Towards a theory of periodic difference equations and population biology

Saber N. Elaydi*, Rafael Luís†, and Henrique Oliveira‡

May 13, 2009

Abstract

Keys Words: Discrete dynamical systems, Nonautonomous periodic difference equation, Cycles, Stability, Bifurcation, Attenuance and resonance

Contents

*Department of Mathematics, Trinity University, San Antonio, Texas, USA. Email: selaydi@trinity.edu

†Center for Mathematical Analysis, Geometry, and Dynamical Systems, Instituto Superior Tecnico, Technical University of Lisbon, Lisbon, Portugal. Email: rafael@ebsaas.com

‡Department of Mathematics, Instituto Superior Tecnico, Technical University of Lisbon, Lisbon, Portugal. Email: holiv@math.ist.utl.pt

1 Introduction

2 Preliminaries

Let X be a topological space and $\mathbb{Z}$ be the set of integers. A discrete dynamical system (X, π) is defined as a map $\pi : X \times \mathbb{Z} \to X$ such that π is continuous and satisfies the following two properties

1. $\pi(x, 0) = x$ for all $x \in X$,
2. $\pi(\pi(x, s), t) = \pi(x, s + t), s, t \in \mathbb{Z}$ and $x \in X$ (the group property).

We say (X, π) is a discrete semidynamical system if $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$, the set of nonnegative integers, and the group property is replaced by the semigroup property.

Notice that (X, π) can be generated by map f defined as $f(x) = \pi(x, 1)$, and $f^n(x) = \pi(x, n)$, where f^n denotes the n^{th} composition of f. Conversely, any map $f : X \to X$ can generate a discrete semidynamical system (X, π) by letting $\pi(x, n) = f^n(x)$. The crucial property here is the semigroup property (ii) above.

A difference equation is called autonomous if it is generated by one map such as

$$x_{n+1} = f(x_n), n \in \mathbb{Z}^+. \tag{1}$$

Notice that for any $x_0 \in X, x_n = f^n(x_0)$. Hence, the orbit $\mathcal{O}(x_0) = \{x_0, x_1, x_2, \ldots\}$ in Eq. (1) is the same as the orbit $\mathcal{O}(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$ under the map f.

A difference equation is called nonautonomous if it is governed by the rule

$$x_{n+1} = F(n, x_n), n \in \mathbb{Z}^+, \tag{2}$$

which may be written in the friendlier form

$$x_{n+1} = f_n(x_n), n \in \mathbb{Z}^+, \tag{3}$$

where $f_n(x) = F(n, x)$.

It should be pointed out here that equation (2) or (3) may not generate a discrete semidynamical system as it may not satisfy the semigroup property. The following example illustrates this point.

Example 1 *Consider the nonautonomous difference equation*

$$x_{n+1} = (-1)^n \left(\frac{n+1}{n+2}\right) x_n, x(0) = x_0. \tag{4}$$

The solution of Eq. (4) is

$$x_n = (-1)^{\frac{n(n-1)}{2}} \frac{x_0}{n+1}.$$

Let $\pi(x_0, n) = x_n$. *Then*

$$\begin{aligned} \pi(\pi(x_0, m), n) &= \pi\left((-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{m+1}, n\right) \\ &= (-1)^{\frac{n(n-1)}{2}} (-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{(n+1)(m+1)} \end{aligned}$$

However,

$$\pi(x_0, m+n) = (-1)^{\frac{(n+m)(n+m-1)}{2}} \frac{x_0}{m+n+1} \neq \pi(\pi(x_0, m), n).$$

3 Skew-product Systems

Consider the nonautonomous difference equation

$$x_{n+1} = F(n, x_n), n \in \mathbb{Z}^+, \tag{5}$$

where $F(n, \cdot) \in C(\mathbb{Z}^+ \times X, X) = C$. The space C is equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z}^+ \times X$. Let $F_t(n, \cdot) = F(t + n, \cdot)$ and $\mathcal{A} = \{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$ be the set of translates of F in C. Then $G(n, \cdot) \in \omega(\mathcal{A})$, the omega limit set of $\mathcal{A}$, if for each $n \in \mathbb{Z}^+$,

$$|F_t(n, x) - G(n, x)| \to 0$$

uniformly for x in compact subsets of X, as $t \to \infty$ along some subsequence $\{t_{n_i}\}$. The closure of $\mathcal{A}$ in C is called the hull of $F(n, \cdot)$ and is denoted by $Y = cl(\mathcal{A}) = \mathcal{H}(F)$.

On the space Y, we define a discrete semidynamical system $\sigma : Y \times \mathbb{Z}^+ \to Y$ by $\sigma(H(n, \cdot), t) = H_t(n, \cdot)$; that is σ is the shift map.

For convenience, one may write equation (5) in the form

$$x_{n+1} = f_n(x_n) \tag{6}$$

with $f_n(x_n) = F(n, x_n)$.

Define the composition operator Φ as follows

$$\Phi_n^i = f_{i+n-1} \circ \ldots \circ f_{i+1} \circ f_i \equiv \Phi_n(F(i, \cdot))$$

When $i = 0$, we write Φ_n^0 as Φ_n.

The skew-product system is now defined as

$$\pi : X \times Y \times \mathbb{Z}^+ \to X \times Y$$

with

$$\pi((x, G), n) = (\Phi_n(G(i, \cdot)), \sigma(G, n)).$$

If $G = f_i$, then $\pi((x, f_i), n) = (\Phi_n^i(x), f_{i+n})$.

The following commuting diagram illustrates the notion of skew-product systems where $\mathcal{P}(a, b) = a$ is the projection map.

$$\begin{array}{ccc} X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\ \Big\downarrow{\scriptstyle \mathcal{P}\times id} & & \Big\downarrow{\scriptstyle \mathcal{P}} \\ Y \times \mathbb{Z}^+ & \xrightarrow[\sigma]{} & Y \end{array}$$

For each $G(n, \cdot) \equiv g_n \in Y$, we define the fiber $\mathcal{F}_g$ over G as $\mathcal{F}_g = \mathcal{P}^{-1}(G)$. If $g = f_i$, we write $\mathcal{F}_g$ as $\mathcal{F}_i$.

Theorem 2 *π is a discrete semidynamical system.*

Example 3 (Example (1) revisited) *Let us reconsider the nonautonomous difference equation*

$$x_{n+1} = (-1)^n \left(\tfrac{n+1}{n+2}\right) x_n, x(0) = x_0.$$

Hence, $F(n, x) = (-1)^n \left(\frac{n+1}{n+2}\right) x = f_n(x)$. Its hull is given by $G(n, x) = (-1)^n x$, that is, g_n is a periodic sequence given by $g_0 = g_{2n}$, $g_1 = g_{2n+1}$, for all $n \in \mathbb{Z}^+$, in which $g_0(x) = x$, and $g_1(x) = -x$.

It is easy to verify that π defined as $\pi((x, f_i), n) = (\Phi_n^i(x), f_{i+n})$ is a semidynamical system.

4 Periodic difference equations I: Periodicity

In this section our focus will be on $p-$periodic difference equations of the form

$$x_{n+1} = f_n(x_n), \tag{7}$$

where $f_{n+p} = f_n$ for all $n \in \mathbb{Z}^+$.

The question that we are going to address is this: What are the permissible periods of the periodic orbits of equation (7)?

We begin by defining an $r-$periodic cycle (orbit).

Definition 4 *An ordered set of points $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ is $r-$periodic in X if*

$$f_{(i+nr)\ mod\ p}(\overline{x}_i) = \overline{x}_{(i+1)\ mod\ r}, n \in \mathbb{Z}^+.$$

In particular,

$$f_i(\overline{x}_i) = \overline{x}_{i+1},\ 0 \leq i \leq r-2,$$

and

$$f_t(\overline{x}_{t\ mod\ r}) = \overline{x}_{(t+1)\ mod\ r}, r-1 \leq t \leq p-1.$$

It should be noted that the $r-$periodic cycle C_r in X generates an $s-$periodic cycle on the skew-product $X \times Y$ of the form $\widetilde{C}_s = \{(\overline{x}_0, f_0), (\overline{x}_1, f_1), ..., (\overline{x}_s, f_{s\ mod\ p})\}$, where $s = lcm[r, p]$ is the least common multiple of r and p.

The r-periodic orbit C_r is called an $r-$geometric cycle, and the $s-$periodic orbit $\widetilde{C_r}$ is called an $s-$complete cycle.

Example 5 *Consider the nonautonomous periodic Beverton-Holt equation*

$$x_{n+1} = \frac{\mu_n K_n x_n}{K_n + (\mu_n - 1)x_n}, \tag{8}$$

with $\mu_n > 1, K_n > 0, K_{n+p} = K_n$, and $\mu_{n+p} = \mu_n$, for all $n \in \mathbb{Z}^+$.

