A Generalization of the Fujisawa-Kuh Global Inversion Theorem

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Abstract

We discuss the problem of global invertibility of nonlinear maps defined on the finite dimensional Euclidean space via differential tests. We provide a generalization of the Fujisawa-Kuh Global Inversion Theorem and introduce a generalized ratio condition which detects when the pre-image of a certain class of linear manifolds is non-empty and connected. In particular, we provide conditions that also detect global injectivity.

Keywords: Global inversion theorem, global diffeomorphism, Fujisawa-Kuh theorem, ratio-condition, global injectivity

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1. Introduction

The problem of deciding whether a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 is a global diffeomorphism (or globally invertible) is a fundamental question in mathematics. It has applications in diverse areas such as differential geometry, differential equations, numerical analysis, algebraic geometry, statistics, general theory of optimization, nonlinear circuit theory, and economics to name a few. In our work we consider results establishing global invertibility by providing analytical conditions that detect it as a topological phenomena.

A classical result in the theory of global invertibility is the Hadamard-Levy-Plastock Theorem (see [5], [6], and [10]) which states that a local diffeomorphism f of class C^1 between two Banach spaces E and F is bijective if it satisfies the following integral condition

$$\int_0^\infty \min_{\|x\|=r} \|f'(x)^{-1}\|^{-1} dr = \infty.$$

This result has been very influential in the theory of global inversion and other areas. The finite dimensional case is due to Hadamard [5], and it was extended to

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infinite dimensional Banach spaces by Levy [6]. The original proofs of Hadamard and Levy were very intuitive and less rigorous. In [10], Plastock gave a proof of the Hadamard-Levy result, in the framework of Banach spaces of arbitrary dimension, based on topological arguments of covering space theory. A self-contained proof of the Hadamard-Levy based on arguments from a first course in mathematical analysis can be found in [13].

An interesting aspect of the Hadamard-Levy-Plastock Theorem is that it provides a simple integral condition to invertibility. However, in applications it can be very difficult and sometimes impossible to check such integral conditions. In fact, there are several global diffeomorphisms that are not detected by the integral condition above. For instance, it fails to detect that the simple planar map $f(x, y) = (x, x^3 + y)$ is invertible. Hence new formulations of Global Invertibility Theorems in terms of other differential tests are desirable. We remark several global invertibility criteria by Radulescu-Radulescu in [12], [14], and [15] that are more suitable to applications.

Meanwhile, there have been several results in the last few years dealing with global invertibility from the point of view of geometry and topology. See for instance, Balreira [1], [2], Nollet-Xavier [7], [8], and [9].

In this paper, we focus on the results of Fujisawa-Kuh [3] who provided an interesting analytical test to detect global invertibility as a result of their work in Nonlinear Circuit Theory. We remark that the differential test below works only in finite dimensional Euclidean spaces.

Theorem 1.1 (Fujisawa-Kuh, [3])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. For $k = 1, 2, \ldots, n$ let $J_k f(\mathbf{x})$ denote the determinant of the matrix consisting of the entries in the first k rows and first k columns of the Jacobian matrix $f'(\mathbf{x})$. Suppose that the following conditions hold:*

(i) $J_k f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in \mathbb{R}^n$ and $k \in \{1, 2, \ldots, n\}$.

(ii) There exists $\epsilon > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ we have:

$$|J_1 f(\mathbf{x})| \geq \epsilon, \left|\frac{J_2 f(\mathbf{x})}{J_1 f(\mathbf{x})}\right| \geq \epsilon, \ldots, \left|\frac{J_n f(\mathbf{x})}{J_{n-1} f(\mathbf{x})}\right| \geq \epsilon$$

Then f is a global diffeomorphism.

Condition (ii) is known as the *ratio condition*. We note that the planar map $f(x, y) = (x, x^3 + y)$ does satisfies condition (i) and (ii) consequently Theorem 1.1 detects its invertibility. Since any map of $\mathbb{R}^n$ onto itself defined by a simple relabeling of the coordinates is a global homeomorphism it follows as a corollary of the above theorem that any nested sequence of n-submatrices of the Jacobian matrix $f'(\mathbf{x})$ can be used as a sequence in condition (ii).

