PERIODIC POINTS OF THE FAMILY OF TENT MAPS

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1. INTRODUCTION. Of interest in this article is the dynamical behavior of the one-parameter family of maps $T_\omega(x) = \omega\,(1/2 - |x - 1/2|)$, $x \in \mathbb{R}$, $\omega > 0$. For each such ω, the map T_ω, which can be described piecewise by

$$T_\omega(x) = \begin{cases} \omega x & \text{if } x \leq 1/2 \\ \omega(1-x) & \text{if } x > 1/2 \end{cases},$$

is continuous, linear on each of the intervals $(-\infty, 1/2]$ and $[1/2, \infty)$ (with respective slopes ω and $-\omega$,) and has the points $(0,0)$ and $(1,0)$ in its graph. The figure below, which shows the graph of the restriction of T_2 to the unit interval $I = [0,1]$, illustrates why the maps of the family T_ω are referred to as tent maps.

If $x \in \mathbb{R}$, the set $\{T_\omega^n(x)\}_{n=0}^{\infty}$, where T_ω^n denotes the n^{th} iterate of T_ω, is called the *orbit* of x. If $T_\omega(x) = x$ then x is called a fixed point, and if $T_\omega^k(x) = x$ for some positive integer k, then x (and its orbit) is called *periodic* or *cyclic.* If in addition $T_\omega^n(x) \neq x$, for $1 \leq n < k$, x is said to have (prime) period k. The dynamical properties of a map T_ω are to some extent determined by the nature of the orbits of points in its domain. For example, that the dynamical behavior of T_ω when $0 < \omega < 1$ is docile can be inferred from the following facts concerning the orbits of T_ω:

(i) $T_\omega(0) = 0$, so 0 is a fixed point of T_ω,
(ii) $T_\omega(1) = 0$, so the point 1 is eventually fixed, and
(iii) for $x \neq 0, 1$, $T_\omega^n(x) \to 0$ as $n \to \infty$.

(The dynamical behavior of T_1 is even easier to understand: if a point is not already fixed it is sent to one by an application of T_1.)

The behavior of T_ω for all other values of ω is far more complicated as we now see. First, let's assume that $\omega > 2$. In this case, by imitating the essential steps of the proof of Thm. ?? in [**?**], we can find a Cantor set $C_\omega \in [0,1]$, invariant under T_ω (that is, for which $T_\omega(C_\omega) = C_\omega$) with these properties:

(1) If $x \notin C_\omega$ then $T^n_\omega(x) \to -\infty$ as $n \to \infty$;
(2) The restriction $T_{\omega|C_\omega}$ of T_ω to C_ω is chaotic.

The term chaotic will be used throughout this paper in the sense of Devaney. Specifically, a continuous dynamical system $f : M \to M$ on the metric space (M, d) is called *chaotic* if:

C1 It is topologically transitive, i.e. given nonempty open sets $U, V \subset M$ there is an $n > 0$ for which $f^n(U) \cap V$ is not empty;
C2 The periodic points are dense in M; and
C3 f has sensitive dependence on initial conditions, i.e.

That T_ω is chaotic on C_ω when $\omega > 2$ is established by finding a symbolic model of $T_{\omega|C_\omega}$ on which the three conditions above are easily verified (see [**?**].) As we will be concerned with periodic ponts, we mention here that this symbolic model also informs us that T_ω has periodic points of all periods and allows us to count their number.

Now assume, as we will for the rest of this paper, that $1 < \omega \leq 2$. We begin with the simple observations that $T_\omega(0) = 0$, $T_\omega(1) = 0$, and that if $x < 0$ or $x > 1$ then $T^n_\omega(x) \to -\infty$ (monotonically) as $n \to \infty$. Consequently, outside the interval $(0, 1)$ the dynamical behavior of T_ω is understood (and very simple.) Also worthy of notice is the fact that the interval $I_\omega = [\omega - \omega^2/2, \omega/2]$ remains invariant under the action of T_ω. Indeed, if $\omega - \omega^2/2 \leq x \leq 1/2$, then $\omega - \omega^2/2 \leq x < T_\omega(x) = \omega x \leq \omega/2$ because $\omega > 1$; the condition $1/2 < x \leq \omega/2$ forces $\omega - \omega^2/2 \leq T_\omega(x) = \omega(1-x) \leq \omega/2$. Furthermore, the points of $(0, 1) \setminus I_\omega$ are eventually mapped into I_ω: For $0 < x < \omega - \omega^2/2$, let n_0 be the smallest nonnegative integer for which $\omega^{n_0+1}x \geq \omega - \omega^2/2$. Then $\omega^{n_0}x < \omega - \omega^2/2 < 1/2$ implies that $\omega - \omega^2/2 \leq T^{n_0+1}_\omega(x) < \omega/2$. On the other hand, if $\omega/2 < x < 1$ then $0 < T_\omega(x) = \omega(1 - x) < \omega - \omega^2/2$. It follows from these considerations that to get a more complete picture of the dynamics of T_ω for $1 < \omega \leq 2$ it is necessary to shed light on the behavior of the restriction $T_{\omega|I_\omega}$. To this task we devote the rest of this paper.