1. *Assume that $\mu_n = \mu > 1$ is constant for all $n \in \mathbb{Z}^+$. Then one may appeal to Corollary 6.5 in [13] to show that (8) has no nontrivial periodic cycles of period less than p. In fact, (8) has a unique globally asymptotically stable cycle of minimal period p.*

2. *Assume that μ_n is periodic. Let $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$, $K_0 = 1$, $K_1 = \frac{6}{17}$, $K_2 = 2$, and $K_3 = \frac{4}{11}$. This leads to a $4-$periodic difference equation. There is, however, a $2-$geometric cycle, namely, $C_2 = \left\{\frac{2}{5}, \frac{2}{3}\right\}$(see Figure 1). This $2-$periodic cycle in the space X generates the following $4-$complete cycle on the skew-product $X \times Y$*

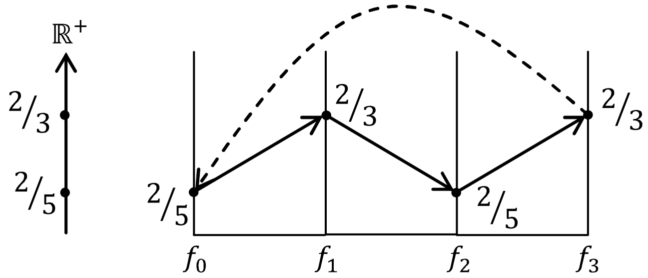

Figure 1: A 2−periodic cycle in a 4−periodic difference equation.

$$\widetilde{C}_4 = \left\{(\tfrac{2}{5}, f_0), (\tfrac{2}{3}, f_1), (\tfrac{2}{5}, f_2), (\tfrac{2}{3}, f_3)\right\},$$

where $f_0(x) = \frac{3x}{1+2x}$, $f_1(x) = \frac{24x}{6+51x}$, $f_2(x) = \frac{4x}{2+x}$, *and* $f_3(x) = \frac{5x}{1+11x}$.

We are going to provide a deeper analysis of the preceding example. Let $d = \gcd(r, p)$ be the greatest common divisor of r and p, $s = lcm[r, p]$ be the least common multiple of r and p, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. The following result is one of two crucial lemmas in this article.

Lemma 6 *[13] Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be a set of points in a metric space X. Then the following statements are equivalent.*

1. *C_r is a periodic cycle of minimal period r.*
2. *For $0 \le i \le r-1$, $f_{(i+nd)\ mod\ p}(\overline{x}_i) = \overline{x}_{(i+1)\ mod\ r}$.*
3. *For $0 \le i \le r-1$, the graphs of the functions*

$$f_i, f_{(i+d)\ mod\ p}, \ldots, f_{(i+(m-1)d) mod\ p}$$

intersect at the ℓ points

$$(\overline{x}_i, \overline{x}_{(i+1)\ mod\ r}), (\overline{x}_{(i+d)\ mod\ r}, \overline{x}_{(i+1+d)\ mod\ r}), \ldots (\overline{x}_{(i+(\ell-1)d)\ mod\ r}, \overline{x}_{(i+(\ell-1)d+1)\ mod\ r}).$$

Corollary 7 *Assume that the one-parameter family $F(\alpha, x)$ is one to one in α. Let $f_n(x_n) = F(\alpha_n, x_n)$. Then if the $p-$periodic difference equation, with minimal period p,*

$$x_{n+1} = f_n(x_n) \tag{9}$$

has a nontrivial periodic cycle of minimal period r, then $r = tp$, $t \in \mathbb{Z}^+$.

Proof. Suppose that equation (9) has a periodic cycle $C_r = \{\overline{x}_0, \overline{x}_1, \dots, \overline{x}_{r-1}\}$ of period $r < p$, and let $d = \gcd(r,p)$, $s = lcm[r,p]$, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. Then by Lemma 6, the graphs of the maps $f_0, f_d, \dots, f_{(m-1)d}$ must intersect at the points $(\overline{x}_0, \overline{x}_1), (\overline{x}_d, \overline{x}_{d+1}), \dots, (\overline{x}_{(\ell-1)d}, \overline{x}_{(\ell-1)d+1})$.

Since $F(\alpha, x)$ is one to one in α, the maps $f_0, f_d, \dots, f_{(m-1)d}$ do not intersect, unless they are all equal. Similarly, one may show that $f_i = f_{i+d} = \dots = f_{i+(m-1)d}$. This shows that equation (9) is of minimal period d, a contradiction. Hence r is equal to p or a multiple of p. ∎

Applying corollary 7 to the periodic Beverton-Holt equation with $K_{n+p} = K_n, \mu_n = \mu$, for all $n \in \mathbb{Z}^+$, shows that the only possible period of a nontrivial periodic cycle is p. However, for the case μ_n and K_n are both periodic of common period p, the situation is murky as was demonstrated by Example 5, case 2.

For the values $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$, $K_0 = 1$, $K_1 = 6/17$, $K_2 = 2$, and $K_3 = 4/11$, we have $f_0(x) = \frac{3x}{1+2x}$, $f_1(x) = \frac{24x}{6+51x}$, $f_2(x) = \frac{4x}{2+x}$, and $f_3(x) = \frac{5x}{1+11x}$. Let $\mathcal{F} = \{f_0, f_1, f_2, f_3\}$. Clearly $x^* = 0$ is a fixed point of the periodic system $\mathcal{F}$. To have a positive fixed point (period 1) or a periodic cycle of period 3, we must have the graphs of f_0, f_1, f_2, f_3 intersect at points $(\overline{x}_0, \overline{x}_1), (\overline{x}_1, \overline{x}_2), \dots, (\overline{x}_{\ell-1}, \overline{x}_\ell)$, where $\ell = 1$ or $\ell = 3$. Simple computation shows that this is not possible. Moreover, one may show that the graphs of f_0 and f_2 intersect at the points $(2/5, 2/3)$ and the graphs of f_1 and f_3 intersect at the points $(2/3, 2/5)$. Hence $C_2 = \{2/5, 2/3\}$ is a 2−periodic cycle. Moreover, the equation has the 4−periodic cycle $\left\{\frac{238}{361}, \frac{119}{298}, \frac{238}{417}, \frac{238}{607}\right\}$.

5 Periodic difference equations II: Stability

Suppose that the p-periodic difference equation

$$x_{n+1} = f_n(x_n), f_{n+p} = f_n, n \in \mathbb{Z}^+ \tag{10}$$

has a periodic cycle of minimal period r. Then the associated skew-product system π has a periodic cycle of period $s = lcm[r,p]$ ($s-$complete cycle). There are p fibers $\mathcal{F}_i = \mathcal{P}^{-1}(f_i)$. Are the s periodic points equally distributed on the fibers? i.e. is the number of periodic points on each fiber equal to $\ell = s/p$?

Before giving the definitive answer to this question, let us examine the diagram present in Figure 2 in which $p = 9$, and $r = 6$.

There are two points $\left(2 = \frac{lcm[6,9]}{9}\right)$ on each fiber. Since $d = \gcd(6,9) = 3$, f_0, f_3, f_6 have the same points; f_1, f_4, f_7 have the same points; and f_2, f_5, f_8 have the same points.

Note that the number of periodic points on each fiber is 2, which is $\ell = \frac{lcm[r,p]}{p}$. The following crucial lemma proves this observation.

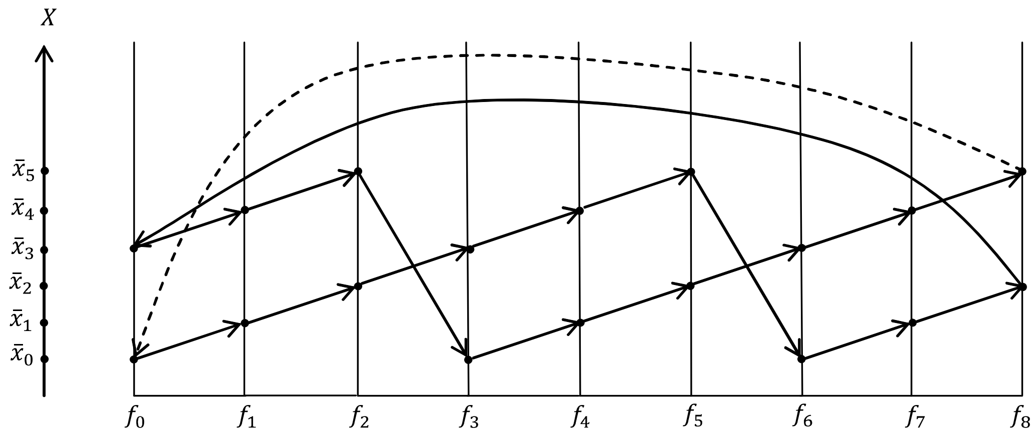

Figure 2: A 6−periodic cycle in a 9−periodic system

Lemma 8 *[12] Let $s = lcm[r, p]$. Then the orbit of $(\overline{x}_i, f_i)$ in the skew-product system intersect each fiber $\mathcal{F}_j$, $j = 0, 1, \ldots, p-1$, in exactly $\ell = s/p$ points and each of these points is periodic under the skew-product π with period s.*

Proof. Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be a periodic cycle of minimal period r. Then the orbit of $(\overline{x}_0, f_0)$ in the skew-product has a minimal period $s = lcm[r, p]$. Now $\mathcal{S} = \mathcal{O}((\overline{x}_0, f_0)) = \{\pi((\overline{x}_0, f_0), n) : n \in \mathbb{Z}^+\} \subset X \times Y$ is minimal, invariant under π and has s distinct points.

For each $i, 0 \leq i \leq p-1$, the maps

$$f_i : \mathcal{S} \cap \mathcal{F}_i \to \mathcal{S} \cap \mathcal{F}_{(i+1)\ mod\ p} \tag{11}$$

are surjective. We now show that it is injective.

Let N_i be cardinality of $\mathcal{S} \cap \mathcal{F}_i$. Then N_i is a non-increasing integer valued function and thus stabilizes at some fixed value from which it follows that N_i is constant. Thus each $\mathcal{S} \cap \mathcal{F}_i$ contains the same number of points, namely s/p. ■

Next we turn our attention to our main topic, namely, stability. We begin by stating the basic definitions of stability.