The proof of Theorem 1.1 given in [3] is very short and it lacks the proper mathematical formalism. Hence a mathematician can consider it only as a hint. Since the original result in [3], no other proof has been given in the mathematical literature. In our work, we provide a generalization of Theorem 1.1 with less restrictive and natural conditions. We also address in our paper the interesting question of what are the properties of the mapping f if some of the inequalities from the ratio condition are missing. We prove that if only a subset of the inequalities from the ratio condition hold then the preimages of a certain class of linear manifolds are non-empty and connected. As an application of our new ideas, we provide a result establishing a differential test to detect global injectivity of maps.

Now, let us state our main result which we refer to as the generalized Fujisawa-Kuh global inversion theorem.

Theorem 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Let $J_0 f(\mathbf{x}) = 1$ and for $k = 1, 2, \ldots, n$ let $J_k f(\mathbf{x})$ denote the determinant of the matrix consisting of the entries in the first k rows and first k columns of the Jacobian matrix $f'(\mathbf{x})$. Suppose that each $k \in \{1, 2, \ldots, n\}$, we have $J_k f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in \mathbb{R}^n$ and that there exists continuous functions $c_k : \mathbb{R}^{n-k+1} \to \mathbb{R}_+$, such that the following conditions hold:*

(i) $\int_{-\infty}^{0} c_k(s, x_{k+1}, x_{k+2}, \ldots, x_n)\, ds = \int_{0}^{\infty} c_k(s, x_{k+1}, x_{k+2}, \ldots, x_n)\, ds = +\infty$ *for every* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and* $k \in \{1, 2, \ldots, n\}$.

(ii) $\left| \frac{J_k f(\mathbf{x})}{J_{k-1} f(\mathbf{x})} \right| \geq c_k(x_k, x_{k+1}, x_{k+2}, \ldots, x_n)$ *for every* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and* $k \in \{1, 2, \ldots, n\}$.

Then f is a C^1 global diffeomorphism.

We note that Theorem 1.1 follows from Theorem 1.2 by simply taking the functions c_k, for $k \in \{1, 2, \ldots, n\}$ to be positive constants. We will show later that Theorem 1.2 is in fact an effective improvement of Theorem 1.1 as there are maps that satisfy the conditions in Theorem 1.2 but do not satisfy Theorem 1.1.

The condition (ii) from the theorem above seems quite arbitrary since c_k depends only on the last $n - k + 1$ variables. Consequently it would be natural to consider the generalization of Theorem 1.1 by the following integral conditions

$$\int_{-\infty}^{0} \left| \frac{J_k f(\mathbf{x})}{J_{k-1} f(\mathbf{x})} \right| dx_k = \int_{0}^{\infty} \left| \frac{J_k f(\mathbf{x})}{J_{k-1} f(\mathbf{x})} \right| dx_k = \infty \tag{1.1}$$

for every $x_1, \ldots x_{k-1}, x_{k+1}, \ldots, x_n \in \mathbb{R}$. The question is if the above condition (1.1) implies that f is a global diffeomorphism. The answer to the above question is negative. The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x-y, \arctan x + \arctan y)$, satisfies (1.1) but f is not a bijection as its second component is a bounded function.

Before we state our next results, we consider a weaker ratio condition, in the sense that we look at a ratio condition that could possibly involve less than the n inequalities as in Theorem 1.2.

Definition 1.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map of class $C^1$and $k \in \{1, 2, \ldots, n\}$. Let $J_0(\mathbf{x}) = 1$ and for $i = 1, 2, \ldots, k$, let $J_i f(\mathbf{x})$ denote the determinant of the matrix consisting of the entries in the first i rows and first i columns of the Jacobian matrix $f'(\mathbf{x})$. We say that f satisfies the *k-ratio condition* if the following hold:

(i) $J_i f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in \mathbb{R}^n$ and $i \in \{1, 2, \ldots, k\}$.

