terms in Eq. (37) is negative. On the other hand, resonance is present if the algebraic sum of the last two terms in Eq. (37) is positive.

Notice that if $\mu_0 = \mu_1 = \mu$ with $p = 2$, then we have

$$\frac{1}{p}\sum_{i=0}^{p-1}\overline{x}_i = \frac{1}{p}\sum_{i=0}^{p-1}K_i - \frac{\mu(K_0 + K_1)(K_1 - K_0)^2}{2\left[\mu K_0^2 + (\mu^2 + 1)K_0K_1 + \mu K_1^2\right]},$$

which gives an exact expression for the difference in the averages.

**Remark 30** *Now for $\mu_0 = 4$, $\mu_1 = 2$, $K_0 = 11$, and $K_1 = 7$, we have resonance as $\frac{1}{2}\sum_{i=0}^{1}\overline{x}_i \approx 9.23$ and $\frac{1}{2}\sum_{i=0}^{1}K_i = 9$. On the other hand, one can show that for $\mu_0 = 2$, $\mu_1 = 4$, $K_0 = 11$, and $K_1 = 7$, we have attenuance as may be seen from (37).*

## 9.5 The signature functions of Franke and Yakubu

In [19], the authors gave a criteria to determine attenuance or resonance for the $2-$periodic difference equation

$$x_{n+1} = x_n g(K_n, \mu_n, x_n), n \in \mathbb{Z}^+, \tag{38}$$

where $K_n = K(1 + \alpha(-1)^n)$, $\mu_n = \mu(1 + \beta(-1)^n)$, and $\alpha, \beta \in (-1, 1)$.

Define the following

$$\omega_1 = \frac{\left(k\frac{\partial^2 g}{\partial x^2} + 2\frac{\partial g}{\partial x}\right)\left(\frac{K^2 \frac{\partial g}{\partial K}}{2+K\frac{\partial g}{\partial x}}\right)^2 + \left(2K\frac{\partial g}{\partial K} + 2K^2\frac{\partial^2 g}{\partial x \partial K}\right) + K^3\frac{\partial^2 g}{\partial K^2}}{-2K\frac{\partial g}{\partial x}}, \tag{39}$$

$$\omega_2 = \frac{-\left(\mu\frac{\partial g}{\partial \mu} + K\mu\frac{\partial^2 g}{\partial x \partial \mu}\right)\left(\frac{-K^2 \frac{\partial g}{\partial K}}{2+K\frac{\partial g}{\partial x}}\right) + K^2\mu\frac{\partial^2 g}{\partial K \partial \mu}}{K\frac{\partial g}{\partial x}}, \tag{40}$$

and

$$\mathcal{R}_d = sign(\alpha(\omega_1\alpha + \omega_2\beta)). \tag{41}$$

**Theorem 31** *[19] If for $\alpha = 0, \beta = 0$, $K$ is hyperbolic fixed point of equation (38), then for all sufficiently small $|\alpha|$ and $|\beta|$, the equation (38), with $\alpha, \beta \in (-1, 1)$, has an attenuant $2-$periodic cycle if $\mathcal{R}_d < 0$ and a resonant $2-$periodic cycle if $\mathcal{R}_d > 0$.*

To illustrate the effectiveness of this theorem, let us to consider the logistic equation

$$x_{n+1} = x_n\left[1 + \mu(1 + \beta(-1)^n)\left(1 - \frac{x_n}{K(1 + \alpha(-1)^n)}\right)\right]. \tag{42}$$

For $0 < \mu < 2$ Eq. (42) has an asymptotically stable $2-$periodic cycle. Using formulas (39) and (40), one obtains

$$\omega_1 = \frac{-8K}{(\mu - 2)^2} \text{ and } \omega_2 = \frac{-4K}{\mu - 2}.$$

Assume that $\alpha > 0$ and $0 < \mu < 2$. Using (41) yields

$$\mathcal{R}_d = sign\left(\frac{2}{\mu - 2}\alpha + \beta\right) = sign\left(\beta - \frac{2}{2 - \mu}\alpha\right).$$

Hence we have attenuance if $\beta < \frac{2}{2-\mu}\alpha$, i.e., if the relative strength of the fluctuation of the demographic characteristic of the species is weaker than $\frac{2}{2-\mu}$ times the relative strength of the fluctuation of the carrying capacity. On the other hand if $\beta > \frac{2}{2-\mu}\alpha$ we obtain resonance.