Definition 9 *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle in the $p-$periodic equation (10) in a metric space (X, ρ) and $s = lcm[r, p]$ be the least common multiple of p and r. Then*

1. *C_r is stable if given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\rho(z, \overline{x}_{i\ mod\ r}) < \delta \ \textit{implies} \ \rho(\Phi_n^i(z), \Phi_n^i(\overline{x}_{i\ mod\ r})) < \epsilon$$

for all $n \in \mathbb{Z}^+$*, and* $0 \leq i \leq p-1$*. Otherwise,* C_r *is said unstable.*

2. C_r *is attracting if there exists* $\eta > 0$ *such that*

$$\rho(z, \overline{x}_{i \ mod \ r}) < \eta \text{ implies } \lim_{n\to\infty} \Phi^i_{ns}(z) = \overline{x}_{i \ mod \ r}.$$

3. *We say that* C_r *is asymptotically stable if it is both stable and attracting. If in addition,* $\eta = \infty$*,* C_r *is said to be globally asymptotically stable.*

Lemma 10 *An* $r-$*periodic cycle* $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ *in equation (10) is*

1. *asymptotically stable if* $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| < 1$,
2. *unstable if* $|\prod_{i=0}^{s} f'_{i \ mod \ p}(\overline{x}_{i \ mod \ r})| > 1$,

where $s = lcm[r, p]$ *is the least common multiple of* p *and* r*.*

Consider the skew-product system π on $X \times Y$ with X a metric space with metric ρ, $Y = \{f_0, f_1, \ldots, f_{p-1}\}$ equipped with the discrete metric $\widetilde{\rho}$, where

$$\widetilde{\rho}(f_i, f_j) = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}.$$

Define a metric D on $X \times Y$ as

$$D((x, f_i), (y, f_j)) = \rho(x, y) + \widetilde{\rho}(f_i, f_j).$$

Let $\pi^1(x, f) = \pi((x, f), 1)$, then $\pi^n(x, f) = \pi((x, f), n)$. Thus $\pi^1 : X \times Y \to X \times Y$ is a continuous map which generates an autonomous system on $X \times Y$. Consequently, the stability definitions of fixed points and periodic cycles follow the standard ones that may be found in [10, 7].

Now we give a definition of stability for a complete periodic cycle in the skew-product system.

Definition 11 *A complete periodic cycle* $\widetilde{C}_s = \{(\overline{x}_0, f_0), ..., (\overline{x}_{s \ mod \ r}, f_{s \ mod \ p})\}$ *is*

1. *stable if given* $\epsilon > 0$*, there exits* $\delta > 0$*, such that*

$$D((z, f_i), (\overline{x}_0, f_0)) < \delta \text{ implies } D(\pi^n(z, f_i), \pi^n(\overline{x}_0, f_0)) < \epsilon, \ \forall n \in \mathbb{Z}^+.$$

   Otherwise, $\widetilde{C}_s$ *is said unstable.*

2. *attracting if there exists* $\eta > 0$ *such that*

$$D((z, f_i), (\overline{x}_0, f_0)) < \eta \textit{ implies } \lim_{n\to\infty} \pi^{ns}(z, f_i) = (\overline{x}_0, f_0).$$

3. *asymptotically stable if it is both stable and attracting. If in addition,* $\eta = \infty$, $\widetilde{C}_s$ *is said to be globally asymptotically stable.*

Since $f_{i+ns} = f_i$ for all n, it follows from the above convergence that $f_i = f_0$. Hence, stability can occur only on each fiber $X \times \{f_i\}, 0 \le i \le p-1$.

We are now ready to state our main result in this survey.

Theorem 12 *[12] Assume that* X *is a connected metric space and each* $f_i \in Y$ *is a continuous map on* X*, with* $f_{i+p} = f_i$*. Let* $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ *be a periodic cycle of minimal period* r*. If* C_r *is globally asymptotically stable, then* r *divides* p*. Moreover,* $r = p$ *if the sequence* $\{f_n\}$ *is a one-parameter family of maps* $F(\mu_n, x)$ *and* F *is one to one with respect to* μ*.*

Proof. The skew-product system π on $X \times Y$ has the periodic orbit

$$\{(\overline{x}_0, f_0), (\overline{x}_1, f_1), \ldots, (\overline{x}_{s \; mod \; r}, f_{smodr})\}$$

which is globally asymptotically stable. But as we remarked earlier, globally stability can occur only on fibers. By Lemma 8, there are $\ell = s/p$ points on each fiber. If $\ell > 1$, we have a globally asymptotically ℓ-periodic cycle in the connected metric space $X \times \{f_i\}$ under the map π^p. This violates Elaydi-Yakubu Theorem [11]. Hence $\ell = 1$ and consequently $r|p$.

Note that by Lemma 6, the graphs of the maps f_i, f_{i+d}, ,$f_{i+(m-1)d}$, $0 \le i \le p-1$, must intersect at ℓ points. However, since $\{f_i\}$ is a one parameter family of maps $F(\mu_n, x)$ where F is one to one with respect to the parameter μ, it follows that $f_i = f_{i+d}$, $0 \le i \le p-1$. This implies that d is the period of our system and since p is the minimal period of the system, this implies that $d = p$. Hence $r = p$. ∎

6 An extension of Singer's Theorem

One of the well known work done by Singer is present in his famous paper [27] and currently known by Singer's theorem. It is a useful tool in finding an upper bound for the number of stable cycles in autonomous difference equations. In this section we present the natural extension of this theorem to the periodic nonautonomous difference equations.

Recall that the Schwarzian derivative, Sf, of a map f at x is defined as

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Let $f : I \to I$ be a C^3 map defined on a closed interval $I = [x_1, x_2]$ such that $Sf(x) < 0, \forall x \in I \setminus \{P_c\}$, where $\{P_c\}$ is the set of the critical points of f. Suppose that f has m critical points in I. Then according to the Singer's theorem [27] we have the following:

1. if $f(x_1) = f(x_2) = x_1$, then f has at most m attracting periodic cycles of any given period,
2. if $f(x_1) > x_1$ and $f(x_2) = x_1$ or $f(x_1) = x_1$ and $f(x_2) > x_1$ or yet $f(x_1) = f(x_2) > x_1$, then we have at most $m+1$ attracting periodic cycle of any given period (we include the orbit of x_1 or x_2 according to the case).
3. if $f(x_1) \neq f(x_2)$ with $f(x_1), f(x_2) > x_1$, then we will have at most $m+2$ attracting periodic cycles of any given (we include the orbits of x_1 and x_2).

In the settings of the autonomous difference equation this means that $x_{n+1} = f(x_n)$ has at most $m+2$ attracting periodic cycles of any given period.

******** YOU WROTE **********

Let $f : I \to I$ be a continuous map with a negative Schwarzian derivative for all $x \in I$, defined on the closed interval I. If f has m critical points in I, then f has at most $m+2$ attracting period cycles of any given period.

******** YOU WROTE *********

Now consider the $p-$periodic system $\mathcal{F} = \{f_0, f_1, f_2, ..., f_{p-1}\}$ of continuous maps defined on a closed interval I.

Assume that there are m_i critical points for the map f_i, $0 \leq i \leq p-1$. On the fiber $\mathcal{F}_0 = I \times f_0$, there are m_0 critical points of f_0, at least m_1 critical points consisting of all the pre-images under f_0 of the m_1 critical points of f_1,... and at least m_{p-1} critical points that consist of all the pre images, under Φ_{p-2}, of the m_{p-1} critical points of f_{p-1}. Since each critical point of Φ_p is mapped, under compositions of our maps, to one of the original critical points of one of the maps f_i, it follows that the number of significant critical points is $\sum_{i=0}^{p-1} m_i$.

By Singer's Theorem, there are at most $\left[\sum_{i=0}^{P-1} m_i + 2\right]$ attracting periodic cycles of any given period. Notice that periodic cycles that appear on fiber $\mathcal{F}_i$ are just phase shifts of periodic cycles that appear on fiber $\mathcal{F}_0$. Hence we conclude that there are at most $\sum_{i=0}^{P-1} m_i + 2$ attracting cycles of any given period (See [2] for details).

So a consequence of this extension, one may show that if the maps are the logistic maps

$$f_i(x) = \mu_i x(1-x), \mu_i > 0,\ 0 \leq i \leq p-1,$$

defined on the interval $[0, 1]$, then the p-periodic system $\{f_0, f_1, ..., f_{p-1}\}$ has at most $p-$attracting cycles of any given period r. Notice that each map f_i has one critical point, $x = 1/2$, and the boundary points 0 and 1 are attracted only to 0.

7 Periodic difference equation III: Bifurcation

The study of various notions of bifurcation in the setting of discrete nonautonomous systems is still in its infancy stage. The main contribution in this area are the papers by Henson [23], Al-Sharawi and Angelos [2], Oliveira and D'Aniello [20], and recently Luís, Elaydi and Oliveira [19].

The main objective in this section is to give the definitions, notions, terminology and results done by Luís et al. in [19] for the various notions of bifurcation in the setting of discrete nonautonomous systems. Though our focus here will be on $2-$periodic systems, the ideas presented can be easily extended to the general periodic case.

In all this section we assume that the maps f and g arise from a one-parameter family of maps such that $f = f_\alpha$ and $g = f_\beta$ with $\beta = q\alpha$ for some real number $q > 0$. Thus one may write, without loss of generality, our system as $\mathcal{F} = \{f, g\}$.

Moreover, we assume that the one-parameter family of maps is one-to-one with respect to the parameter. Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of $\mathcal{F}$. Then the latter assumption implies by corollary 7 that $r = 2m$, $m \geq 1$.