(ii) For each $i \in \{1, 2, \ldots, k\}$, there exists a continuous function $c_i : \mathbb{R}^{n-i+1} \to \mathbb{R}_+$ such that for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\int_{-\infty}^{0} c_i(s, x_{i+1}, x_{i+2}, \ldots, x_n)\, ds = \int_{0}^{\infty} c_i(s, x_{i+1}, x_{i+2}, \ldots, x_n)\, ds = +\infty$$

(iii) For each $i \in \{1, 2, \ldots, k\}$ and for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have

$$\left| \frac{J_i f(\mathbf{x})}{J_{i-1} f(\mathbf{x})} \right| \geq c_i(x_i, x_{i+1}, x_{i+2}, \ldots, x_n)$$

Simply put, we say that f satisfies the k-ratio condition if it satisfies the first k inequalities from the ratio condition. Geometrically, we note that when the k-ratio condition is satisfied, then the map f can be viewed as a local diffeomorphism in the first k coordinates. In view of this remark, we obtain an interesting result describing the underlying topological phenomena. For $k \in \{1, 2, \ldots, n-1\}$, let us consider the decomposition of $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Let the canonical projections onto the first k coordinates and onto the last $n-k$ coordinates be given by $\rho_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $\rho_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$, respectively. For $\mathbf{v} \in \mathbb{R}^k$, we say that $L(\mathbf{v})$ is the linear manifold *spanned by the last* $(n-k)$*-axes* orthogonal to $\mathbf{v}$ if $L(\mathbf{v}) = \left\{(\mathbf{v}, \mathbf{w}) \in \mathbb{R}^n \mid \mathbf{w} \in \mathbb{R}^{n-k}\right\}$, that is, $L(\mathbf{v}) = \rho_1^{-1}(\mathbf{v})$. Our next result is as follows.

Proposition 1.4. *For $n \geq 2$, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, $k \in \{1, \ldots, n-1\}$, and L be a linear manifold spanned by the last $(n-k)$-axes. Suppose that f satisfies the k-ratio condition, then $f^{-1}(L)$ is non-empty and connected. In fact, $f^{-1}(L)$ is diffeomorphic to L.*

By applying Proposition 3.1 and Lemma 2.2 we prove the following important result that detects global injectivity.

Corollary 1.5. *For $n \geq 2$, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be local diffeomorphism of class C^1. Suppose f satisfies the $(n-1)$-ratio condition, then f is injective.*

The statement from the above corollary appeared for the first time in [15] without a proof. We also note that under definition 1.3, Theorem 1.2 can be simply stated as follows: A C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the n ratio condition is a global diffeomorphism.

We emphasize that the proof of Proposition 1.4 will be a consequence of Theorem 1.2 where we look at the map f restricted to linear manifolds of dimension k. However, we now are able to reveal how the analytical conditions are in fact a manifestation of a topological phenomena that can also be used to establish injectivity. This type of argument is related to the work in [2] where analytical conditions involving the rows of the Jacobian matrix are used to establish topological properties of pre-images of linear manifolds under a local diffeomorphism. In fact, the results in [2] also establish a global inversion theorem and a global injectivity result.

2. Proof of the Main Result

In this section, we provide the proof of our main result in Theorem 1.2 on global invertibility and give an example to show that we indeed have an improvement over the Fujisawa-Kuh global inversion theorem. The main idea of the proof is to understand how the Jacobian matrix of a map changes when we restrict it to an implicitly defined submanifold. In fact, using a modified form of the implicit function theorem, we show that the Jacobian matrix changes in a surprisingly elementary way. Before we embark in its proof, we need some preliminaries results.

Lemma 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 map. Suppose that there exist a continuous function $c : \mathbb{R}^n \to \mathbb{R}_+$ such that*

(i) $\displaystyle\int_{-\infty}^{0} c(s, \mathbf{y})ds = \int_{0}^{\infty} c(s, \mathbf{y})ds = +\infty$ *for every* $\mathbf{y} \in \mathbb{R}^{n-1}$

(ii) $f'_x(x, \mathbf{y}) \neq 0$ *for every* $x \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^{n-1}$

(iii) $|f'_x(x, \mathbf{y})| \geq c(x, \mathbf{y})$ *for every* $x \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^{n-1}$

Then there exists a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(\cdot, \mathbf{y})$ is differentiable for every $\mathbf{y} \in \mathbb{R}^{n-1}$ and $f(\phi(x, \mathbf{y}), \mathbf{y}) = x$ for every $x \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^{n-1}$.