Notice that if $\alpha = 0$ (the carrying capacity is fixed), then we have resonance if $\beta > 0$ and we have attenuance if $\beta < 0$. For the case that $\beta = 0$ (the intrinsic growth rate is fixed), we have attenuance.

Finally, we note that Franke and Yakubu extended their study to periodically forced Leslie model with density-dependent fecundity functions [18]. The model is of the form

$$\begin{aligned} x^1_{n+1} &= \sum_{i=1}^{s} x^i_n g^i_n(x^i_n) = \sum_{i=1}^{s} f^i_n(x^i_n) \\ x^2_{n+1} &= \lambda_1 x^1_n \\ &\vdots \\ x^s_{n+1} &= \lambda_{s-1} x^{s-1}_n, \end{aligned}$$

where $f^i_n$ is of the Beverton-Holt type. Results similar to the one-dimensional case where each $f^i_n$ is under compensatory, i.e.,

$$\frac{\partial f^i_n(x_i)}{\partial x_i} > 0, \frac{\partial^2 f^i_n(x_i)}{\partial x_i^2} < 0,$$

and $\lim\limits_{x_i \to \infty} f^i_n(x_i)$ exists for all $n \in \mathbb{Z}^+$.

# 10 Almost periodic difference equations

In this section we extend our study to the almost periodic case. This is particularly important in applications to biology in which habitat's fluctuations are not quite periodic.

But in order to embark on this endeavor, one needs to almost reinvent the wheel. The problem that we encounter here is that the existing literature deals exclusively with almost periodic fluctuations (sequences) on the real line $\mathbb{R}$ (on the integers $\mathbb{Z}$). To have meaningful applications to biology, we need to study almost periodic fluctuations or sequence on $\mathbb{Z}^+$ (the set of nonnegative integers). Such a program has been successfully implemented in [10]. Our main objective here is to report to the reader a brief but through exposition of these results.

We start with the following definitions from [17, 21].

**Definition 32** *An $\mathbb{R}^k-$valued sequence $x = \{x_n\}_{n\in\mathbb{Z}^+}$ is called Bohr almost periodic if for each $\epsilon > 0$, there exists a positive integer $T_0(\epsilon)$ such that among any $T_0(\epsilon)$ consecutive integers, there exists at least one integer $\tau$ with the following property:*

$$\| x_{n+\tau} - x_n \| < \epsilon, \forall n \in \mathbb{Z}^+.$$

*The integer* $\tau$ *is then called an* $\epsilon-$*period of the sequence* $x = \{x_n\}_{n\in\mathbb{Z}^+}$.

**Definition 33** *An* $\mathbb{R}^k-$*value sequence* $x = \{x_n\}_{n\in\mathbb{Z}^+}$ *is called Bochner almost periodic if for every sequence* $\{h(n)\}_{n\in\mathbb{Z}^+}$ *of positive integers there exists a subsequence* $\{h_{n_i}\}$ *such that* $\{x_{n+n_i}\}_{n_i\in\mathbb{Z}^+}$ *converges uniformly in* $n \in \mathbb{Z}^+$.

In [10] it was shown that the notions of Bohr almost periodicity and Bochner almost periodicity are equivalent.

Now a sequence $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is called almost periodic in $n \in \mathbb{Z}^+$ uniformly in $x \in \mathbb{R}^k$ if for each $\varepsilon > 0$, there exists $T_0(\varepsilon) \in \mathbb{Z}^+$ such that among $T_0(\varepsilon)$ consecutive integers there exists at least one integer $s$ with

$$\| f(n+s, x) - f(n, x) \| < \varepsilon$$

for all $x \in \mathbb{R}^k$, and $s \in \mathbb{Z}^+$.