With $\Psi = f \circ g$, one may write the orbit of $\overline{x}_0$ as (see figure 3)

$$\mathcal{O}(\overline{x}_0) = \left\{\overline{x}_0, g(\overline{x}_0), \Psi(\overline{x}_0), g \circ \Psi(\overline{x}_0), \Psi^2(\overline{x}_0), ..., \Psi^{m-1}(\overline{x}_0), g \circ \Psi^{m-1}(\overline{x}_0)\right\} \tag{12}$$

or equivalently

$$\mathcal{O}(\overline{x}_1) = \left\{f \circ \widetilde{\Psi}^{m-1}(\overline{x}_1), \overline{x}_1, f(\overline{x}_1), \widetilde{\Psi}(\overline{x}_1), f \circ \widetilde{\Psi}(\overline{x}_1), ..., f \circ \widetilde{\Psi}^{m-2}(\overline{x}_1), \widetilde{\Psi}^{m-1}(\overline{x}_1)\right\} \tag{13}$$

where $\widetilde{\Psi} = g \circ f$. Hence the order of the composition is irrelevant to the dynamics of the system.

The dynamics of $\mathcal{F}$ depends very much on the parameter and the qualitative structure of the dynamical system changes as the parameter changes. These qualitative changes in the dynamics of the system are called bifurcation and the parameter values at which they occur are called bifurcation points. For autonomous systems or single maps the bifurcation analysis may be found in Elaydi [10].

In a one-dimensional systems generated by a one-parameter family of maps f_α, bifurcation at a fixed point x^* occurs when $\frac{\partial f}{\partial x}(\alpha^*, x^*) = 1$ or -1 at a bifurcation point α^*. The former case leads to a saddle-node bifurcation, while the latter case leads to a period-doubling bifurcation.

Now we are going to extend this analysis to $2-$periodic difference equations or $\mathcal{F} = \{f, g\}$. To simplify the notation we write $\Psi(\alpha, x)$ instead of $\Psi(x)$ and $\widetilde{\Psi}(\alpha, x)$ instead of $\widetilde{\Psi}$. Then $\Psi^m(\overline{x}_{2i}) = \overline{x}_{(2i)modr}$ and $\widetilde{\Psi}^m(\overline{x}_{2i+1}) = \overline{x}_{(2i+1)modr}$, $1 \leq i \leq m$. In general, we have $\Psi^{nm}(\overline{x}_{2i}) = \overline{x}_{(2i)modr}$ and $\widetilde{\Psi}^{nm}(\overline{x}_{2i+1}) = \overline{x}_{(2i+1)modr}$, $n \geq 1$.

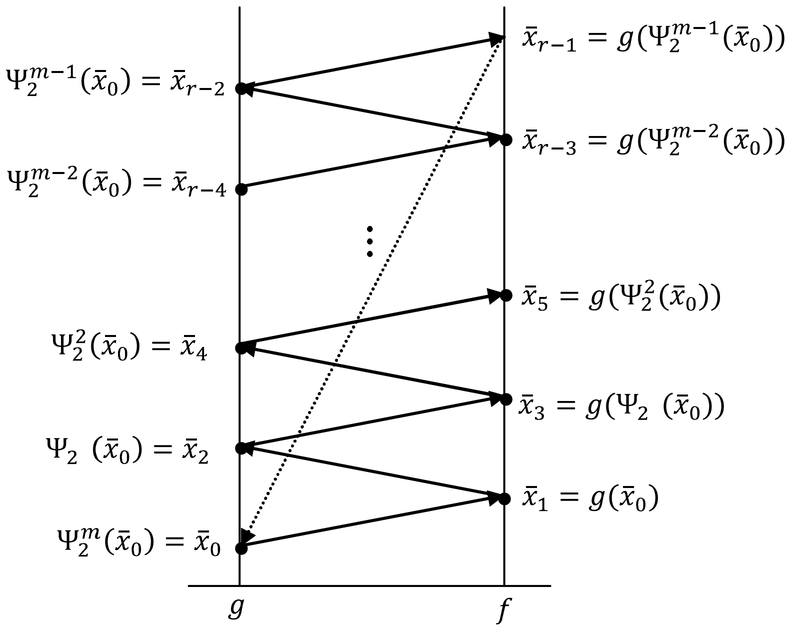

Figure 3: Sequence of the periodic points $\{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ in the 2−periodic system $\mathcal{F} = \{f, g\}$ illustrate in the fibers g and f, where $\Psi = f \circ g$ and $r = 2m, m \geq 1$.

Assuming $\frac{\partial \Psi^m}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ at a bifurcation point $\overline{\alpha}$, by the chain rule, we have

$$\frac{\partial \Psi}{\partial x}(\overline{\alpha}, \overline{x}_{2m-2}) \frac{\partial \Psi}{\partial x}(\overline{\alpha}, \overline{x}_{2m-4}) ... \frac{\partial \Psi}{\partial x}(\overline{\alpha}, \overline{x}_2) \frac{\partial \Psi}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$$

or

$$f'_{\overline{\alpha}}(\overline{x}_{2m-1}) g'_{\overline{\alpha}}(\overline{x}_{2m-2}) f'_{\overline{\alpha}}(\overline{x}_{2m-3}) g'_{\overline{\alpha}}(\overline{x}_{2m-4}) ... f'_{\overline{\alpha}}(\overline{x}_3) g'_{\overline{\alpha}}(\overline{x}_2) f'_{\overline{\alpha}}(\overline{x}_1) g'_{\overline{\alpha}}(\overline{x}_0) = 1 \qquad (14)$$

Applying $g_{\overline{\alpha}}$ on both sides of the identity $\Psi^m(\overline{\alpha}, \overline{x}_0) = \overline{x}_0$, yields $\widetilde{\Psi}^m(\overline{\alpha}, \overline{x}_1) = \overline{x}_1$. Differentiating both sides of this equation we get

$$\frac{\partial \widetilde{\Psi}}{\partial x}(\overline{\alpha}, \overline{x}_{2m-1}) \frac{\partial \widetilde{\Psi}}{\partial x}(\overline{\alpha}, \overline{x}_{2m-3}) ... \frac{\partial \widetilde{\Psi}}{\partial x}(\overline{\alpha}, \overline{x}_3) \frac{\partial \widetilde{\Psi}}{\partial x}(\overline{\alpha}, \overline{x}_1) = 1$$

or equivalently

$$g'_{\overline{\alpha}}(\overline{x}_0) f'_{\overline{\alpha}}(\overline{x}_{2m-1}) g'_{\overline{\alpha}}(\overline{x}_{2m-2}) f'_{\overline{\alpha}}(\overline{x}_{2m-3}) ... g'_{\overline{\alpha}}(\overline{x}_4) f'_{\overline{\alpha}}(\overline{x}_3) g'_{\overline{\alpha}}(\overline{x}_2) f'_{\overline{\alpha}}(\overline{x}_1) = 1. \qquad (15)$$

Hence Eq. (14) is equivalent to Eq. (15). More generally the following relation yields

$$\frac{\partial \Psi^m}{\partial x}(\overline{\alpha}, \overline{x}_{2j}) = \frac{\partial \widetilde{\Psi}^m}{\partial x}(\overline{\alpha}, \overline{x}_{2j-1}), j \in \{1, 2, ..., m\}. \qquad (16)$$

Now we are ready to write the two main results of this section.

Theorem 13 (Saddle-node Bifurcation for 2−periodic systems [19]) *Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a periodic $r-$cycle of $\mathcal{F}$. Suppose that both $\frac{\partial^2 \Psi}{\partial x^2}$ and $\frac{\partial^2 \Psi}{\partial^2}$ exist and are continuous in a neighborhood of a periodic orbit such that $\frac{\partial \Psi^m}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ for the periodic point $\overline{x}_0$. Assume also that*

$$A = \frac{\partial \Psi^m}{\partial \alpha}(\overline{\alpha}, \overline{x}_0) \neq 0 \text{ and } B = \frac{\partial^2 \Psi^m}{\partial x^2}(\overline{\alpha}, \overline{x}_0) \neq 0.$$

Then there exists an interval J around the periodic orbit and a C^2-map $\alpha = h(x)$, where $h : J \to \mathbb{R}$ such that $h(\overline{x}_0) = \overline{\alpha}$, and $\Psi^m(x, h(x)) = x$. Moreover, if $AB < 0$, the periodic points exists for $\alpha > \overline{\alpha}$, and, if $AB > 0$, the periodic points exists for $\alpha < \overline{\alpha}$.

When $\frac{\partial \Psi^m}{\partial x}(\overline{\alpha}, \overline{x}_0) = 1$ but $\frac{\partial \Phi^m}{\partial \alpha}(\overline{\alpha}, \overline{x}) = 0$, two types of bifurcation appear. The first is called transcritical bifurcation which appears when $\frac{\partial^2 \Psi^m}{\partial x^2}(\overline{\alpha}, \overline{x}_0) \neq 0$ and the second called pitchfork bifurcation which appears when $\frac{\partial^2 \Psi^m}{\partial x^2}(\overline{\alpha}, \overline{x}_0) = 0$. For more details about this two types of bifurcation see table 2.1 in [10, pp. 90] and [20]. In the former work the author present many cases for autonomous maps while in the latter article the authors study the pitchfork bifurcation for nonautonomous 2−periodic systems in which the maps have negative Schwarzian derivative.

The next result gives the conditions for the period-doubling bifurcation.

Theorem 14 (Period-Doubling Bifurcation for 2−periodic systems [19]) *Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a periodic $r-$cycle of $\mathcal{F}$. Assume that both $\frac{\partial^2 \Psi}{\partial x^2}$ and $\frac{\partial \Psi}{\partial \alpha}$ exist and are continuous in a neighborhood of a periodic orbit, $\frac{\partial \Psi^m}{\partial x}(\overline{\alpha}, \overline{x}_0) = -1$ for the periodic point $\overline{x}_0$ and $\frac{\partial^2 \Psi^{2m}}{\partial \alpha \partial x}(\overline{\alpha}, \overline{x}_0) \neq 0$. Then, there exists an interval J around the periodic orbit and a function $h : J \to \mathbb{R}$ such that $\Psi^m(x, h(x)) \neq x$ but $\Psi^{2m}(x, h(x)) = x$.*

Now we are going to apply these two results with an interesting example from [19]. First we need the following definition.

