Journal of Difference Equations and Applications
Vol. 00, No. 00, January 2011, 1–5

Open problems in some competition models

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(*Received 00 Month 20xx; in final form 00 Month 20xx*)

We present open problems and conjectures for some two-dimensional competition models, namely the logistic competition model and a Ricker-type competition model.

Keywords: Ricker competition model, Logistic competition model, Stability.

1. Ricker competition model

In [2], the authors considered the Ricker competition model given by

$$\begin{cases} x_{n+1} = x_n e^{K - x_n - a y_n} \\ y_{n+1} = y_n e^{L - y_n - b x_n} \end{cases}, \tag{1}$$

where $(x, y) \in \mathbb{R}^2_+$. The parameters $K, L > 0$ are the carrying capacities of species x and y, respectively. Moreover, the competition parameters a and b are assumed to be positive real numbers.

System (1) is based on the classical Ricker competition model given by

$$\begin{cases} u_{n+1} = u_n \exp(K - c_{11} u_n - c_{12} v_n) \\ v_{n+1} = v_n \exp(L - c_{21} u_n - c_{22} v_n) \end{cases},$$

where the parameters K and L are assumed to be positive real numbers and $c_{ij} \in (0, 1)$, $1 \le i, j \le 2$.

System (1) possesses four fixed points $(0, 0)$, $(K, 0)$, $(0, L)$, and (x^*, y^*), where

$$(x^*, y^*) = \left(\frac{aL - K}{ab - 1}, \frac{bK - L}{ab - 1} \right).$$

The point (x^*, y^*) is "positive", i.e., it is a coexistence equilibrium, if and only if either

$$aL < K \text{ and } bK < L \tag{2}$$

or

$$al > K \text{ and } bK > L. \tag{3}$$

Note that

$$(2) \text{ implies } ab < 1$$

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ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/10236198.YYYY.CATSid
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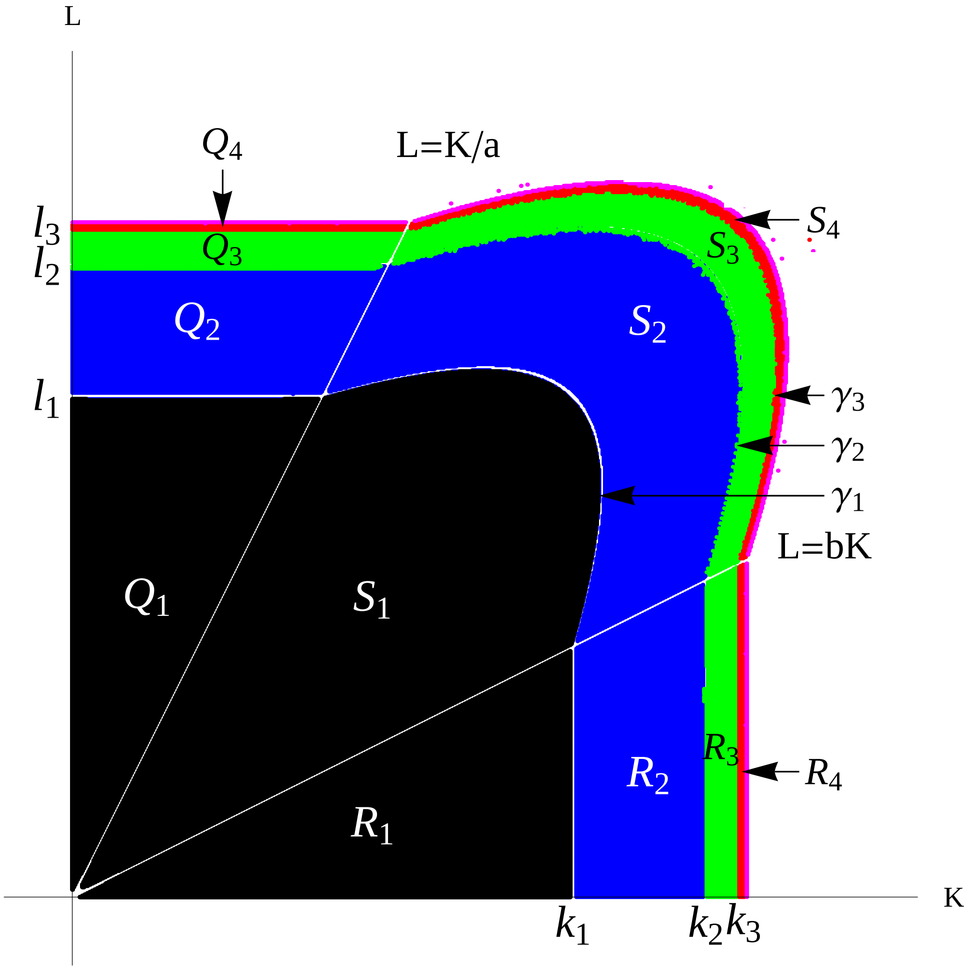

Figure 1. The stability regions and the bifurcation scenario, of the Ricker competition equation in the parameter space $K - L$

and

$$(3) \text{ implies } ab > 1.$$

The case $ab = 1$ is discarded since in this case the two isoclines are parallel and no coexistence fixed point is present. Notice that the case $ab > 1$ leads to a celebrated scenario in classic competition theory, the saddle exclusion case.