Now consider the almost periodic difference equations

$$x_{n+1} = A_n x_n \quad (43)$$

$$y_{n+1} = A_n y_n + f(n, y_n), \quad (44)$$

where $A_n$ is a $k \times k$ almost periodic matrix on $\mathbb{Z}^+$, and $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is almost periodic.

Let $\Phi(n, s) = \prod_{r=s}^{n-1} A_r$ be the state transition matrix of equation (43). Then equation (43) is said to posses a regular exponential dichotomy [23] if there exist a $k \times k$ projection matrix $P_n$, $n \in \mathbb{Z}^+$, and positive constants $M$ and $\beta \in (0, 1)$ such that the following properties hold:

1. $A_n P_n = P_{n+1} A_n$;
2. $\| X(n, r) P_r x \| \leq M\beta^{n-r} \| x \|$, $0 \leq r \leq n, x \in \mathbb{R}^k$;
3. $\| X(r, n)\,(I - P_n)\,x \| \leq M\beta^{n-r} \| x \|$, $0 \leq r \leq n, x \in \mathbb{R}^k$;
4. The matrix $A_n$ is an isomorphism from $R\,(I - P_n)$ onto $R\,(I - P_{n+1})$, where $R(B)$ denotes the range of the matrix $B$.

We are now in a position to state the main stability result for almost periodic systems.

**Theorem 34** *Suppose that Eq. (43) possesses a regular exponential dichotomy with constant $M$ and $\beta$ and $f$ is a Lipschitz with a constant Lipschitz $L$. Then Eq. (44) has a unique globally asymptotically stable almost periodic solution provided*

$$\frac{M\beta L}{1-\beta} < 1.$$

**Proof.** Let $AP(\mathbb{Z}^+)$ be the space of almost periodic sequences on $\mathbb{Z}^+$ equipped with the topology of the supremum norm. Define the operator $\Gamma$ on $AP(\mathbb{Z}^+)$ by letting

$$(\Gamma\varphi)_n = \sum_{r=0}^{n-1}\left(\prod_{s=r}^{n-1}\right) A_s f(r,\varphi_r).$$

Then $\Gamma : AP(\mathbb{Z}^+) \to AP(\mathbb{Z}^+)$ is well defined. Moreover $\Gamma$ is a contraction. Using the Banach fixed point theorem, we obtain the desired conclusion. ■

The preceding result may be applied to many populations models. However, we will restrict our treatment here on the almost periodic Beverton-Holt equation with overlapping generations

$$x_{n+1} = \gamma_n x_n + \frac{(1-\gamma_n)\mu K_n x_n}{(1-\gamma_n)K_n + (\mu - 1\gamma_n)x_n} \tag{45}$$

with $K_n > 0$ and $\gamma_n \in (0,1)$ are almost periodic sequences, and $\mu > 1$. As before $\mu$ and $K$ denote the intrinsic growth rate and the carrying capacity of the population, respectively, while $\gamma$ is the survival rate of the population from one generation to ne next.

The following result follows from theorem 34

**Theorem 35** *Eq. (45) has a unique globally asymptotically stable almost periodic solution provided that*

$$\sup\left\{\gamma_n : n \in \mathbb{Z}^+\right\} < \frac{1}{1+\mu}$$

To this end, we have addressed the question of stability and existence of almost periodic solution of almost periodic difference equation. We now embark on the task of the determination of whether a system is attenuant or resonant.

Let $\{\mu_n\}_{n\in\mathbb{Z}^+}$ be an almost periodic sequence on $\mathbb{Z}^+$. Then we define its mean value as

$$M(\mu_n) = \lim_{n\to\infty}\frac{1}{m}\sum_{r=1}^{m}\mu_{n+r} \tag{46}$$

It may be shown that $M(\mu_n)$ exists [10].

Let $\{\overline{x}_n\}$ be the almost periodic solution of a given almost periodic system. Then we say that the system is

1. attenuant if $M(\overline{x}_n) < M(K_n)$,
2. resonant if $M(\overline{x}_n) > M(K_n)$.