Definition 15 *A unimodal map is said to have the Allee[1] effect if it has three fixed points $x_1^* = 0$, $x_2^* = A$, and $x_3^* = K$, with $0 < A < K$, in which x_1^* is asymptotically stable, x_2^* is unstable, and x_3^* may be stable or unstable.*

[1]The Allee effect is a phenomenon in population dynamics attributed to the American biologist Warder Clayde Allee 1885-1955 [1]. Allee proposed that the per capita birth rate declines at low density or population sizes. In the languages of dynamical systems or difference equations, a map representing the Allee effect must have tree fixed points, an asymptotically stable zero fixed point, a small unstable fixed point, called the threshold point, and a bigger positive fixed point, called the carrying capacity, that is asymptotically stable at least for smaller values of the parameters.

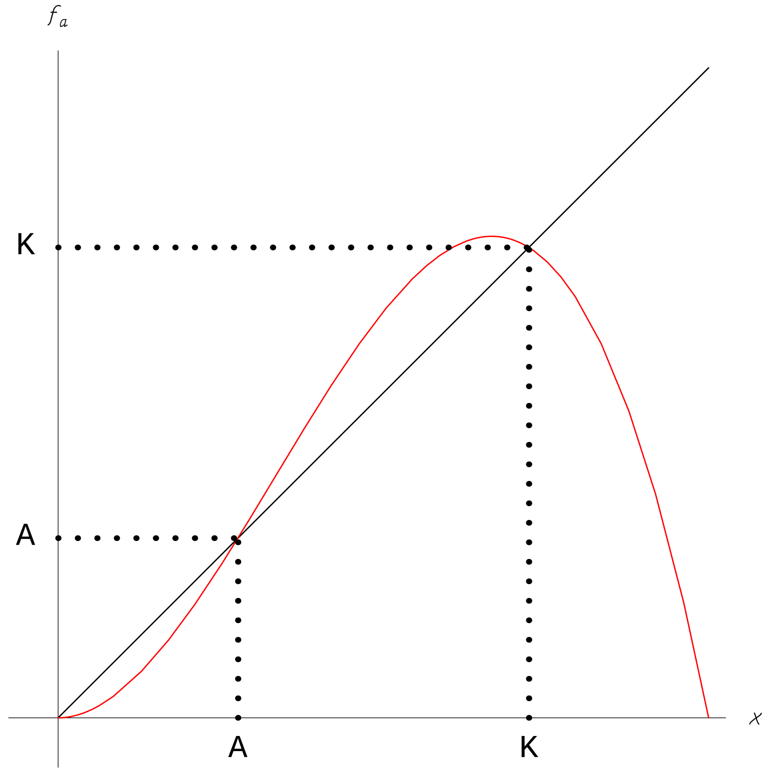

Figure 4: A unimodal Allee map with three fixed points 0, A and k.

Remark 16 *Note that if $\mathcal{F}$ is a periodic set formed by unimodal Allee maps, neither the zero fixed point nor the threshold point can contribute to bifurcation, since the former is always asymptotically stable and the latter is always unstable. Hence bifurcation may only occur at the carrying capacity of $\mathcal{F}$.*

Example 17 *[19] Consider the 2−periodic system $\mathcal{W} = \{f_0, f_1\}$, where*

$$f_i(x) = a_i x^2(1 - x),\ i = 0, 1$$

in which $x \in [0, 1]$ and $a_i > 0$, $i = 0, 1$. For an individual map f_i, if $a_i < 4$ we have a globally asymptotically stable zero fixed point and no other fixed point. At $a_i = 4$ an unstable fixed point is born after which f_i becomes a unimodal map with an Allee effect (see Figure 4). Henceforth, we will assume that $a_0, a_1 > 4$.

Since 0 is the only fixed point under the system $\mathcal{W}$, we focus our attention on 2−periodic cycles $\{\overline{x}_0, \overline{x}_1\}$ with $f_0(\overline{x}_0) = \overline{x}_1$, and $f_1(\overline{x}_1) = \overline{x}_0$.

A Saddle-node bifurcation occurs when $\Psi'(\overline{x}_0) = 1$, and a period-doubling bifurcation occurs when $\Psi'(\overline{x}_0) = -1$, where $\Psi = f_1 \circ f_0$.

For the saddle-node bifurcation we then solve the equations

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(F_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = 1 \end{cases} \tag{17}$$

and for the period-doubling bifurcations we solve the equations

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(f_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = -1 \end{cases} \tag{18}$$

Using the command "resultant" in Mathematica or Maple Software, we eliminate the variable $\overline{x}_0$ in equations (17) and (18). Eq. (17) yields

$$16777216 + 16384a_0a_1 - 576000a_0^2a_1 + 84375a_0^3a_1 - 576000a_0a_1^2 + 914a_0^2a_1^2 - 350a_0^3a_1^2 + 84375a_0a_1^3 - 350a_0^2a_1^3 + 19827a_0^3a_1^3 - 2916a_0^4a_1^3 - 2916a_0^3a_1^4 + 432a_0^4a_1^4 = 0$$

while Eq. (18) yields

$$100000000 - 120000a_0a_1 - 2998800a_0^2a_1 + 453789a_0^3a_1 - 2998800a_0a_1^2 - 4598a_0^2a_1^2 + 2702a_0^3a_1^2 + 453789a_0a_1^3 + 2702a_0^2a_1^3 + 89765a_0^3a_1^3 - 13500a_0^4a_1^3 - 13500a_0^3a_1^4 + 2000a_0^4a_1^4 = 0$$

For each one of these last two equations we invoke the implicit function theorem to plot, in the $(a_0, a_1)-$plane, the bifurcation curves (see figure 5). The black curves are the solution of the former equation at which saddle-node bifurcation occurs, while the gray curves are the solution of the latter equation at which period-doubling bifurcations occurs. The black cusp is the curve of pitchfork bifurcation. In the regions identified by letters one can conclude the following.

- *If $a_0, a_1 \in A$ then the fixed point $x^* = 0$ is globally asymptotically stable.*
- *If $a_0, a_1 \in B \backslash D$ then there are two $2-$periodic cycles, one attracting and one unstable.*
- *If $a_0, a_1 \in D$ then there are two attracting $2-$periodic cycles (from the pitchfork bifurcation) and two unstable $2-$periodic cycles.*
- *If $a_0, a_1 \in (C_1 \cup C_2) \backslash (D_1 \cup D_2)$ then there is an attracting $4-$periodic cycle (from the period doubling bifurcation) and two unstable $2-$periodic cycles.*
- *If $a_0, a_1 \in D_1 \cup D_2$ then there are two attracting $4-$periodic cycles (from pitchfork bifurcation) and two unstable $2-$periodic cycles.*
- *If $a_0, a_1 \in E$ then there are two attracting $8-$periodic cycles (from period doubling bifurcation), two attracting $4-$periodic cycles (from pitchfork bifurcation), and four unstable $2-$periodic cycles.*

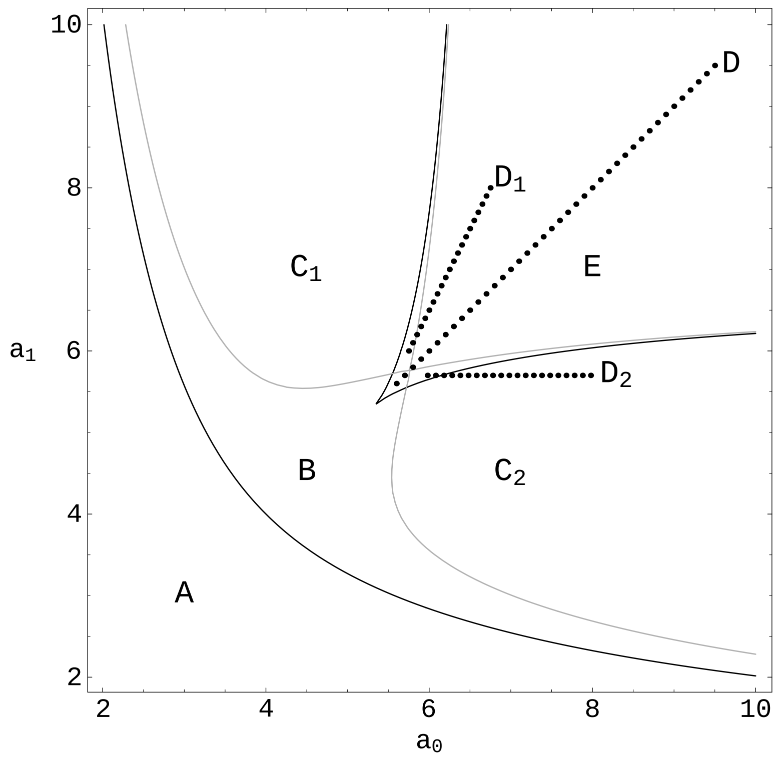

Figure 5: Bifurcations curves for the 2−periodic nonautonomous difference equation with Allee effects $x_{n+1} = a_n x_n^2(1 - x_n)$, in the $(a_0, a_1)-$plane, where $a_{n+2} = a_n$ and $x_{n+2} = x_n$.

Finally, we end this section presenting the following example from [20] where the system is formed by quadratic maps and cubic maps.

