Asymptotic Stability of Linear Difference Equations of Advanced Type

Fozi M. Dannan
Department of Mathematics, Faculty of Science,
Qatar University, Doha, Qatar
Saber N. Elaydi
Department of Mathematics,Trinity University,
San Antonio, Texas 78212

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Abstract

Necessary and sufficient conditions are obtained for the asymptotic stability of difference equations of advanced typen of the form

$$x(n) - ax(n+1) + bx(n+k) = 0, n = 0, 1, ..$$

where a and b are arbitrary real numbers and $k > 1$.
For $a = 1$, we establish an analogue of a result by Levin and May.

1 Introduction

Delay idfference equations of the form

$$x(n+1) - ax(n) + bx(n-k) = 0, \quad n = 0, 1, ... \, ; (a \neq 0)\,, \tag{1}$$

where $k > 1$ is an integer and $a, b \in \Re$, have been extensively studied in the last decade, see for example Clark[1], Elaydi[2], Levin and May[6], Kuruklis[5], Kocic and Ladas [4]. When a=1 and b is an arbitrary real number, and k is a positive integer, Levin and May showed that the zero solution of Eq(1) is asymptotically stable if and only if

$$0 < b < 2\cos[k\pi/(2k+1)]. \tag{2}$$

Matsunaga and Hara [?] extended this result to the two dimensional system $x(n+1)-x(n)+Bx(n-k)=0$, where B is a 2×2 constant matrix. For the general case when a is any real number, Clark[1] gave an elegant proof to the following result: if $|a|+|b|\leq 1$, then the zero solution of Eq(1) is asymptotically stable. Later, Kuruklis [5] gave necessary and sufficient conditions for the zero solution of Eq(1) to be asymptotically stable. Moreover, his result includes as a special case the result of May and Levin cited above.

The main objective of this paper is to extend the above work to linear difference equations of advanced type of the form

$$x(n) - ax(n+1) + bx(n+k) = 0, \quad n = 0, 1, 2, ... \tag{3}$$

where a and b are arbitrary reals and $k > 1$ is an integer . These difference equations appeared in the book of Gyori and Ladas [?]. They may represent a mathematical model of species whose kth generation depends on the present and next generations. Moreover, difference equations of advanced type are usually associated with the study of differential equations with piecewise continuous argument such as

$$y' = Ay(t) + By([t+k]) \tag{4}$$

, where [ ] denotes the greatest integer function. If we let $y_n(t)$ to be the solution of Eq(4) on the interval $[n, n+1)$ and $x(n) = y_n(n)$, then Eq(4) may be transformed to Eq(3) (for more details see [?]. It is well known that the zero solution of Eq.(4) is asymptotically stable if and only if all the roots of its characteristic equation

$$b\lambda^k - a\lambda + 1 = 0 \tag{5}$$

are inside the unit disk. Equation (5) can be written equivalently as

$$c\mu^k - \mu + 1 = 0 \tag{6}$$

where $c = b/a^k \quad (a \neq 0)$ and $\mu = a\lambda$. Hence all the roots of Eq.(5) are inside the unit desk if and only if all the roots of Eq.(6) are inside the disk $|\mu| < |a|$.

Our main result is Theorem 3.8 which provides necessary and sufficient conditions for the asymptotic stability of the zero solution of Eq.(3). As a consequence of this theorem we obatin an analogue of Levin and May's above celebrated result for advanced difference equations.