**Theorem 36** *[10] Suppose that $\{K_n\}_{n\in\mathbb{Z}^+}$ is almost periodic, $K_n > 0$, $\mu > 1$, and $\gamma_n = \gamma \in (0,1)$. Then*

1. $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} x_m \le \limsup\limits_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} K_m$ *for any solution $x_n$ of Eq. (45),*
2. $M(\overline{x}_n) \le M(K_n)$ *if $\overline{x}_n$ is the unique almost periodic solution of Eq. (45).*

# 11 Stochastic difference equations

In [22] the authors investigated the stochastic Beverton-Holt equation and introduced new notions of attenuance and resonance in the mean.

Following on the same lines [6] the authors investigated the stochastic Beverton-Holt equation with overlapping generations.

In this section, we will consider the latter study and consider the equation

$$x_{n+1} = \gamma_n x_n + \frac{(1-\gamma_n)\mu K_n x_n}{(1-\gamma_n)K_n + (\mu - 1 + \gamma_n)x_n}. \tag{47}$$

Let $L^1(\Omega, \upsilon)$ be the space of integrable functions on a measurable space $(\Omega, \mathcal{F}, \upsilon)$ equipped with its natural norm given by

$$\| f \|_1 = \int_\Omega f(x) d\upsilon.$$

Let

$$\mathcal{D}(E) := \{f \in L^1(E, \upsilon) : f \ge 0 \text{ and } \textstyle\int_\Omega f d\upsilon\}$$

be the space of all densities on $\Omega$.

**Definition 37** *Let $\mathcal{Q} : L^1(\Omega, \upsilon) \to L^1(\Omega, \upsilon)$ be a Markov operator. Then $\{\mathcal{Q}^n\}$ is said to be asymptotically stable if there exists $f^* \in \mathcal{D}$ for which*

$$\mathcal{Q} f^* = f^*$$

*and for all $f \in \mathcal{D}$,*

$$\lim_{n\to\infty} \| \mathcal{Q}^n f - f^* \|_1 = 0.$$

We assume that both the carrying capacity $K_n$ and the survival rate $\gamma_n$ are random and for all $n$, $(K_n, \gamma_n)$ is chosen independently of $(x_0, K_0, \gamma_0)$, $(x_1, K_1, \gamma_1)$, ..., $(x_{n-1}, K_{n-1}, \gamma_{n-1})$ from a distribution with density $\Phi(K, \gamma)$.

The joint density of $x_n, K_n, \gamma_n$ is $f_n(x)\Phi(K, \gamma)$, where $f_n$ is the density of $x_n$. Furthermore, we assume that

$$E|K_n| < \infty, E|x_0| < \infty$$

and $K^2\Phi(K, \gamma)$ is bounded above independently of $\gamma$ and that $\Phi$ is supported on the product interval

$$[K_{\min}, \infty) \times [\gamma_{\min}, \infty),$$

for some $K_{\min} > 0$ and $\gamma_{\min} > 0$.

Moreover, we assume there exists an interval $(K_l, K_u) \subset \mathbb{R}^+$ on which $\Phi$ is positive everywhere for all $\gamma$.

Let $h$ be an arbitrary bounded and measurable function on $\mathbb{R}^+$ and define $b(K_n, \gamma_n, x_n)$ to be equal to the right-hand side of equation (47). The expected value of $h$ at time $n + 1$ is then given by

$$E[h(x_{n+1})] = \int_0^\infty h(x) f_{n+1}(x) dx. \tag{48}$$

Furthermore, because of (47) and the fact that the joint density of $x_n$, and $\gamma_n$ is just $f_n(x)\Phi(K, \gamma)$, we also have

$$\begin{aligned} E[h(x_{n+1})] &= E[h(b(K_n, \gamma_n, x_n))] \\ &= \int_0^\infty \int_0^1 \int_0^\infty h(b(K, \gamma, y)) f_n(y) \Phi(K, \gamma) dy d\gamma dy. \end{aligned}$$