Example 18 *HENRIQUe -¿ Your example with EMMa:*** quadratic and cubic ****

8 Attenuance and resonance

8.1 The Beverton-Holt equation

In [6] Cushing and Henson conjectured that a nonautonomous $p-$periodic Beverton-Holt equation with periodically varying carrying capacity must be attenuant. This means that if $C_p = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{p-1}\}$ is its $p-$periodic cycle, and K_i, $0 \leq i \leq p-1$

are the carrying capacities, then

$$\frac{1}{p}\sum_{i=0}^{p-1}\overline{x}_i < \frac{1}{p}\sum_{i=0}^{p-1}K_i. \tag{19}$$

Since the periodic cycle C_p is globally asymptotically stable on $(0,\infty)$, it follows that for any initial population density x_0, the time average of the population density x_n is eventually less than the average of the carrying capacities, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_i < \frac{1}{p}\sum_{i=0}^{p-1}K_i. \tag{20}$$

Eq. (20) gives a justification for the use of the word "attenuance" to describe the phenomenon in which a periodically fluctuation carrying capacity of the Beverton-Holt equation has a deleterious effect on the population. This conjecture was first proved by Elaydi and Sacker in [13, 9, 12] and independently by Kocic [28] and Kon [29]. The following theorem summarizes our findings.

Theorem 19 *[13] Consider the $p-$periodic Beverton-Holt equation*

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu-1)x_n}, n \in \mathbb{Z}^+, \tag{21}$$

where $\mu > 1$, $K_{n+p} = K_n$, and $K_n > 0$. Then Eq. (21) has a globally asymptotically stable $p-$periodic cycle. Moreover, Eq. (21) is attenuant.

Kocic [28], however gave the most elegant proof for the presence of attenuance. Utilizing effectively the Jensen's inequality, he was able to give the following more general result.

Theorem 20 *[28] Assume that $\mu > 1$ and $\{K_n\}$ is a bounded sequence of positive numbers*

$$0 < \alpha < K_n < \beta < \infty.$$

Then for every positive solution $\{x_n\}$ of Eq. (21) we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\overline{x}_i \le \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}K_i. \tag{22}$$

8.2 Neither attenuance nor resonance

By a simple trick, Sacker [25] showed that neither attenuance nor resonance occurs when periodically forcing the Ricker maps

$$R(x) = xe^{p-x}.$$

So consider the $k-$periodic system

$$x_{n+1} = x_n e^{p_n - x_n}, p_{n+k} = p_n, n \in \mathbb{Z}^+. \tag{23}$$

If $0 < p_n < 2$, Eq. (23) has a globally asymptotically stable $k-$periodic cycle [25].

Let $C_k = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{k-1}\}$ be this unique $k-$periodic cycle. Then

$$\begin{aligned} \overline{x}_0 &= \overline{x}_k = \overline{x}_{k-1} e^{p_{k-1} - \overline{x}_{k-1}} \\ &= \overline{x}_{k-2} e^{p_{k-2} - \overline{x}_{k-2}} e^{p_{k-1} - \overline{x}_{k-1}}, \end{aligned}$$

and by iteration we get

$$\overline{x}_0 = \overline{x}_0 e^{\sum_{i=0}^{k-1} p_i - \sum_{i=0}^{k-1} \overline{x}_i}.$$

Hence

$$\tfrac{1}{k} \sum_{i=0}^{k-1} p_i = \tfrac{1}{k} \sum_{i=0}^{k-1} \overline{x}_i,$$

i.e., neither attenuance nor resonance.

8.3 An extension: monotone maps

Using an extension to monotone maps, Kon [29] considered a $p-$periodic difference equation of the form

$$x_{n+1} = g\left(x_n / K_n\right) x_n, n \in \mathbb{Z}^+, \tag{24}$$

where $K_{n+p} = K_n$, $K_n > 0$, $x_0 \in [0, \infty)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which satisfies the following properties

- $g(1) = 1$,
- $g(x) > 1$ for all $x \in (0, 1)$,
- $g(x) < 1$ for all $x \in (1, \infty)$.

Theorem 21 *[29] Let $C_r = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{r-1}\}$ be a positive $r-$periodic cycle of Eq. (24) such that $K_i \neq K_{i+1}$ for some $0 \leq i \leq p-1$. Assume that $zg(z)$ is strictly concave on an interval (a, b), $0 < a < b$ containing all points $\frac{\overline{x}_i}{K_i} \in (a, b)$, $1 \leq i \leq rp$. Then the cycle C_r is attenuant.*

This theorem provides an alternative proof of the attenuance of the periodic Bevertob-Holt equation (21).

Consider the equation [29]

$$x_{n+1} = \left(\frac{x_n}{K_n}\right)^{a-1}, 0 < a < 1, \tag{25}$$

where $K_{n+p} = K_n$, $n \in \mathbb{Z}^+$, and $K_i \neq K_{i+1}$ for some $i \in \mathbb{Z}^+$. The maps belong to the class $\mathcal{K}$ and satisfy the assumption of the preceding theorem. Consequently, Eq. (25) has a globally asymptotically stable $p-$periodic cycle that is attenuant.

8.4 The loss of attenuance: resonance.

Consider the periodic Beverton-Holt equation (21) in which both parameters μ_n and K_n are periodic of common period p. This equation may be attenuant or resonant. In fact, when $p = 2$, Elaydi and Sacker [13] showed that

$$\overline{x} = \overline{K} + \sigma\frac{K_0 - K_1}{2} - \Delta\frac{(\mu_0 - 1)(\mu_1 - 1)}{2(\mu_0\mu_1 - 1)}(K_0 - K_1)^2 \tag{26}$$

where

$$\overline{x} = \frac{\overline{x}_0 + \overline{x}_1}{2} \text{ and } \overline{K} = \frac{K_0 + K_1}{2},$$

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0\mu_1 - 1}, 0 \leq |\sigma| < 1,$$

and

$$\Delta = \frac{\mu_0(\mu_1^2 - 1)K_0 + \mu_1(\mu_0^2 - 1)K_1}{\mu_0(\mu_1 - 1)^2K_0^2 + (\mu_0 - 1)(\mu_1 - 1)(\mu_0\mu_1 + 1)K_0K_1 + \mu_1(\mu_0 - 1)^2K_1^2} > 0.$$

It follows that attenuance is present if either $(\mu_1 - \mu_0)(K_0 - K_1) < 0$ (out of phase) or the algebraic sum of the last two terms in Eq. (26) is negative. On the other hand, resonance is present if the algebraic sum of the last two terms in Eq. (26) is positive.

Notice that if $\mu_0 = \mu_1 = \mu$ with $p = 2$, then we have

$$\frac{1}{p}\sum_{i=0}^{p-1}\overline{x}_i = \frac{1}{p}\sum_{i=0}^{p-1}K_i - \frac{\mu(K_0 + K_1)(K_1 - K_0)^2}{2\left[\mu K_0^2 + (\mu^2 + 1)K_0K_1 + \mu K_1^2\right]},$$

which gives an exact expression for the difference in the averages.

Remark 22 *Now for $\mu_0 = 4$, $\mu_1 = 2$, $K_0 = 11$, and $K_1 = 7$, we have resonance as $\frac{1}{2}\sum_{i=0}^{1}\overline{x}_i \approx 9.23$ and $\frac{1}{2}\sum_{i=0}^{1}K_i = 9$. On the other hand, one can show that for $\mu_0 = 2$, $\mu_1 = 4$, $K_0 = 11$, and $K_1 = 7$, we have attenuance as may be seen from (26).*

8.5 The signature functions of Franke and Yakubu

In [18], the authors gave a criteria to determine attenuance or resonance for the 2−periodic difference equation

$$x_{n+1} = x_n g(K_n, \mu_n, x_n), n \in \mathbb{Z}^+, \tag{27}$$

where $K_n = K(1+\alpha(-1)^n)$, $\mu_n = \mu(1+\beta(-1)^n)$, and $\alpha, \beta \in (-1, 1)$.

Define the following

$$\omega_1 = \frac{\left(k\frac{\partial^2 g}{\partial x^2} + 2\frac{\partial g}{\partial x}\right)\left(\frac{K^2\frac{\partial g}{\partial K}}{2+K\frac{\partial g}{\partial x}}\right)^2 + \left(2K\frac{\partial g}{\partial K} + 2K^2\frac{\partial^2 g}{\partial x \partial K}\right) + K^3\frac{\partial^2 g}{\partial K^2}}{-2K\frac{\partial g}{\partial x}}, \tag{28}$$

$$\omega_2 = \frac{-\left(\mu\frac{\partial g}{\partial \mu} + K\mu\frac{\partial^2 g}{\partial x \partial \mu}\right)\left(\frac{-K^2\frac{\partial g}{\partial K}}{2+K\frac{\partial g}{\partial x}}\right) + K^2\mu\frac{\partial^2 g}{\partial K \partial \mu}}{K\frac{\partial g}{\partial x}}, \tag{29}$$

and

$$\mathcal{R}_d = sign(\alpha(\omega_1\alpha + \omega_2\beta)). \tag{30}$$

Theorem 23 *[18] If for $\alpha = 0, \beta = 0$, K is hyperbolic fixed point of equation (27), then for all sufficiently small $|\alpha|$ and $|\beta|$, the equation (27), with $\alpha, \beta \in (-1, 1)$, has an attenuant 2−periodic cycle if $\mathcal{R}_d < 0$ and a resonant 2−periodic cycle if $\mathcal{R}_d > 0$.*

To illustrate the effectiveness of this theorem, let us to consider the logistic equation

$$x_{n+1} = x_n\left[1 + \mu(1+\beta(-1)^n)\left(1 - \frac{x_n}{K(1+\alpha(-1)^n)}\right)\right]. \tag{31}$$

For $0 < \mu < 2$ Eq. (31) has an asymptotically stable 2−periodic cycle. Using formulas (28) and (29), one obtains

$$\omega_1 = \frac{-8K}{(\mu-2)^2} \text{ and } \omega_2 = \frac{-4K}{\mu-2}.$$

Assume that $\alpha > 0$ and $0 < \mu < 2$. Using (30) yields

$$\mathcal{R}_d = sign\left(\frac{2}{\mu-2}\alpha + \beta\right) = sign\left(\beta - \frac{2}{2-\mu}\alpha\right).$$

Hence we have attenuance if $\beta < \frac{2}{2-\mu}\alpha$, i.e., if the relative strength of the fluctuation of the demographic characteristic of the species is weaker than $\frac{2}{2-\mu}$ times the relative strength of the fluctuation of the carrying capacity. On the other hand if $\beta > \frac{2}{2-\mu}\alpha$ we obtain resonance.