2 preliminary Lemmas

Lemma 2.1 . *Let $k > 1$ be an integer and $a \neq 0$ be an arbitrary real number . Then the following inequality holds true*

$$\frac{|a| - 1}{|a|^k} \leq \frac{(k-1)^{k-1}}{k^k} = \beta_k \ . \tag{7}$$

Proof . The equality sign holds for $a = \pm\frac{k}{k-1}$. Define the function $f(a) = \frac{(k-1)^{k-1}}{k^k}a^k - a + 1$. Hence $f'(a) = \left(\frac{k-1}{k}\right)^{k-1} a^{k-1} - 1$, $f''(a) = \frac{(k-1)^k}{k^{k-1}} a^{k-2}$. Since $f(\frac{k}{k-1}) = f'(\frac{k}{k-1}) = 0$, it follows that $a = \frac{k}{k-1}$ is a double root of $f(a) = 0$. Now if k is an even number , then $f''(a) > 0$ (for $a > 0$ or $a < 0$) . Therefore $f(a) > 0$ and thus Inequality (7) holds. On the other hand if k is an odd number and $a > 0$, then $f''(a) > 0$ and consequently, $f(a) > 0$ and thus Inequality (7) holds . For $a < 0$ and k is an odd number , we have $f''(a) < 0$. Since $f'\left(-\frac{k}{k-1}\right) = 0$,it follows that $f\left(-\frac{k}{k-1}\right) = 2$ is the maximum value of $f(a)$. Hence $f(a) \leq 2$. This implies

$$\frac{(k-1)^{k-1}}{k^k}a^k \leq a + 1$$

and

$$\frac{a+1}{a^k} \leq \frac{(k-1)^{k-1}}{k^k} \ .$$

If we put $a = -|a|$, then (7) follows.

Lemma 2.2 . *Let ξ be real root of Eq.(6) . The following statements hold true :*

(i) c increases as ξ increases if either $0 < \xi < \frac{k}{k-1}$ or $\xi < 0$ and k is an odd number .

(ii) c increases as ξ decreases if either $\xi > \frac{k}{k-1}$ or $\xi < 0$ and k is an even number .

Proof. Since ξ is a root of Eq. (6) , we have $c = (\xi - 1)/\xi^k$ and

$$\frac{dc}{d\xi} = \xi^{-k-1}\left[((1-k)\xi) + k\right]. \tag{8}$$

(i) If $0 < \xi < \frac{k}{k-1}$ or $\xi < 0$ and k is an odd number , then $\frac{dc}{d\xi} > 0$ and ξ increases with c .

(ii) In this case we get from (8) that $\frac{dc}{d\xi} < 0$. Thus ξ decreases as c increases .

Lemma 2.3 *Let $m \neq n$ be positive integers and $f(\theta) = \sin m\theta / \sin n\theta$. Then*

(i) $f(\theta)$ decreases in $(0, \frac{\pi}{m})$ for $m > n$ and in $(0,\pi)\backslash\{0, \frac{\pi}{n}, \frac{2\pi}{n}, ..., \pi\}$ for $m = n+1$.

(ii)$f(\theta)$ increases in $(0, \frac{\pi}{n})$ for $m < n$ and in $(0,\pi)\backslash\{0, \frac{\pi}{n}, \frac{2\pi}{n}, ..., \pi\}$ for $m = n-1$.

Proof. We have that

$$f'(\theta) = (m \cos m\theta \sin n\theta - n \cos n\theta \sin m\theta) / \sin^2 n\theta \ .$$

Letting

$$G(\theta) = 2\left(m \cos m\theta \sin n\theta - n \cos n\theta \sin m\theta\right),$$

then

$$G(\theta) = (m-n)\sin(m+n)\theta + (m+n)\sin(m-n)\theta, \tag{9}$$

and

$$G'(\theta) = 2\left(n^2 - m^2\right)\sin m\theta \sin n\theta \,. \tag{10}$$

(i) : Since $m > n$, we have that $G'(\theta) < 0$ *for* $0 < \theta < \frac{\pi}{m}$ *, but* $G(0) = 0.$ *Thus* $f'(\theta) < 0$ *in* $(0, \frac{\pi}{m})$ and $f(\theta)$ is a decreasing function in $(0, \frac{\pi}{m})$. Now , let $m = n+1$ Eq.(9) becomes

$$G(\theta) = \sin(2n+1)\theta - (2n+1)\sin\theta \ .$$

Since $G(\theta) < 0$ for $0 < \theta < \pi$, it follows that $f'(\theta) < 0$ and $f(\theta) = \frac{\sin(n+1)\theta}{\sin n\theta}$ is a decreasing function on $0 < \theta < \pi$. The second part (ii) can be proved similarly .