2. $1 < \omega \leq 2$. From now on we confine our attention to $T_{\omega|I_\omega}$, $1 < \omega \leq 2$ which we will simply denote by T_ω. We will further simplify our writing by finding models of these T_ω's in a common interval as follows: Let I denote $[0, 1]$. We define the map $\overline{T}_\omega : I \to I$ as $\overline{T}_\omega = h^{-1}_\omega \circ T_\omega \circ h_\omega$, where $h_\omega : I \to I_\omega$ is given by

$$h_\omega(x) = (\omega - \frac{\omega^2}{2})(1 - x) + \frac{\omega}{2}x.$$

Because h_ω is a homeomorphism, the relation between the maps T_ω and $\overline{T}_\omega$ is that of *conjugacy*. We justify the introduction of the $\overline{T}_\omega$'s by reminding the reader that conjugate maps share dynamical properties. In particular, T_ω is chaotic if and only if $\overline{T}_\omega$ is. The graph of a typical $\overline{T}_\omega$ is shown in figure 2.

We now claim that $\overline{T}_\omega$ (and thus T_ω) is chaotic if $\sqrt{2} < \omega \leq 2$. To verify this claim all we need to check is that corresponding to any nondegenerate interval $J \subset I$ there is an integer n for which $\overline{T}^n_\omega(J) = I$, for this condition obviously implies that $\overline{T}_\omega$ is topologically transitive and, as shown in [**?**], on intervals **C1** implies both **C2** and **C3**. Thus, we just prove the following:

Lemma 1. *If $\sqrt{2} < \omega \leq 2$ and $J \subset I$ is a non degenerate interval then there is a positive integer n for which $\overline{T}^n_\omega(J) = I$.*

Proof. Set $a = \omega/(\omega+1)$, the fixed point of $\overline{T}_\omega$. We leave to the reader the verification that if J contains a in its interior then J is eventually mapped onto I.

If $a \notin \operatorname{int}(J)$ there are several possibilities. First, we could have $J = [a, b]$, $a < b$. Then $\overline{T}_\omega(J) = [c, a]$, $c < a$, and if $(\omega-1)/\omega \notin [c, a]$ then $|[c, a]| = \omega|[a, b]|$. (Here, $|\cdot|$ denotes length.) Further applications of $\overline{T}_\omega$ eventually yield m such that $\overline{T}^m_\omega(J) = [d, a]$ and $(\omega-1)/\omega \in [d, a]$. Therefore, $\overline{T}^{m+1}_\omega(J) \supset [a, 1]$, $\overline{T}^{m+2}_\omega(J) \supset [0, a]$, $\overline{T}^{m+3}_\omega(J) \supset [2-\omega, 1]$ and $[2-\omega, 1]$ has a in its interior because $2-\omega < a = \omega/(\omega+1)$ for $\omega > \sqrt{2}$. But we are now in the situation contemplated in the previous paragraph. A second possibility is that $J = [c, a]$, $c < a$. In this case, either $\overline{T}_\omega(J) = [a, b]$, $a < b$ (the previous case), or plainly $\overline{T}_\omega(J)$ contains a in its interior. Again, we are in one of the previous two cases. Finally, it could happen that $a \notin J$. If in addition $a \notin \overline{T}_\omega(J)$ then $|\overline{T}^2_\omega(J)| \geq \frac{\omega^2}{2}|J| > |J|$. Therefore, repeated application of $\overline{T}_\omega$ eventually yields m such that $a \in \overline{T}^m_\omega(J)$ and again we get the desired conclusion by invoking the earlier cases. $\square$

Now assume that $1 < \omega \leq \sqrt{2}$. It will be convenient to examine the map $\overline{T}_{\omega}^{2}$. The graph of one such map is sketched in the figure below.