For the case $ab < 1$, the following result is the main stability theorem in [2].

Theorem 1.1. *[2] Suppose that $ab < 1$ and let $\hat{S} = Int(S_1) \cup \gamma_1$, where $Int(S_1)$ denotes the interior of S_1. Then the coexistence fixed point*

$$(x^*, y^*) = \left(\frac{aL - K}{ab - 1}, \frac{bK - L}{ab - 1}\right)$$

of the Ricker equation (1) is asymptotically stable if

$$4(ab-1) + 2(1-a)L + 2(1-b)K \leq (aL - K)(bK - L) < (1-a)L + (1-b)K. \quad (4)$$

Equivalently, the coexistence fixed point is asymptotically stable if $(K, L) \in \hat{S}$.

The region S_1, represents (4) in the parameter space $K - L$ and is depicted in Figure 1.

Now we have the following conjecture

Conjecture 1.2 Under condition (4), the coexistence fixed point (x^*, y^*) is globally asymptotically stable in the positive first quadrant, for all $(K, L) \in Int(S_1)$.

Open Problem 1.3 Determine the basin of attraction of the periodic cycles of period 2^{n-1}, $n = 2, 3, \ldots$ in regions S_n.

For the exclusion fixed points $(K, 0)$ and $(0, L)$ we have the following result from [2]

THEOREM 1.4. *[2] For the Ricker competition equation (1), the following statements hold true:*

(1) $(K, 0)$ *is asymptotically stable if* $0 < K \leq 2$ *and* $L < bK$,
(2) $(0, L)$ *is asymptotically stable if* $0 < L \leq 2$ *and* $L > K/a$.

Open Problem 1.5

(1) Determine the stability of the fixed point $(K, 0)$ when $K = 2$ and $L = bK$. In this case the eigenvalues of the Jacobian of the map of Eq. (1) are $\lambda_1 = -1$ and $\lambda_2 = 1$.
(2) Determine the stability of the fixed point $(0, L)$ when $L = 2$ and $L = K/a$. In this case the eigenvalues of the Jacobian of the map of Eq. (1) are $\lambda_1 = 1$ and $\lambda_2 = -1$.

2. Logistic competition model

In [1], the authors considered the following logistic competition model

$$\begin{cases} x_{n+1} = \frac{ax_n(1-x_n)}{1+cy_n} \\ y_{n+1} = \frac{by_n(1-y_n)}{1+dx_n} \end{cases}, \tag{5}$$

where $x, y \in [0, 1]$, $c, d \in (0, 1)$ and $a, b \in (0, 4]$. The parameters a and b are called the intrinsic growth rates of species x and y, respectively, and c, d denote the competition of the species.

System (5) has the fixed points $(0, 0)$, $(\frac{a-1}{a}, 0)$, $(0, \frac{b-1}{b})$ and (x^*, y^*), where

$$x^* = \frac{b(a-1) - c(b-1)}{ab - cd}, y^* = \frac{a(b-1) - d(a-1)}{ab - cd}.$$

We make the assumption that

$$b > 1 + \frac{d(a-1)}{a} \text{ and } a > 1 + \frac{c(b-1)}{b} \tag{6}$$

which insures that the fixed point (x^*, y^*) lies in the positive first quadrant.

THEOREM 2.1. *[1] The positive fixed point* (x^*, y^*) *of the Logistic competition equation (5) is asymptotically stable if the following conditions hold:*

$$\frac{-c(b-c+bc)d^2 + a^3b^2(2-b+2d) + a(b-c+bc)d(3b+c+cd)}{ab(-(1+c)d + a(1+d))(-b(1+c) + c(1+d))}$$
$$+\frac{a^2b\left(2b^2(1+c) + 3c(1+d) - b(3+5d+c(5+4d))\right)}{ab(-(1+c)d + a(1+d))(-b(1+c) + c(1+d))} < 0, \tag{7}$$