Lemma 2.4 *Let* $k > 1$ *be positive integer , and* $0 < \theta < \pi$. *If* $|a| \geq \frac{k}{k-1}$, *then*

$$\frac{\sin k\theta}{\sin(k-1)\theta} - |a| = 0 \tag{11}$$

has exactly $(k-2)$ *roots* $\theta_i\,(i = 2, 3, ..., k-1)$ *such that* $\theta_i \in I_i = \left(\frac{(i-1)\pi}{k-1}, \frac{i\pi}{k}\right) = (a_i, b_i)$. *For* $|a| < \frac{k}{k-1}$, *Eq.(11) has an additional root in the interval* $\left(0, \frac{\pi}{k}\right)$.

Proof. By Lemma 2.1, the function $f(\theta) = \frac{\sin k\theta}{\sin(k-1)\theta}$ decreases in $\left(0, \frac{\pi}{k}\right)$. If $|a| < \frac{k}{k-1}$, then from $\lim_{\theta \to 0^+} f(\theta) = \frac{k}{k-1}$, it follows that there exists $\varepsilon(k) > 0$ such that $f(\varepsilon) - |a| > 0$ and $f\left(\frac{\pi}{k}\right) - |a| < 0$. Therefore, Eq.(11) has only one root in $\left(0, \frac{\pi}{k}\right)$. For $|a| \geq \frac{k}{k-1}$, it is clear that $f(0^+) < \frac{k}{k-1} \leq |a|$ and $f\left(\frac{\pi}{k}\right) < \frac{k}{k-1} \leq |a|$. Since $f(\theta)$ decreases in $\left(0, \frac{\pi}{k}\right)$, it follows that $f(\theta) - \frac{k}{k-1} = 0$ has no roots in $\left(0, \frac{\pi}{k}\right)$.

Now we show that every interval $I_i\,(i = 2, 3, ..., k-1)$ contains only one root of Eq.(11) . We notice that for any $|a|$, there exists a number $\tau \in I_i$ such that $f(\tau) - |a| > 0$ (this follows from $\lim_{\theta \to a_i} f(\theta) = \infty$), also $f(b_i) - |a| < 0$. Therefore, I_i contains at least one root of equation Eq.(11) . To complete the proof , it is enough to show that $f(\theta)$ decreases in any interval I_i . Following the steps of the proof of Lemma 2.3. , where $m = k$ and $n = k-1$,then Eq.(10) becomes

$$G'(\theta) = -2(2k-1)\sin k\theta \sin(k-1)\theta\ . \tag{12}$$

In the interval I_i we have

$$(i-1)\pi < \frac{(i-1)k\pi}{k-1} < k\theta < i\pi$$

and

$$(i-1)\pi < \frac{(i-1)k\pi}{k-1} < k\theta < i\pi$$

and

$$(i-1)\pi < (k-1)\theta < \frac{(k-1)i\pi}{k} < i\pi$$

Therefore, $\frac{\sin k\theta}{\sin(k-1)\theta} > 0$ in $I_i\,(i = 2, 3, ..., k-1)$ and $G'(\theta) < 0$. It follows that $f'(\theta) < 0$ in I_i and $f(\theta)$ decreases in I_i and the root of $f(\theta) - |a| = 0$ is unique in I_i .