It can be checked by straightforward computations that the interval $Q_1 = [0, \overline{T}_{\omega}^{2}(0)] = [0, \omega^2 - \omega]$ remains invariant under the action of $\overline{T}_{\omega}^{2}$, i.e. that if $x \in [0, \overline{T}_{\omega}^{2}(0)]$ then $\overline{T}_{\omega}^{2}(x) \in [0, \overline{T}_{\omega}^{2}(0)]$. Furthermore, if we define the new function $F_\omega : I \to I$ by $F_\omega = g^{-1} \circ \overline{T}_{\omega}^{2} \circ g$, where $g : I \to Q_1$ is the homeomorphism $g(x) = (1-x)\overline{T}_{\omega}^{2}(0)$ (so F_ω and $(\overline{T}_{\omega}^{2})_{|Q_1}$ are conjugate) then, by another elementary computation, we see that F_ω (a piecewise linear function) has the graph shown in Figure 4 below. It is apparent from this

graph that $F_\omega = \overline{T}_{\omega}^{2}$. Therefore, we have proved the following:

Theorem 1. *$(\overline{T}_{\omega}^{2})_{|Q_1}$ is conjugate to $\overline{T}_{\omega^2}$.*

Using Theorem 1 along with some of our previous results we can draw some conclusions. For example, if $\omega < \sqrt{2}$ and $\omega^2 > \sqrt{2}$ then $\overline{T}_{\omega|Q_1}^{2}$, being conjugate to $\overline{T}_{\omega^2}$, is chaotic. Furthermore, for such ω, if we set $Q_2 = \overline{T}_\omega(Q_1)$ then Q_1 and Q_2 are disjoint intervals, $Q_2 = \overline{T}_\omega(Q_1)$ (so $Q_1 \cup Q_2$ is $\overline{T}_\omega$-invariant), $\overline{T}_\omega$ is chaotic on $Q_1 \cup Q_2$, and every point in $I \setminus Q_1 \cup Q_2$ is eventually mapped by $\overline{T}_\omega$ into $Q_1 \cup Q_2$. In other words, $\left(\overline{T}_\omega\right)_{|Q_1 \cup Q_2}$ is a chaotic attractor.

More generally, the following result can be obtained from judicious, repeated application of Theorem 1.

Corollary 1. *Assume that $1 < \omega^k < \sqrt{2}$, $1 \leq k \leq n-1$, and $\omega^n > \sqrt{2}$. Then there exist disjoint subintervals $Q_1, Q_2 \ldots Q_{2^n}$ of I permuted by $\overline{T}_\omega$ such that $Q = Q_1 \cup Q_2 \cup \ldots \cup Q_{2^n}$ is a chaotic attractor of $\overline{T}_\omega$. That is, $\overline{T}_{\omega|Q}$ is chaotic, and every point in $I \setminus Q$ is eventually mapped by $\overline{T}_\omega$ to Q.*

Similarly, if $\omega = \sqrt{2}$ then $Q_1 = [0, a]$ where $a = \omega/(\omega+1)$ is the fixed point of $\overline{T}_\omega$, $Q_2 = \overline{T}_\omega(Q_1) = [a, 1]$ (so $Q_1 \cup Q_2 = I$), and $\overline{T}_\omega$ permutes Q_1 and Q_2. Applying Theorem 1 once again we obtain:

Corollary 2. *$\overline{T}_{\sqrt{2}}$ is chaotic on $I = [0,1]$.*

3. PERIODIC POINTS. In the previous section we learnt that when $1 < \omega \leq 2$ there is a set $Q_\omega \subseteq I_\omega$ on which T_ω is chaotic. We know then, from **C2** of the definition of chaos, that there is a dense (hence infinite) subset of Q_ω consisting of T_ω-periodic points. However, in contrast to the case $\omega > 2$, we don't have as yet any specific information about them. Myriad interesting questions can be asked regarding the periodic points of T_ω, and we devote this section to answering some of them.