Notice that if $\alpha = 0$ (the carrying capacity is fixed), then we have resonance if $\beta > 0$ and we have attenuance if $\beta < 0$. For the case that $\beta = 0$ (the intrinsic growth rate is fixed), we have attenuance.

Finally, we note that Franke and Yakubu extended their study to periodically forced Leslie model with density-dependent fecundity functions [17]. The model is of the form

$$\begin{aligned} x_{n+1}^1 &= \sum_{i=1}^{s} x_n^i g_n^i(x_n^i) = \sum_{i=1}^{s} f_n^i(x_n^i) \\ x_{n+1}^2 &= \lambda_1 x_n^1 \\ &\vdots \\ x_{n+1}^s &= \lambda_{s-1} x_n^{s-1}, \end{aligned}$$

where f_n^i is of the Beverton-Holt type. Results similar to the one-dimensional case where each f_n^i is under compensatory, i.e.,

$$\frac{\partial f_n^i(x_i)}{\partial x_i} > 0, \frac{\partial^2 f_n^i(x_i)}{\partial x_i^2} < 0,$$

and $\lim_{x_i \to \infty} f_n^i(x_i)$ exists for all $n \in \mathbb{Z}^+$.

9 Almost periodic difference equations

In this section we extend our study to the almost periodic case. This is particularly important in applications to biology in which habitat's fluctuations are not quite periodic.

But in order to embark on this endeavor, one needs to almost reinvent the wheel. The problem that we encounter here is that the existing literature deals exclusively with almost periodic fluctuations (sequences) on the real line $\mathbb{R}$ (on the integers $\mathbb{Z}$). To have meaningful applications to biology, we need to study almost periodic fluctuations or sequence on $\mathbb{Z}^+$ (the set of nonnegative integers). Such a program has been successfully implemented in [8]. Our main objective here is to report to the reader a brief but through exposition of these results.

We start with the following definitions from [16, 21].

Definition 24 *An $\mathbb{R}^k-$valued sequence $x = \{x_n\}_{n\in\mathbb{Z}^+}$ is called Bohr almost periodic if for each $\epsilon > 0$, there exists a positive integer $T_0(\epsilon)$ such that among any $T_0(\epsilon)$ consecutive integers, there exists at least one integer τ with the following property:*

$$\| x_{n+\tau} - x_n \| < \epsilon, \forall n \in \mathbb{Z}^+.$$

The integer τ *is then called an* $\epsilon-$*period of the sequence* $x = \{x_n\}_{n \in \mathbb{Z}^+}$.

Definition 25 *An* $\mathbb{R}^k-$*value sequence* $x = \{x_n\}_{n\in\mathbb{Z}^+}$ *is called Bochner almost periodic if for every sequence* $\{h(n)\}_{n\in\mathbb{Z}^+}$ *of positive integers there exists a subsequence* $\{h_{n_i}\}$ *such that* $\{x_{n+n_i}\}_{n_i\in\mathbb{Z}^+}$ *converges uniformly in* $n \in \mathbb{Z}^+$.

In [8] it was shown that the notions of Bohr almost periodicity and Bochner almost periodicity are equivalent.

Now a sequence $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is called almost periodic in $n \in \mathbb{Z}^+$ uniformly in $x \in \mathbb{R}^k$ if for each $\varepsilon > 0$, there exists $T_0(\varepsilon) \in \mathbb{Z}^+$ such that among $T_0(\varepsilon)$ consecutive integers there exists at least one integer s with

$$\| f(n+s, x) - f(n, x) \| < \varepsilon$$

for all $x \in \mathbb{R}^k$, and $s \in \mathbb{Z}^+$.

Now consider the almost periodic difference equations

$$x_{n+1} = A_n x_n \tag{32}$$

$$y_{n+1} = A_n y_n + f(n, y_n), \tag{33}$$

where A_n is a $k \times k$ almost periodic matrix on $\mathbb{Z}^+$, and $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is almost periodic.

Let $\Phi(n, s) = \prod_{r=s}^{n-1} A_r$ be the state transition matrix of equation (32). Then equation (32) is said to posses a regular exponential dichotomy [24] if there exist a $k \times k$ projection matrix P_n, $n \in \mathbb{Z}^+$, and positive constants M and $\beta \in (0, 1)$ such that the following properties hold:

1. $A_n P_n = P_{n+1} A_n$;
2. $\| X(n, r) P_r x \| \leq M\beta^{n-r} \| x \|$, $0 \leq r \leq n, x \in \mathbb{R}^k$;
3. $\| X(r, n) (I - P_n) x \| \leq M\beta^{n-r} \| x \|$, $0 \leq r \leq n, x \in \mathbb{R}^k$;
4. The matrix A_n is an isomorphism from $R(I - P_n)$ onto $R(I - P_{n+1})$, where $R(B)$ denotes the range of the matrix B.

We are now in a position to state the main stability result for almost periodic systems.

Theorem 26 *Suppose that Eq. (32) possesses a regular exponential dichotomy with constant M and β and f is a Lipschitz with a constant Lipschitz L. Then Eq. (33) has a unique globally asymptotically stable almost periodic solution provided*

$$\frac{M\beta L}{1-\beta} < 1.$$

Proof. Let $AP(\mathbb{Z}^+)$ be the space of almost periodic sequences on $\mathbb{Z}^+$ equipped with the topology of the supremum norm. Define the operator Γ on $AP(\mathbb{Z}^+)$ by letting

$$(\Gamma\varphi)_n = \sum_{r=0}^{n-1}\left(\prod_{s=r}^{n-1}\right) A_s f(r,\varphi_r).$$

Then $\Gamma : AP(\mathbb{Z}^+) \to AP(\mathbb{Z}^+)$ is well defined. Moreover Γ is a contraction. Using the Banach fixed point theorem, we obtain the desired conclusion. ■

The preceding result may be applied to many populations models. However, we will restrict our treatment here on the almost periodic Beverton-Holt equation with overlapping generations

$$x_{n+1} = \gamma_n x_n + \frac{(1-\gamma_n)\mu K_n x_n}{(1-\gamma_n)K_n + (\mu - 1\gamma_n)x_n} \tag{34}$$

with $K_n > 0$ and $\gamma_n \in (0,1)$ are almost periodic sequences, and $\mu > 1$. As before μ and K denote the intrinsic growth rate and the carrying capacity of the population, respectively, while γ is the survival rate of the population from one generation to ne next.

The following result follows from theorem 26

Theorem 27 *Eq. (34) has a unique globally asymptotically stable almost periodic solution provided that*

$$\sup\left\{\gamma_n : n \in \mathbb{Z}^+\right\} < \frac{1}{1+\mu}$$

To this end, we have addressed the question of stability and existence of almost periodic solution of almost periodic difference equation. We now embark on the task of the determination of whether a system is attenuant or resonant.

Let $\{\mu_n\}_{n\in\mathbb{Z}^+}$ be an almost periodic sequence on $\mathbb{Z}^+$. Then we define its mean value as

$$M(\mu_n) = \lim_{n\to\infty} \frac{1}{m}\sum_{r=1}^{m} \mu_{n+r} \tag{35}$$

It may be shown that $M(\mu_n)$ exists [8].

Let $\{\overline{x}_n\}$ be the almost periodic solution of a given almost periodic system. Then we say that the system is

1. attenuant if $M(\overline{x}_n) < M(K_n)$,
2. resonant if $M(\overline{x}_n) > M(K_n)$.

Theorem 28 *[8] Suppose that $\{K_n\}_{n\in\mathbb{Z}^+}$ is almost periodic, $K_n > 0$, $\mu > 1$, and $\gamma_n = \gamma \in (0,1)$. Then*

1. $\limsup\limits_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1} x_m \leq \limsup\limits_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1} K_m$ *for any solution x_n of Eq. (34),*
2. $M(\overline{x}_n) \leq M(K_n)$ *if $\overline{x}_n$ is the unique almost periodic solution of Eq. (34).*

10 Stochastic difference equations

In [22] the authors investigated the stochastic Beverton-Holt equation and introduced new notions of attenuance and resonance in the mean.

Following on the same lines [4] the authors investigated the stochastic Beverton-Holt equation with overlapping generations.

In this section, we will consider the latter study and consider the equation

$$x_{n+1} = \gamma_n x_n + \frac{(1-\gamma_n)\mu K_n x_n}{(1-\gamma_n)K_n + (\mu - 1 + \gamma_n)x_n}. \tag{36}$$

Let $L^1(\Omega, \upsilon)$ be the space of integrable functions on a measurable space $(\Omega, \mathcal{F}, \upsilon)$ equipped with its natural norm given by

$$\| f \|_1 = \int_\Omega f(x) d\upsilon.$$

Let

$$\mathcal{D}(E) := \{f \in L^1(E, \upsilon) : f \geq 0 \text{ and } \textstyle\int_\Omega f d\upsilon\}$$

be the space of all densities on Ω.