Lemma 2.5 *The number of real roots of Eq.(6) is given as follows :*

(i) Two (one) roots for $c < 0$ *and* k *is an even (odd) number .*

(ii) No roots (one root) for $c > \beta_k = \frac{(k-1)^{k-1}}{k^k}$ *and* k *is an even (odd) number .*

(iii) Two (three) roots for $c < \beta_k$ *and* k *is an even (odd) number .*

Proof. The validity of this lemma can be shown graphically from the graphs of $\eta = c\mu^k$ and $\eta = \mu - 1$. Note that when $c = \beta_k$, the line $\eta = \mu - 1$ is tangent to the curve $\eta = c\mu^k$ The analytical proof will appear in the sequel. If we write complex roots of Eq.(6) in the form $\mu = r(\cos\theta + i\sin\theta)$, we get the following equations

$$cr^k \cos k\theta - r\cos\theta + 1 = 0 \tag{13}$$

$$cr^k \sin k\theta - r\sin\theta = 0. \tag{14}$$

From Eqs.(13) and (8) , it follows that

$$r = \frac{\sin k\theta}{\sin(k-1)\theta} \tag{15}$$

$$c = \frac{\sin\theta}{r^{k-1}\sin k\theta} \ . \tag{16}$$

Let $\theta_i\,(i = 1, 2, ..., k-1)$ be the solutions of Eq.(11) , it follows from (15) and (16) that

$$c = \frac{\sin\theta_i}{|a|^{k-1}\sin k\theta_i} \ . \tag{17}$$

The number of the complex roots of Eq.(6) corresponding to θ_i will be discussed in the following .

Lemma 2.6 . *The number of complex roots of Eq.(6) equals the number of solutions of Eq.(11) .*

Proof. It is clear that if θ_i is a solution of Eq.(11) in $(0,\pi)$, then $2\pi-\theta_i$ is also a solution. This means that θ_i and $2\pi-\theta_i$ are corresponding to complex roots of Eq.(6) that are conjugate pairs . Thus if Eq.(14) has N roots in $(0,\pi)$, then it has $2N$ roots in $(0,2\pi)$.

Case 1 : $c<0$. By Lemma 2.2, Eq.(11) has $k-1$ roots for $|a|<\frac{k}{k-1}$ and $k-2$ roots for $|a|>\frac{k}{k-1}$. We choose those values of θ_i such that $c<0$. From Eq.(16) we conclude that $c<0$ when $\sin k\theta_i<0$. From Lemma 2.2. we have

$$\frac{(i-1)\,\pi}{k-1}<\theta_i<\frac{i\pi}{k}\ ,$$

where $i=1,2,...,k-1$ for $|a|<\frac{k}{k-1}$ and $i=2,3,...,k-1$ for $|a|>\frac{k}{k-1}$. Hence

$$(i-1)\,\pi<\frac{k\,(i-1)\,\pi}{k-1}<k\theta_i<i\pi\ . \tag{18}$$

If k is even , then $\sin k\theta_i<0$ for $i=2,4,6,...,k-2$ and the number of roots in $(0,\pi)$ is $\frac{k-2}{2}$ and consequently $k-2$ in $(0,2\pi)$. If k is odd , then $\sin k\theta_i<0$ for $i=2,4,6,...,k-2$ and the number of roots in $(0,2\pi)$ is $k-1$.

Case 2 may be proved similarly.

3 Main Results

In what follows we give theorems which guarantee that all roots of Eq.(6) are inside the disk $|\mu|<|a|$.

Theorem 3.1 *Let $k>1$ be an integer, and c an arbitrary real . Then all complex roots of Eq.(6) are inside the disk $|\mu|<|a|$ if and only if*

$$|c|>\frac{\left(1+a^2-2\,|a|\cos\phi\right)^{\frac{1}{2}}}{|a|^k}\ , \tag{19}$$

where ϕ is the solution in $I=\left(\frac{(k-2)\pi}{k-1},\frac{(k-1)\pi}{k}\right)$ of $\frac{\sin k\theta}{\sin(k-1)\theta}=|a|$.