Let us define $K = K(x, \gamma, y)$ by the equation

$$x = \frac{(1-\gamma)\mu K y}{(1-\gamma)K + (\mu - 1 + \gamma)y} + \gamma y. \tag{49}$$

Solving explicitly this equation for $K$ yields

$$K = \frac{(\mu - 1 + \gamma)y(x - \gamma y)}{(1-\gamma)[\mu y - (x - \gamma y)]}. \tag{50}$$

By a change of variables, this can be written as

$$E[h(x_{n+1})] = \iiint_{\{(x,\gamma,y): 0 < x - \gamma y < \mu y\}} h(x) f_n(y) \Phi(K, \gamma) \frac{dk}{db(K, \gamma, y)} dx d\gamma dy.$$

A simple calculation yields

$$E[h(x_{n+1})] = \mu \int_0^{\infty} \left\{ \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f_n(y) K^2 \Phi(K,\gamma) d\gamma dy \right\} dx,$$

where

$$A = \{(\gamma, y) : 0 < x - \gamma y < \mu y\}. \tag{51}$$

Equating the above equations, and using the fact that $h$ was an arbitrary, bounded, measurable function, we immediately obtain

$$f_{n+1}(x) = \mu \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f_n(y) K^2 \Phi(K,\gamma) d\gamma dy.$$

Let $\mathcal{P} : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ be defined by

$$\mathcal{P}f(x) = \mu \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f(y) K^2 \Phi(K,\gamma) d\gamma dy, \tag{52}$$

where $k = K(x, \gamma, y)$ is defined by (50) and $A$ in (51).

We can now state the main theorem of this section

**Theorem 38** *[6] The Markov operator $\mathcal{P} : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ defined by equation (52) is asymptotically stable.*

For the case when $\gamma_n = \gamma$ is a constant and $K_n$ is a random sequence, the following attenuance result was obtain.

For almost every $w \in \Omega$ and $x \in \mathbb{R}^+$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i(w, x) < \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} K_i(w),$$

that is we have attenuance in the mean.

It is still an open problem to determine the attenuance or resonance when both $\gamma_n$ and $K_n$ are random sequences on $\mathbb{Z}^+$.

# References


[1] W. C. Allee, *The Social Life of Animals*, William Heinemann, London 1938.

[2] Z. AlSharawi and J. Angelos, *On the periodic logistic equation*, Appl. Math. Comput., Vol. 180, No 1, pp. 342-352, 2006.

[3] Z. AlSharawi, J. Angelos, S. Elaydi and L. Rakesh, *An extension of Sharkovsky's theorem to periodic difference equations*, J. Math. Anal. Appl., J. Math. Anal. Appl., Vol. 316, 128-141, 2006.

[4] Emma D'Aniello; Henrique Oliveira, *Swallowtail bifurcation for non-autonomous interval maps,* Preprint CAMGSD, 2009.

[5] Emma D'Aniello; Henrique Oliveira, *A note on the bifurcation equations of the periodic non autonomous systems*, Preprint, Center for Mathematical Analysis, Geometry, and Dynamical Systems, Instituto Superior Tecnico, Lisbon, 2009.

[6] H. Bezandry, T. Diagana, and S. Elaydi, *On the stochastic Beverton-Holt equation with survival rates* ,Journal of Difference Equations and Applications, Vol. 14, No. 2, pp. 175190, 2007

[7] J. M. Cushing, *Oscillations in age-structured population models with an Allee effect*, Journal of Computational and Applied Mathematics 52, 71-80, 1994.

[8] J. M. Cushing and S. M. Henson, *Aperiodically forced Beverton-Holt equation*, Journal of Difference Equations and Applications, Vol. 8, 1119-1120, 2002.

[9] R. L. Devaney, *An introduction to Chaotic Dynamical Systems*, Perseus Books Publishing, L.L.C., Second edition, 1989.

[10] T. Diagana, S. Elaydi, A. Yakubu, *Populations models in almost periodic environments*, Journal of Difference Equations and Applications, Vol. 13, No. 4, pp. 289-308, 2007

[11] S. Elaydi, *Discrete Chaos: With Applications in Science and Engineering*, Chapman and Hall/CRC, Second Edition, 2008.