Definition 29 *Let $\mathcal{Q} : L^1(\Omega, \upsilon) \to L^1(\Omega, \upsilon)$ be a Markov operator. Then $\{\mathcal{Q}^n\}$ is said to be asymptotically stable if there exists $f^* \in \mathcal{D}$ for which*

$$\mathcal{Q} f^* = f^*$$

and for all $f \in \mathcal{D}$,

$$\lim_{n\to\infty} \| \mathcal{Q}^n f - f^* \|_1 = 0.$$

We assume that both the carrying capacity K_n and the survival rate γ_n are random and for all n, (K_n, γ_n) is chosen independently of (x_0, K_0, γ_0), (x_1, K_1, γ_1), ..., $(x_{n-1}, K_{n-1}, \gamma_{n-1})$ from a distribution with density $\Psi(K, \gamma)$.

The joint density of x_n, K_n, γ_n is $f_n(x)\Psi(K, \gamma)$, where f_n is the density of x_n. Furthermore, we assume that

$$E|K_n| < \infty, E|x_0| < \infty$$

and $K^2\Psi(K, \gamma)$ is bounded above independently of γ and that Ψ is supported on the product interval

$$[K_{\min}, \infty) \times [\gamma_{\min}, \infty),$$

for some $K_{\min} > 0$ and $\gamma_{\min} > 0$.

Moreover, we assume there exists an interval $(K_l, K_u) \subset \mathbb{R}^+$ on which Ψ is positive everywhere for all γ.

Let h be an arbitrary bounded and measurable function on $\mathbb{R}^+$ and define $b(K_n, \gamma_n, x_n)$ to be equal to the right-hand side of equation (36). The expected value of h at time $n + 1$ is then given by

$$E[h(x_{n+1})] = \int_0^\infty h(x) f_{n+1}(x) dx. \tag{37}$$

Furthermore, because of (36) and the fact that the joint density of x_n, and γ_n is just $f_n(x)\Psi(K, \gamma)$, we also have

$$\begin{aligned} E[h(x_{n+1})] &= E[h(b(K_n, \gamma_n, x_n))] \\ &= \int_0^\infty \int_0^1 \int_0^\infty h(b(K, \gamma, y)) f_n(y) \Psi(K, \gamma) dy d\gamma dy. \end{aligned}$$

Let us define $K = K(x, \gamma, y)$ by the equation

$$x = \frac{(1-\gamma)\mu K y}{(1-\gamma)K + (\mu - 1 + \gamma)y} + \gamma y. \tag{38}$$

Solving explicitly this equation for K yields

$$K = \frac{(\mu - 1 + \gamma)y(x - \gamma y)}{(1-\gamma)[\mu y - (x - \gamma y)]}. \tag{39}$$

By a change of variables, this can be written as

$$E[h(x_{n+1})] = \iiint_{\{(x,\gamma,y): 0 < x - \gamma y < \mu y\}} h(x) f_n(y) \Psi(K, \gamma) \frac{dk}{db(K, \gamma, y)} dx d\gamma dy.$$

A simple calculation yields

$$E[h(x_{n+1})] = \mu \int_0^\infty \left\{ \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f_n(y) K^2 \Psi(K,\gamma) d\gamma dy \right\} dx,$$

where

$$A = \{(\gamma, y) : 0 < x - \gamma y < \mu y\}. \tag{40}$$

Equating the above equations, and using the fact that h was an arbitrary, bounded, measurable function, we immediately obtain

$$f_{n+1}(x) = \mu \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f_n(y) K^2 \Psi(K,\gamma) d\gamma dy.$$

Let $\mathcal{P} : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ be defined by

$$\mathcal{P} f(x) = \mu \iint_A \frac{1-\gamma}{(\mu - 1 + \gamma)} \frac{1}{(x-\gamma y)^2} f(y) K^2 \Psi(K,\gamma) d\gamma dy, \tag{41}$$

where $k = K(x, \gamma, y)$ is defined by (39) and A in (40).

We can now state the main theorem of this section

Theorem 30 *[4] The Markov operator $\mathcal{P} : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ defined by equation (41) is asymptotically stable.*

For the case when $\gamma_n = \gamma$ is a constant and K_n is a random sequence, the following attenuance result was obtain.

For almost every $w \in \Omega$ and $x \in \mathbb{R}^+$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i(w, x) < \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} K_i(w),$$

that is we have attenuance in the mean.

It is still an open problem to determine the attenuance or resonance when both γ_n and K_n are random sequences on $\mathbb{Z}^+$.

References

[1] W. C. Allee, *The Social Life of Animals*, William Heinemann, London 1938.

[2] Z. Al-Sharawi and J. Angelos, *On the periodic logistic equation*, Appl. Math. Comput., Vol. 180, No 1, pp. 342-352, 2006.

[3] Z. Al-Sharawi, J. Angelos, S. Elaydi and L. Rakesh, *An extension of Sharkovsky's theorem to periodic difference equations*, J. Math. Anal. Appl., J. Math. Anal. Appl., Vol. 316, 128-141, 2006.

[4] H. Bezandry, T. Diagana, and S. Elaydi, *On the stochastic Beverton-Holt equation with survival rates* ,Journal of Difference Equations and Applications, Vol. 14, No. 2, pp. 175190, 2007

[5] J. M. Cushing, *Oscillations in age-structured population models with an Allee effect*, Journal of Computational and Applied Mathematics 52, 71-80, 1994.

[6] J. M. Cushing and S. M. Henson, *Aperiodically forced Beverton-Holt equation*, Journal of Difference Equations and Applications, Vol. 8, 1119-1120, 2002.

[7] R. L. Devaney, *An introduction to Chaotic Dynamical Systems*, Perseus Books Publishing, L.L.C., Second edition, 1989.

[8] T. Diagana, S. Elaydi, A. Yakubu, *Populations models in almost periodic environments*, Journal of Difference Equations and Applications, Vol. 13, No. 4, pp. 289-308, 2007

[9] S. Elaydi, and R. Sacker, *Global stability of periodic orbits of nonautonomous difference equations and population biology*, Journal of Differential Equations, Vol. 208, pp. 258-273 (2005).

[10] S. Elaydi, *Discrete Chaos: With Applications in Science and Engineering*, Chapman and Hall/CRC, Second Edition, 2008.

[11] S. Elaydi and A. Yakubu, *Global stability of cycles: Lotka-Volterra competition model with stocking*, Journal of Difference Equations and Applications, 8, No. 6, 2002, pp. 537-549

[12] S. Elaydi and R. Sacker, *Global stability of periodic orbits of nonautonomous difference equations and populations biology*, J. Differential Equations 208, pp. 258-273, 2005.

[13] S. Elaydi and R. Sacker, *Nonautonomous Beverton-Holt equations and the Cushing-Henson conjectures*, Journal of Difference Equations and Applications, 11, No. 4-5, 2005, pp. 337-346(10)

[14] S. Elaydi and R. Sacker, *Periodic difference equations, populations biology and the Cushing-Henson conjecture*, Mathematical Biosciences 201, pp. 195-2007, 2006.

[15] S. Elaydi, and R. Sacker, *Skew-product dynamical systems: Applications to difference equations*, Proceedings of the Second Annual Celebration of Mathematics, United Arab Emirates, 2005.

[16] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, New York-Berlin: Springer-Verlag, 377, 1974.

[17] J. Franke and A. Yakubu, *Globally attracting attenuant versus resonant cycles in periodic compensatory Leslie models* , Mathematical Biosciences, Vol. 204, 1, pp. 1-20, 2006

[18] J. Franke and A. Yakubu, *Using a signature function to determine resonant and attenuant 2-cycles in the SmithSlatkin population model*, Journal of Difference Equations and Applications, Vol. 13, No. 4, pp. 289308, 2007

[19] R. Luís, S.Elaydi, and H.Oliveira, *Nonautonomous periodic systems with Allee effects*, Journal of Difference Equations and Applications, to appear

[20] H. Oliveira and E. D'Aniello, *Pitchfork bifurcation for non autonomous interval maps*, Journal of Difference Equations and Applications, 15, No. 3, pp. 291-302, 2009

[21] A. Halanay, and V. Rasvan, *Stability and stable oscillations in discrete time systems*, Advances in Discrete Mathematics and Applications (Gordon and Breach Science Publication), Vol. 2., 2000

[22] C. Haskell,and R. Sacker, *The Stochastic Beverton-Holt Equation and the M. Neubert Conjecture*, Journal of Dynamics and Differential Equations/Elsevier, Vol. V. 17, No. 4, pp. pp. 825-842, 2005

[23] S. Henson, *Multiple attractors an resonance in periodically forced population models*, Physica D. 140, pp. 33-49, 2000.

[24] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, New York-Berlin: Springer-Verlag, 1981,.

[25] R. Sacker, *A Note on Periodic Ricker Maps*, J. Difference Equations and Applications, Vol. 13, No. 1, pp. 89-92, 2007

[26] R. Sacker and S. Elaydi, *Population models with Allee effects: A new model*, Preprint, 2009.

[27] D. Singer, *Stable orbits and bifurcation of maps of the interval*, SIAM Journal Appl. Math. 35 (1978), pp. 260-267

[28] V. Kocic, *A note on the nonautonomous Beverton-Holt model*,Journal of Difference Equations and Applications, Vol. 11, No. 4-5, pp. 415422, 2005.

[29] R. Kon, *A note on attenuant cycles of population models with periodic carrying capacity*, J. Difference Equations and Applications, Vol. 10, 8, pp. 791-793, 2004.