Proof. The complex roots of (6) , $\mu = r(\cos\theta + i\sin\theta)$,where $0 < \theta < 2\pi$ and $r > 0$, can be obtained from (13) and (11) . Applying simple operations on (13) and (11) , one obtains the following :

$$|c| = \frac{1}{r^{k-1}}\left|\frac{\sin\theta}{\sin k\theta}\right| .$$

The level curves of $F(c, r, \theta) = |c|\, r^{k-1} - \left|\frac{\sin\theta}{\sin k\theta}\right|$ are given by θ =constant . Assume that $|a| < \frac{k}{k-1}$ (the case $|a| > \frac{k}{k-1}$ can be treated similarly) and $\theta_1 < \theta_2 < ... < \theta_{k-1}$ are the solutions of $|a| = \frac{\sin k\theta}{\sin(k-1)\theta}$ in $(0, \pi)$. The equations of the level curves in the (r, c) plane are given by

$$|c(r)| = \frac{1}{r^{k-1}}\left|\frac{\sin\theta_i}{\sin k\theta_i}\right| , \ i = 1, 2, ..., k-1 .$$

It is clear that $|c(r)|$ is a decreasing function. Since $|c(|a|)| = \frac{\left(1+a^2-2|a|\cos\theta\right)^{\frac{1}{2}}}{|a|^k}$ is an increasing function of θ in $(0, \pi)$, $\left(\frac{d|c|}{d\theta} > 0\right)$, then for $r = |a|$ and $\theta_1 < \theta_2 < ... < \theta_{k-1}$, we get the corresponding values

$$|c_i(|a|)| = \frac{(1 + a^2 - 2|a|\cos\theta_i)^{\frac{1}{2}}}{|a|^k} , \ i = 1, 2, ..., k-1$$

that satisfy $|c_1| < |c_2| < ... < |c_{k-1}|$.

It is not difficult to show that $r < |a|$ if and only if $|c| > |c_{k-1}(|a|)|$. In fact since $|c(r)|$ is a decreasing function, $|c(r)| > |c_{k-1}(|a|)|$ implies $r < |a|$. Also $r < |a|$ implies that $|c(r)| > c_{k-1}(|a|)$, for if we assume the contrary , we obtain $r \geq |a|$ .This is a contradiction. Since $c_{k-1}(|a|)$ is the corresponding value to $\theta_{k-1} = \phi$ that lies in $\left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$, the proof is complete .

Theorem 3.2 . *Let* $k > 1$ *be an odd integer and* $c > 0$. *Then all roots of Eq.(6) are inside the disk* $|\mu| < |a|$ *if and only if*

$$c > \frac{|a| + 1}{|a|^k} \tag{20}$$

$\frac{|a|+1}{|a|^k}$

Proof. First we deal with the roots of Eq.(6) . Set $F(\mu) = c\mu^k - \mu + 1$ and note that $F(0) > 0$, $F(-\infty) < 0$ and $F''(\mu) < 0$ for $\mu < 0$ it follows that Eq.(6) has one negative root. If $c = \beta_k = \frac{(k-1)^{k-1}}{k^k}$, then Eq.(6) has a double root $\mu = \frac{k}{k-1}$ Putting $F_t(\mu) = \beta_k \mu^k - \mu + 1$, we see that if $c > \beta_k$ then $F\left(\frac{k}{k-1}\right) > F_t\left(\frac{k}{k-1}\right) = 0$, and so Eq.(6) has no positive roots . If $c < \beta_k$, then $F\left(\frac{k}{k-1}\right) < F_t\left(\frac{k}{k-1}\right) = 0$, and so Eq.(6) has exactly two positive roots .

Now , consider the equation

$$F_a(\mu) \equiv c_a \mu^k - \mu + 1 = 0 \tag{21}$$

where
$c_a = \frac{|a|+1}{|a|^k}$

Case 1 : $c_a > \beta_k$. There are no positive roots and $c > c_a$ is a sufficient condition.