Proof. From conditions (i)-(iii) above it follows that for every $\mathbf{y} \in \mathbb{R}^{n-1}$ the function $f(\cdot, \mathbf{y}) : \mathbb{R} \to \mathbb{R}$ is a bijection. Hence there exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $f(\phi(x, \mathbf{y}), \mathbf{y}) = x$ for every $x \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^{n-1}$. It follows from the implicit function theorem that ϕ is continuous and $\phi(\cdot, \mathbf{y})$ is differentiable for every $\mathbf{y} \in \mathbb{R}^{n-1}$. □

For easiness of notation, given M a $n\times n$ matrix, for $k\in\{1,\ldots,n\}$ let M_k denote the $k\times k$ submatrix obtained from the first k rows and columns of M.

Lemma 2.2. *Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a C^1 map and $p\in\{1,2,\ldots,n\}$. Assume that for every $\mathbf{y}=(x_{p+1},\ldots x_n)\in\mathbb{R}^{n-p}$ the map $\varphi_{\mathbf{y}}:\mathbb{R}^p\to\mathbb{R}^p$ given by $\varphi_{\mathbf{y}}(x_1,x_2,\ldots,x_p)=(f_1(\mathbf{x}),\ldots,f_p(\mathbf{x}))$ is a global diffeomorphism where $\mathbf{x}=(x_1,\ldots,x_n)$. Then the map $g:\mathbb{R}^n\to\mathbb{R}^n$ given by $g(\mathbf{x})=(f_1(\mathbf{x}),\ldots,f_p(\mathbf{x}),x_{p+1},\ldots x_n)\,,\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ is a global diffeomorphism. Moreover, if we denote $h=f\circ g^{-1}$, then h is the identity of the first p coordinates, that is $h(\mathbf{x})=(x_1,x_2,\ldots,x_p,h_{p+1}(\mathbf{x}),...,h_n(\mathbf{x}))$ and $J_kh\,(\mathbf{x})=\det\left(h'(\mathbf{x})\right)_k=1$ for $k=1,\ldots,p$ and for $k=p+1,\ldots n$, we have*

$$J_kh\,(\mathbf{x})=\frac{J_kf\left(g^{-1}\left(\mathbf{x}\right)\right)}{J_pf\left(g^{-1}\left(\mathbf{x}\right)\right)}. \tag{2.1}$$

Proof. In order to prove that g is a global diffeomorphism consider $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbb{R}^n$ and the equation $g(\mathbf{x})=\mathbf{w}$. Let $\mathbf{y}=(w_{p+1},\ldots,w_n)$. Then $x_i=w_i$ for every $i\in\{p+1,...,n\}$ and $f_i(\mathbf{x})=w_i$ for every $i\in\{1,...,p\}$. One can easily see that $g^{-1}\left(\mathbf{w}\right)=\left(\varphi_{\mathbf{y}}^{-1}\left(w_1,\ldots,w_p\right),w_{p+1},\ldots,w_n\right)$ and g is a global diffeomorphism.

Next, let us write the Jacobian matrix of g as a block matrix as follows:

$$g'(\mathbf{x})=\begin{pmatrix}(f'(\mathbf{x}))_p & B(\mathbf{x})\\ 0 & 1_q\end{pmatrix} \tag{2.2}$$

where $(f'(\mathbf{x}))_p$ is the submatrix of $f'(\mathbf{x})$ formed by the first p rows and p columns of $f'(\mathbf{x})$, 1_q is the identity matrix of order $q=n-p$, and $B(\mathbf{x})$ is the $p\times q$ submatrix of $f'(\mathbf{x})$ formed by the first p rows and the last q columns. For simplicity, let us write $(f'(\mathbf{x}))_p=A$ and $B(\mathbf{x})=B$. In this notation, we can compute the inverse of $g'(\mathbf{x})$ as

$$(g'(\mathbf{x}))^{-1}=\begin{pmatrix}A^{-1} & -A^{-1}B\\ 0 & 1_q\end{pmatrix} \tag{2.3}$$

observing that by hypotheses A is invertible. Let us also write $f'(\mathbf{x})=P$ in block notation as

$$P=f'(\mathbf{x})=\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{2.4}$$

where C is a $q\times p$ matrix and D is a $q\times q$ matrix.