Naturally, when considering such questions, the well-known theorem of Šarkovskii which establishes that the presence of certain periodic orbits forces the existence of certain other periodic orbits will be very useful. The precise statement of Šarkovskii's theorem is the following:

Theorem 2 ([**?**])**.** *Let $(\mathbb{N}, \prec)$ be the following transitive ordering of the positive integers: $3 \prec 5 \prec 7 \prec \ldots \prec 2\cdot 3 \prec 2\cdot 5 \prec 3\cdot 7 \prec \ldots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec 2^2\cdot 7 \prec \ldots \prec 2^n\cdot 3 \prec 2^n\cdot 5 \prec 2^n\cdot 7 \prec \ldots\ldots \prec 2^n \prec 2^{n-1} \prec \ldots \prec 2^2 \prec 2^1 \prec 2^0$. Assume that J is an interval and that $f : J \to J$ is continuous. If f has a periodic point of period k then it has a periodic point of period n for all $k \prec n$.*

We will find it convenient to make the following definition: $\overline{\omega}_k = \inf\{\omega > 1 | T_\omega \text{ has a point of period } k\}$. A few of the $\overline{\omega}_k$'s can be found by explicit algebraic computation. For example, $\overline{\omega}_2 = 1$, $\overline{\omega}_3 = (1+\sqrt{5})/2$, $\overline{\omega}_4 = 1$, $\overline{\omega}_5 \approx 1.5129$, and $\overline{\omega}_6 = \sqrt{(1+\sqrt{5})/2}$. The relation observed between $\overline{\omega}_3$ and $\overline{\omega}_6$ is an instance of a general property which is consequence of our work above. In fact, several simple properties of $\overline{\omega}_k$ can be found easily using the geometry of T_ω, Šarkovskii's theorem, and our previous results. For example, the reader can check that if $\omega > \overline{\omega}_k$ then T_ω has a periodic point of period k, and that $k \prec n$ implies $\overline{\omega}_n \leq \overline{\omega}_k$, i.e. that the map $\overline{\omega} : (\mathbb{N}, \prec) \to (\mathbb{R}, \leq)$ is order-reversing. Also, as we saw in the proof of Theorem 1, when $1 < \omega \leq \sqrt{2}$ there is a set Q_1 which intersects all the periodic orbits of T_ω, for which $Q_2 = T_\omega(Q_1)$ and Q_1 intersect in a set containing no more than one point, and for which $(T^2_\omega)_{|Q_1}$ is conjugate to T^2_ω (on an appropriate set.) These facts yield the following:

Proposition 1. *(i) $\overline{\omega}_{2n+1} \geq \sqrt{2}$; (ii) $\overline{\omega}_k = \overline{\omega}^2_{2k}$*

Proof. (i) The properties of Q_1 imply that T_ω has no periodic points of odd period; (ii) by the conjugacy referred to before the statement of the proposition, corresponding to a T_{ω^2}-periodic point of period $n \neq 1$ there is a point x in Q_1 with T_ω^2-period n. Since Q_1 and its T_ω-image Q_2 intersect at at most one point, x must have T_ω-period $2n$. □

We now describe a class of polynomials which will appear often in our discussion. We say that a polynomial *alternates* if it has the form $p(x) = x^{a_1} - x^{a_2} + x^{a_3} - \ldots \pm x^{a_i}$, where $a_1 > a_2 > a_3 > \cdots > a_i \geq 1$, including the zero polynomial. We will denote arbitrary alternating polynomials of degree less than or equal to k by p_k; similarly, by q_k we will denote arbitrary alternating polynomials of degree exactly k. To indicate that p_k or q_k have an even number of nonzero terms we will write $\hat{p}_k$ or $\hat{q}_k$; and to indicate that they have an odd number of nonzero terms we will write $\tilde{p}_k$ or $\tilde{q}_k$. The relevance of alternating polynomials arises from the fact that $T_\omega^k(x)$ can be expressed in terms of them. The following result states this fact precisely.