Case 2 : $c_a < \beta_k$.For $c_a < c < \beta_k$, Eq.(6) has two positive roots $\xi_1 < \frac{k}{k-1}$, $\xi_2 > \frac{k}{k-1}$ and a negative root ξ_3 . Lemma 2.1. implies that $|\xi_3| < |a|$.

To prove that ξ_1 and ξ_2 are less than $|a|$, it is enough to show that $\xi_2 < |a|$ for $c_a < c < \beta_k$. For this aim we show that $F(\mu) > 0$ for $\mu \geq |a|$. In fact , for $\mu \geq |a|$, we have that $F_a'(\mu) \geq \frac{k(1+|a|)}{|a|} - 1 > 0$, and $F_a(|a|) > 0$, and so $F_a(\mu) > 0$. Since $F(\mu) > F_a(\mu)$, then $F(\mu) > 0$. Therefore the real roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if $c > c_a$. Theorem 3.1 implies that all complex roots of Eq.(6) are inside the unit disk $|\mu| < |a|$ if and only if

$$|c| > \frac{\left(1 + a^2 - 2|a|\cos\phi\right)^{\frac{1}{2}}}{|a|^k},$$

where ϕ is defined in Theorem 3.1

Theorem 3.3 . *Let $k > 1$ be an odd integer and $c < 0$. Then all roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if*

$$c < -\frac{\left(1 + a^2 - 2|a|\cos\phi\right)^{\frac{1}{2}}}{|a|^k}, \tag{22}$$

where ϕ is defined in Theorem 3.1

Proof. Since $F(\mu) = c\mu^k - \mu + 1$ satisfies the following properties : $F(0)F(1) < 0$, $F(\mu) < 0$ for $\mu \geq 1$, $F(\mu) > 0$ for $\mu \leq 0$, and $F'(\mu) < 0$ for $0 < \mu < 1$, then Eq.(6) has one positive root $\xi < 1$. If $|a| > 1$, then the positive root $\xi < |a|$. It follows from Theorem 3.1 that all the roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if (12) holds . If $|a| < 1$, then the equation

$$\frac{|a| - 1}{|a|^k}\mu^k - \mu + 1 = 0$$

has the positive root $\mu = |a|$. Applying Lemma 2.1, we conclude that the positive root ξ of Eq.(6) satisfies $\xi < |a|$ if $c < \frac{|a|-1}{|a|^k}$. Using Theorem 3.2. and the fact that

$$-\frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} < \frac{|a| - 1}{|a|^k}$$

for $|a| < 1$, yields the desired result.

Theorem 3.4 . *Let $k > 1$ be an even integer and $c > 0$. Then the following statements hold true :*

(i) if $c > \frac{(k-1)^{k-1}}{k^k} = \beta_k$, then all roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if

$$c > \frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} \tag{23}$$

(ii) if $|a| > \frac{k}{k-1}$, then all roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if

$$\frac{(1 + a^2 - 2a\cos\phi)^{\frac{1}{2}}}{|a|^k} < c < \beta_k \tag{24}$$

where ϕ is defined in Theorem 3.1

Proof. (i) If $c > \beta_k$, then Eq.(6) has no real roots and Theorem 3.1 implies that all complex roots are inside the disk $|\mu| < |a|$ if and only if (23) holds .

(ii) If $c < \beta_k$, then Eq.(6) has two positive roots $0 < \xi_1 < \frac{k}{k-1}$ and $\xi_2 > \frac{k}{k-1}$. The proof is similar to that of Theorem 3.3. For $|a| < \frac{k}{k-1}$, it follows that $\xi_2 > |a|$. For $|a| > \frac{k}{k-1}$, the equation

$$\frac{|a|-1}{|a|^k}\mu^k - \mu + 1 = 0$$

has the positive root $\mu = |a|$. Applying Lemma 2.1. we conclude that if $\frac{|a|-1}{|a|^k} < c < \beta_k$, then $|a| > \xi_2 > \frac{k}{k-1}$. Since

$$\frac{|a|-1}{|a|^k} < \frac{\left(1 + a^2 - 2\,|a|\cos\phi\right)^{\frac{1}{2}}}{|a|^k} \quad .$$

it follows by Theorem 3.1 that all the roots of Eq.(6) are inside the disk $|\mu| < |a|$ if and only if (23) holds .