We now define the map $h:\mathbb{R}^n\to\mathbb{R}^n$ given by $h(\mathbf{y})=(f\circ g^{-1})(\mathbf{y})$. It is clear that h is the identity on the first p coordinates of $\mathbf{y}$ and therefore $J_kh\,(\mathbf{x})=1$ for $k=1,2,\ldots,p$. In the case $k\in\{p+1,\ldots,n\}$ we can compute $J_kh\,(\mathbf{x})$ as follows. Since $h\circ g=f$, by the chain rule we have $h'\left(g(\mathbf{x})\right)\cdot g'(\mathbf{x})=f'(\mathbf{x})$, thus $h'\left(g(\mathbf{x})\right)=f'(\mathbf{x})\cdot(g'(\mathbf{x}))^{-1}$. From (2.3) and (2.4), we have

$$h'\left(g(\mathbf{x})\right)=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}A^{-1} & -A^{-1}B\\ 0 & 1_q\end{pmatrix}=\begin{pmatrix}1_p & 0\\ CA^{-1} & D-CA^{-1}B\end{pmatrix} \tag{2.5}$$

We note that the block matrix $D-CA^{-1}B$ is the *Schur complement* of A in $f'(x)$, see [11]. We denote it by $(P|A)$. Applying the determinant operator to the above identity we obtain:

$$\det(P|A)=\frac{\det P}{\det A}. \tag{2.6}$$

In fact, we can now easily relate $J_k h\left(\mathbf{x}\right)$ to $J_k f(g^{-1}\left(\mathbf{x}\right))$. Indeed, a simple computation shows that for $k \geq p$ we have $(P_k|A)=(P|A)_{k-p}$, and using (2.6) we have

$$J_k h\left(\mathbf{x}\right)=\det\left(h'\left(\mathbf{x}\right)\right)_k=\det\left(f'\left(g^{-1}\left(\mathbf{x}\right)\right)|A\right)_{k-p}=\frac{\det(f'(g^{-1}\left(\mathbf{x}\right)))_k}{\det A}=\frac{J_k f(g^{-1}\left(\mathbf{x}\right))}{J_p f(g^{-1}\left(\mathbf{x}\right))}$$

which is the desired conclusion. □

We are now ready to prove our main theorem which will follow from the previous two lemmas.

Proof of Theorem 1.2. We shall prove the result by induction. For $n=1$ we see that the statement of Theorem 1.2 is as follows. Let $f:\mathbb{R}\to\mathbb{R}$ be a map of class C^1 and $c:\mathbb{R}\to\mathbb{R}_+$ be a continuous function such that $f'\left(x\right)\neq 0$ for every $x\in\mathbb{R}$, $\left|f'\left(x\right)\right|\geq c\left(x\right)$ for every $x\in\mathbb{R}$ and $\int_{-\infty}^{0}c\left(s\right)ds=\int_{0}^{\infty}c\left(s\right)ds=\infty$. Therefore applying Lemma 2.1, the map ϕ is the inverse of the function f, and this establishes the result for $n=1$.

Let us suppose that the statement of Theorem 1.2 holds for every function $f:\mathbb{R}^n\to\mathbb{R}^n$ satisfying the hypotheses of the theorem. We now consider a map $f=(f_1,f_2,\ldots,f_n,f_{n+1}):\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ of class C^1 and continuous functions $c_k:\mathbb{R}^{n+2-k}\to\mathbb{R}_+$, for $k\in\{1,2,\ldots,n+1\}$ such that the following conditions hold:

(i) $\displaystyle\int_{-\infty}^{0} c_k\left(s,x_{k+1},x_{k+2},...,x_{n+1}\right)ds=\int_{0}^{\infty} c_k\left(s,x_{k+1},x_{k+2},...,x_{n+1}\right)ds=\infty$ for every $x_2,x_3,\ldots,x_{n+1}\in\mathbb{R}$ and $k\in\{1,2,\ldots,n+1\}$.