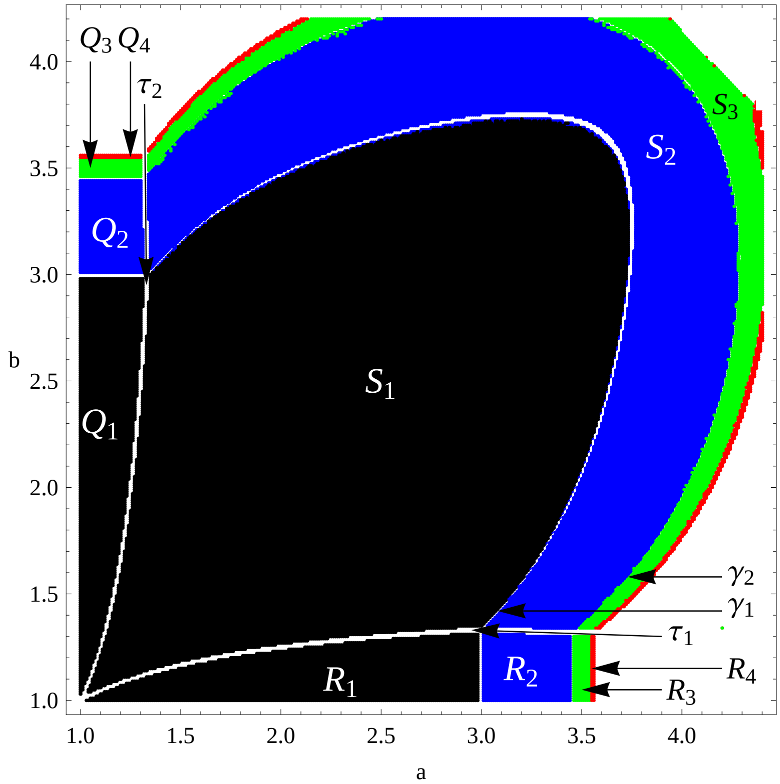

Figure 2. The stability regions and the bifurcation scenario of the competition logistic model in the parameter space $a - b$.

and

$$\frac{-c(b-c+bc)d^2+a^3b^2(3-b+3d)-ad\left(-b^2\left(9+14c+5c^2\right)+c^2(1+d)\right)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))}-$$

$$\frac{adbc\,(8+4c+4d+3cd)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))}+$$

$$\frac{a^2b\left(3b^2(1+c)+c\left(9+14d+5d^2\right)-3b(3+4d+4c(1+d))\right)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))}>0. \tag{8}$$

Note that inequality (21) in [1], i.e.

$$\frac{(b(-1+a-c)+c)(a(-1+b-d)+d)(ab-cd)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))}<0, \tag{9}$$

holds true under condition (6). This observation has not been noted in [1].

Equivalently, the positive fixed point (x^*, y^*) of Eq. (5) is asymptotically stable if $(a,b) \in Int(S_1)$, where S_1 is the region depicted in Figure 2. Note that the curves τ_1 and τ_2 in Figure 2 are defined as

$$\tau_1 = \{(a,b) \in \mathbb{R}^2_+ : b = 1 + \tfrac{d(a-1)}{a}\} \text{ and } \tau_2 = \{(a,b) \in \mathbb{R}^2_+ : a = 1 + \tfrac{c(b-1)}{b}\}.$$

Conjecture 2.2 The positive fixed point (x^*, y^*) of the logistic competition model (5) is globally asymptotically stable in the positive first quadrant if $(a,b) \in Int(S_1)$.

Open Problem 2.3 Determine the basin of attraction of the periodic cycles of period 2^{n-1}, $n = 2, 3, \ldots$ in regions S_n in Figure 2.

For the fixed points $(\frac{a-1}{a}, 0)$ and $(0, \frac{b-1}{b})$, we have the following result:

Theorem 2.4. *[1] The following statements holds true:*

(1) *The fixed point* $(\frac{a-1}{a}, 0)$ *of Eq. (5) is asymptotically stable if* $1 < a \leq 3$ *and* $1 < b < 1 + \frac{d(a-1)}{a}$ *and is unstable if* $1 < a < 3$ *and* $b = 1 + \frac{d(a-1)}{a}$,

(2) *The fixed point* $(0, \frac{b-1}{b})$ *of Eq. (5) is asymptotically stable if* $1 < b \leq 3$ *and* $1 < a < 1 + \frac{c(b-1)}{b}$ *and is unstable if* $1 < b < 3$ *and* $a = 1 + \frac{c(b-1)}{b}$.

Open Problem 2.5

(1) Determine the stability of the fixed point $(\frac{a-1}{a}, 0)$ of Eq. (5) if $a = 3$ and $b = 1 + \frac{d(a-1)}{a}$,

(2) Determine the stability of the fixed point $(0, \frac{b-1}{b})$ of Eq. (5) if $b = 3$ and $a = 1 + \frac{c(b-1)}{b}$.

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