[12] S. Elaydi and A. Yakubu, *Global stability of cycles: Lotka-Volterra competition model with stocking*, Journal of Difference Equations and Applications, 8, No. 6, 2002, pp. 537-549

[13] S. Elaydi and R. Sacker, *Global stability of periodic orbits of nonautonomous difference equations and populations biology*, J. Differential Equations 208, pp. 258-273, 2005.

[14] S. Elaydi and R. Sacker, *Nonautonomous Beverton-Holt equations and the Cushing-Henson conjectures*, Journal of Difference Equations and Applications, 11, No. 4-5, 2005, pp. 337-346(10)

[15] S. Elaydi and R. Sacker, *Periodic difference equations, populations biology and the Cushing-Henson conjecture*, Mathematical Biosciences 201, pp. 195-2007, 2006.

[16] S. Elaydi, and R. Sacker, *Skew-product dynamical systems: Applications to difference equations*, Proceedings of the Second Annual Celebration of Mathematics, United Arab Emirates, 2005.

[17] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, New York-Berlin: Springer-Verlag, 377, 1974.

[18] J. Franke and A. Yakubu, *Globally attracting attenuant versus resonant cycles in periodic compensatory Leslie models* , Mathematical Biosciences, Vol. 204, 1, pp. 1-20, 2006

[19] J. Franke and A. Yakubu, *Using a signature function to determine resonant and attenuant 2-cycles in the SmithSlatkin population model*, Journal of Difference Equations and Applications, Vol. 13, No. 4, pp. 289308, 2007

[20] M. Grinfeld, P.A. Knight, H. Lamba, *On the periodically perturbed logistic equation*, J. Phys. A: Math. Gen. 29 (1996) 80358040

[21] A. Halanay, and V. Rasvan, *Stability and stable oscillations in discrete time systems*, Advances in Discrete Mathematics and Applications (Gordon and Breach Science Publication), Vol. 2., 2000

[22] C. Haskell,and R. Sacker, *The Stochastic Beverton-Holt Equation and the M. Neubert Conjecture*, Journal of Dynamics and Differential Equations/Elsevier, Vol. V. 17, No. 4, pp. pp. 825-842, 2005

[23] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, New York-Berlin: Springer-Verlag, 1981,.

[24] S. Henson, *Multiple attractors an resonance in periodically forced population models*, Physica D. 140, pp. 33-49, 2000.

[25] W. Johnson, *The Curious History of Faà di Bruno's Formula.* Amer. Math. Monthly 109, 217-234, 2002.

[26] V. Kocic, *A note on the nonautonomous Beverton-Holt model*,Journal of Difference Equations and Applications, Vol. 11, No. 4-5, pp. 415422, 2005.

[27] R. Kon, *A note on attenuant cycles of population models with periodic carrying capacity*, J. Difference Equations and Applications, Vol. 10, 8, pp. 791-793, 2004.

[28] Silvestre-François Lacroix, *Traité du Calcul différentiel et du Calcul intégral*, 2e Édition, Revue et Augmentée, vol. 1, Chez Courcier, Paris, 1810; vol. 2, Mme. Courcier, Paris, 1814; vol. 3, Mme. Courcier, Paris, 1819.

[29] R. Luís, S.Elaydi, and H.Oliveira, *Nonautonomous periodic systems with Allee effects*, Journal of Difference Equations and Applications, to appear

[30] H. Oliveira and E. D'Aniello, *Pitchfork bifurcation for non autonomous interval maps*, Journal of Difference Equations and Applications, 15, No. 3, pp. 291-302, 2009

[31] R. Sacker, *A Note on Periodic Ricker Maps*, J. Difference Equations and Applications, Vol. 13, No. 1, pp. 89-92, 2007

[32] R. Sacker and S. Elaydi, *Population models with Allee effects: A new model*, Preprint, 2009.

[33] D. Singer, *Stable orbits and bifurcation of maps of the interval*, SIAM Journal Appl. Math. 35 (1978), pp. 260-267