(ii) $J_k\left(\mathbf{x}\right)\neq 0$ for every $\mathbf{x}=(x_1,x_2,\ldots,x_{n+1})\in\mathbb{R}^{n+1}$ and $k\in\{1,2,\ldots,n+1\}$

(iii) For every $\mathbf{x}=(x_1,x_2,\ldots,x_{n+1})\in\mathbb{R}^{n+1}$ and $k\in\{1,2,\ldots,n+1\}$, we have $\left|\dfrac{J_k\left(\mathbf{x}\right)}{J_{k-1}\left(\mathbf{x}\right)}\right|\geq c_k\left(x_k,x_{k+1},x_{k+2},\ldots,x_{n+1}\right)$

For each $y \in \mathbb{R}$ we consider the map $\varphi_y : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\varphi_y(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n, y), \ldots f_n(x_1, \ldots, x_n, y)) .$$

Then for $k \in \{2, \ldots, n\}$, it is trivial to see that

$$\frac{\det(\varphi_y'(x_1, \ldots, x_n))_k}{\det(\varphi_y'(x_1, \ldots, x_n))_{k-1}} = \frac{J_k f(x_1, \ldots, x_n, y)}{J_{k-1} f(x_1, \ldots, x_n, y)}$$

From the induction hypotheses, we have that φ_y is a global diffeomorphism. Now we can apply Lemma 2.2 and obtain that the map $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$g(x_1, \ldots, x_n, x_{n+1}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}), x_{n+1})$$

is a global diffeomorphism. Let $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be given by $h = f \circ g^{-1}$. Note that $h(\mathbf{x}) = (x_1, \ldots, x_n, h_{n+1}(\mathbf{x}))$ and $h_{n+1}(\mathbf{x}) = f_{n+1}\left(g^{-1}\left(\mathbf{x}\right)\right)$. Since the last component of the map $\mathbf{x} \rightarrow g^{-1}\left(\mathbf{x}\right)$ is the identity it follows that

$$\left| \frac{\partial h_{n+1}}{\partial x_{n+1}}(\mathbf{x}) \right| = |J_{n+1} h(\mathbf{x})| = \left| \frac{J_{n+1} f(g^{-1}\left(\mathbf{x}\right))}{J_n f(g^{-1}\left(\mathbf{x}\right))} \right| \geq c_{n+1}(x_{n+1}) \tag{2.7}$$

Therefore, an application of the obvious modification of Lemma 2.1 to the map h_{n+1} with condition (2.7) shows that exists a differentiable map $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$

$$h_{n+1}(x_1, \ldots, x_n, \psi(x_1, \ldots, x_n, x)) = x$$

Hence, given $\mathbf{w} = (w_1, \ldots, w_n, w_{n+1}) \in \mathbb{R}^{n+1}$, we have that

$$h(w_1, \ldots, w_n, \psi(w_1, \ldots, w_n, w_{n+1})) = \mathbf{w}.$$

This shows that h is a diffeomorphism and since $f = h \circ g$ we have that f is a bijection as we wished to prove.

$\square$

In order to see that our generalized Fujisawa-Kuh Theorem (Theorem. 1.2) is in fact a generalization of the original Fujisawa-Kuh Theorem (Theorem. 1.1), we provide in the example below a map that satisfies Theorem. 1.2 but does not satisfy Theorem. 1.1.

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f\left(x, y\right) = \left(\phi\left(x\right), \phi\left(y\right)\right)$ where $\phi(x) = \ln\left(x + \sqrt{1 + x^2}\right)$, $x \in \mathbb{R}$. Consider the functions $c_1 : \mathbb{R}^2 \to \mathbb{R}_+$, $c_1(x, y) = \dfrac{1}{\sqrt{1 + x^2}}$ and $c_2 : \mathbb{R} \to \mathbb{R}_+$, $c_2(y) = \dfrac{1}{\sqrt{1 + y^2}}$. Note that $J_1\left(x, y\right) = \phi'(x)$, and $J_2\left(x, y\right) / J_1\left(x, y\right) = \phi'\left(y\right)$ have both 0 as infimum, hence the conditions from the original Fujisawa-Kuh

Theorem are not satisfied. Note also that $|J_1(x,y)| \geq c_1(x,y)$, $|J_2(x,y)/J_1(x,y)| \geq c_2(y)$, $x, y \in \mathbb{R}$ and

$$\int_{-\infty}^{0} c_1(s,y)\,ds = \int_{0}^{\infty} c_1(s,y)\,ds = +\infty \quad \text{for every } y \in \mathbb{R}$$

$$\int_{0}^{\infty} c_2(s)\,ds = +\infty$$

Thus f verifies the conditions of Theorem 1.2.