Theorem 3.5 . *Let* $k > 1$ *be an even integer and* $c < 0$. *Then all roots of Eq.(6) are inside the disk* $|\mu| < |a|$ *if and only if*

$$c < -\frac{|a|+1}{|a|^k}$$

Proof. For $F(\mu) = c\mu^k - \mu + 1$,we notice that $F(0)\,F(1) < 0$ and for $0 \leq \mu \leq 1$ we have $F'(\mu) = kc\mu^{k-1} - 1 < 0$. Therefore Eq.(6) has exactly one positive root in $(0,1)$. Ia is clear that $F(\mu) < 0$ for $\mu > 1$, and so Eq.(6) has no other positive roots. Similarly it is not difficult to show that Eq.(6) has only one negative root . Consider the equation

$$-\frac{|a|+1}{|a|^k}\mu^k - \mu + 1 = 0$$

that has a negative root $\mu = -|a|$. If $c < -\frac{|a|+1}{|a|^k}$, then Lemma 1. implies that the negative root ξ_1 satisfies $|\xi_1| < |a|$. Clearly that the positive root $\xi_2 < |a|$ for $|a| > 1$. If $|a| < 1$, then from $F(-\infty) < 0$ and $F(-\xi_2) = c\xi_2^k + 1 + \xi_2 = 2\xi_2 > 0$, we conclude that $\xi_1 < -\xi_2$, hence $\xi_2 < |\xi_1| < |a|$. The proof would be complete if we apply Theorem 3.1 to complex roots and observing that

$$-\frac{|a|+1}{|a|^k} < -\frac{(1+a^2-2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} \quad .$$

Next we present necessary and sufficient conditions for the roots of Eq.(5) to be inside the unit disk .

Theorem 3.6 . *Let* $k>1$ *be an odd integer . Then all the roots of Eq.(6) are inside the unit disk if and only if one of the following conditions hold.*
(i) $b>a+1$ *and* $a>0$.
(ii) $b<a-1$ *and* $a<0$.
(iii) $b<-\ (1+a^2-2a\cos\phi)^{\frac{1}{2}}$ *and* $a>0$.
(iv) $b>\ (1+a^2-2|a|\cos\phi)^{\frac{1}{2}}$ *and* $a<0$,
where ϕ *is the solution in* $I=\left(\frac{(k-2)\pi}{k-1},\frac{(k-1)\pi}{k}\right)$ *of* $\frac{\sin k\theta}{\sin(k-1)\theta}=|a|$.

Proof. We recall first that the roots of Eq.(5) are inside the unit disk if and only if the roots of Eq.(6) are inside the disk $|\mu|<|a|$.

(i) Since $a>0$ and $b>0$, then $c>0$ and Theorem 3.2. implies that all roots of Eq.(5) are inside the unit disk if and only if

$$c=\frac{b}{a^k}>\frac{a+1}{a^k}$$

i.e. $b>a+1$.

(ii) Here also we have that $c>0$. Theorem 3.2 implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k}>\frac{-a+1}{(-a)^k}=\frac{a-1}{a^k}$$

or $b<a-1$. Note that $a^k<0$.