3. Topological Result

In this section, we prove Proposition 1.4 and establish Corollary 1.5 detecting global injectivity. We consider maps that satisfy the k-ratio condition and apply Lemma 2.2 to obtain topological information on the pre-images of hyperplanes. A geometrical interpretation of the proof of Lemma 2.2 reveals how the image of the map $f: \mathbb{R}^n \to \mathbb{R}^n$ can be decomposed as $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and hence we obtained the desired topological conclusions. These ideas are related to the work in [1] and [2] where topological hypotheses on the pre-images of hyperplanes imply global invertibility and injectivity. First, let us state the following result.

Proposition 3.1. *For $n \geq 2$ let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, $k \in \{1, \ldots, n\}$. Assume that f satisfies the k-ratio condition. Then there is a global diffeomorphism $g: \mathbb{R}^n \to \mathbb{R}^n$ such that $h = f \circ g^{-1}$ is the identity on the first k coordinates, i.e. $h(\mathbf{x}) = (x_1, \ldots, x_k, h_{k+1}(\mathbf{x}), \ldots, h_n(\mathbf{x}))$.*

Proof. For each fixed $\mathbf{y} = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$ we define the following function $\varphi_{\mathbf{y}}: \mathbb{R}^k \to \mathbb{R}^k$ given by

$$\varphi_{\mathbf{y}}(x_1, \ldots, x_k) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})), \qquad \mathbf{x} = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in \mathbb{R}^n$$

Since f satisfies the k-ratio condition, then $\varphi_{\mathbf{y}}$ also satisfies the k-ratio condition. An application of Theorem 1.2 to the function $\varphi_{\mathbf{y}}$ shows that $\varphi_{\mathbf{y}}$ is a global diffeomorphism. From Lemma 2.2 the conclusion follows. □

We are now ready to provide a simple proof for our result.

Proof of Proposition 1.4. From Proposition 3.1, we see that for $\mathbf{v} \in \mathbb{R}^k$ and $L(\mathbf{v}) = \left\{(\mathbf{v}, \mathbf{w}) |\ \mathbf{w} \in \mathbb{R}^{n-k}\right\}$, we consider the diffeomorphism $g: \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}), x_{k+1}, \ldots, x_n).$$

Then $f^{-1}(L(\mathbf{v})) = g^{-1}(L(\mathbf{v}))$ which is diffeomorphic to $L(\mathbf{v})$ via g. This is the desired conclusion. □

Next, the proof of Corollary 1.5 follows directly from Proposition 3.1 and Lemma 2.2. Indeed, suppose f is a local diffeomorphism that satisfies the $(n-1)$-ratio condition. From Proposition 3.1 there is a global diffeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $h = f \circ g^{-1}$ is the identity on the first $n-1$ coordinates. Now, apply Lemma 2.2 for $k = n$ and $p = n-1$. From (2.1) we obtain

$$\left|\frac{\partial h_n}{\partial x_n}(\mathbf{x})\right| = |J_n h(\mathbf{x})| = \left|\frac{J_n f(g^{-1}(\mathbf{x}))}{J_{n-1} f(g^{-1}(\mathbf{x}))}\right| \neq 0.$$

It follows that for every $\mathbf{y} = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ the function $u_{\mathbf{y}}(t) = h_n(\mathbf{y}, t)$, $t \in \mathbb{R}$ is injective. This implies h is injective. Since $f = h \circ g$ it follows that f is also injective. We note that we can prove injectivity of f in more general hypotheses. For instance, one can prove that f is injective if f satisfies the $(n-1)$-ratio-condition, $J_n f(\mathbf{x})$ does not change the sign, and its set of zeros contains no non-trivial segment parallel to the last coordinate axis.

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