(iii) Since $a>0$ and $b<0$, then $c<0$ and Theorem 3.3 implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k}<-\frac{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}{a^k} \quad .$$

(iv) Since $a<0$ and $b>0$, then $c<0$ and Theorem 3.3. implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} < -\frac{(1+a^2-2|a|\cos\phi)^{\frac{1}{2}}}{-a^k}$$

or equivalently

$$b > \left(1+a^2-2|a|\cos\phi\right)^{\frac{1}{2}} .$$

Theorem 3.7 . *Let $k>1$ be an even integer . Then all the roots of Eq.(6) are inside the unit disk if and only if one of the following conditions holds :*

(i) $b < -|a| - 1$

(ii) $b > (1+a^2-2|a|\cos\phi)^{\frac{1}{2}}$ *and either* $b > \frac{(k-1)^{k-1}}{k^k}a^k$ *or* $b < \frac{(k-1)^{k-1}}{k^k}a^k$ *and* $|a| > \frac{k}{k-1}$,

where ϕ *is the solution in* $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$ *of* $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$.

Proof . (i) Since $b<0$, then $c<0$ and Theorem 3.5. implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} < -\frac{|a|+1}{|a|^k} .$$

The result follows if we notice that $a^k > 0$.

(ii) Since $b>0$ and k is an even integer , then $c = \frac{b}{a^k} > 0$ Applying Theorem 3.5. (i) we obtain that if $\frac{b}{a^k} > \frac{(k-1)^{k-1}}{k^k}$, then all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} > \frac{(1+a^2-2|a|\cos\phi)^{\frac{1}{2}}}{-a^k} .$$

Thus the first part of (iii) follows directly . The second part of (iii) can be proved similarly using Theorem 3.4. (ii) .

Using Theorems 3.6 and 3.7 we have th following fundamental result..

Theorem 3.8 . *Let* $k > 1$ *be an integer and let* a, b *be arbitrary reals . Equation (3) is asymptotically stable if and only if Conditions (i) - (iv) of Theorem 3.6. holds for even* k *or Conditions (i)- (ii) of Theorem 3.7. for odd* k *.*

Figures 1 and 2 show the domains of (a, b) *for which the roots of* $b\lambda^k - a\lambda + 1 = 0$ *with* $a \neq 0$ *and* $k > 1$ *, are inside the unit disk and consequently for which the difference equation (3) is asymptotically stable .* Finally we establish the counterpart of the result of Levin and May for difference equations of advanced type.

Theorem 3.9 *Let a=1 and b is an arbitrary real number, and* $k > 1$ *is a positive integer. Then the zero solution of Eq(3) is asymptotically stable if and only if either* $b > 2$ *or*

Corollary 3.10

$$b < -2|\cos[(2k-5)\pi/(4k-2)]| \tag{25}$$

Proof. From Theorems 3.6 and 3.7, the zero solution of Eq(3) is asymptotically stable if and only if

(i) $b > 2$ or $b < -\sqrt{(2 - 2\cos\phi)}$ if k is odd.

(ii) $b < -2$ or $b > \sqrt{(2 - 2\cos\phi)}$ and $b > \frac{(k-1)^{k-1}}{k^k}$

where ϕ is the solution in $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$ of $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$. We first observe that $b > 2$ implies $b > \sqrt{(2 - 2\cos(\phi))}$ which in turn implies that $b > \frac{(k-1)^{k-1}}{k^k}$. Furthermore, $b < \sqrt{(2 - 2\cos\phi)}$ implies that $b < -2$. Hence the zero solution of Eq(3) is asymptotically stable if and only if $b > 2$ or

$$b < -\sqrt{(2 - 2\cos\phi)} = -2\sin(\phi). \tag{26}$$

Since $\sin(k\phi) = \sin(k-1)\phi$, it follows that either $(k-1)\phi + k\phi = (2n+1)\pi$ or $(k-1)\phi = k\phi + 2n\pi, n \in Z$. But the second option is invalid for $\phi \in I$. The first option yields $n = k - 2$. Hence

$$\phi = \left(\frac{((2k-3)\pi)}{2k-1}\right). \tag{27}$$

For $k = 2, \phi = \pi/3$ lies in the first quadrant and for $k > 2, \phi$ lies in the second quadrant. Hence

$$-2\sin(\phi) = -2|\cos\left(\frac{((2k-5)\pi)}{4k-2}\right) \tag{28}$$

which establishes the theorem.

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