Lemma 2. *$T_\omega^k(x) = \tilde{p}_k(\omega) - \omega^k x$ or $T_\omega^k(x) = -\hat{p}_k(\omega) + \omega^k x$. Furthermore, in the expressions above we can make sure that deg(p_k) $< k$ if $x \leq 1/2$ and that deg(p_k) $= k$ if $x \geq 1/2$.*

Proof. The proof is by induction. If $k = 1$ then $T_\omega(x)$ is ωx or $\omega - \omega x$ depending on whether $x \leq 1/2$ or $x \geq 1/2$. In either case, the lemma is valid for $k = 1$. Now assume that the statement of the lemma is true for $k = n - 1$, so

$$T_\omega^{n-1}(x) = \tilde{p}_{n-2}(\omega) - \omega^{n-1}x \text{ or } T_\omega^{n-1}(x) = -\hat{p}_{n-2}(\omega) + \omega^{n-1}x$$

if $x \leq 1/2$, and

$$T_\omega^{n-1}(x) = \tilde{q}_{n-1}(\omega) - \omega^{n-1}x \text{ or } T_\omega^{n-1}(x) = -\hat{q}_{n-1}(\omega) + \omega^{n-1}x$$

if $x \geq 1/2$. To verify the lemma for $k = n$ we consider different cases. For example, if $x \leq 1/2$ and $T_\omega^{n-1}(x) \leq 1/2$ then $T_\omega^n(x)$ is $\omega T_\omega^{n-1}(x)$, that is,

$$T_\omega^n(x) = \omega(\tilde{p}_{n-2}(\omega) - \omega^{n-1}x) = \omega\tilde{p}_{n-2}(\omega) - \omega^n x = \tilde{p}_{n-1}(\omega) - \omega^n x$$

or

$$T_\omega^n(x) = \omega(-\hat{p}_{n-2}(\omega) + \omega^{n-1}x) = -\omega\hat{p}_{n-2}(\omega) + \omega^n x = -\hat{p}_{n-1}(\omega) + \omega^n x.$$

Or if $x \geq 1/2$ and $T_\omega^{n-1}(x) \geq 1/2$ then $T_\omega^n(x)$ is $\omega - \omega T_\omega^{n-1}(x)$, that is,

$$T_\omega^n(x) = \omega - \omega(\tilde{q}_{n-1}(\omega) - \omega^{n-1}x) = \omega - \omega\tilde{q}_{n-1}(\omega) + \omega^n x = -\hat{q}_n(\omega) + \omega^n x$$

or

$$T_\omega^n(x) = \omega - \omega(-\hat{q}_{n-1}(\omega) + \omega^{n-1}x) = \omega + \omega\hat{q}_{n-1}(\omega) - \omega^n x = \tilde{q}_n(\omega) - \omega^n x.$$

We see that in the two cases considered the statements of the lemma are verified. There are two more cases to consider, and we leave them to the diligent reader. In each case the statements of the lemma are again validated for $k = n$, and the proof is complete. □

Corollary 3. *Assume that x is a point of period k. If $x \leq 1/2$ then it can be represented in the form*

$$x = \frac{\hat{p}_{k-1}(\omega)}{\omega^k - 1} \ \textit{or} \ \frac{\tilde{p}_{k-1}(\omega)}{\omega^k + 1},$$

and if $x \geq 1/2$ then it can be represented in the form

$$\frac{\hat{q}_k(\omega)}{\omega^k - 1} \ \textit{or} \ \frac{\tilde{q}_k(\omega)}{\omega^k + 1}.$$

Proof. If x has T_ω-period k then $T_\omega^k(x) = x$. Now use the expressions for $T_\omega^k(x)$ found in the preceding lemma and solve for x. $\square$

This last corollary raises hope that the problem of clarifying the nature of the periodic orbits of T_ω may be aided by the study of expressions of the form

$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ and } \frac{\tilde{p}_k(\omega)}{\omega^k + 1}.$$

Such hope is further strenghtened by the following observations:
(a) Given k and ω, the (finite) set $\left\{\frac{\hat{p}_k(\omega)}{\omega^k-1}\right\}$ is closed under the following operation R:

$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \longmapsto \omega\frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ if } \deg(\hat{p}_k) < k, \text{ and}$$
$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \longmapsto \omega - \omega\frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ if } \deg(\hat{p}_k) = k.$$

(b) Given k and ω, the (finite) set $\left\{\frac{\tilde{p}_k(\omega)}{\omega^k+1}\right\}$ is closed under the following operation R:

$$\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \longmapsto \omega\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \text{ if } \deg(\hat{p}_k) < k, \text{ and}$$
$$\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \longmapsto \omega - \omega\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \text{ if } \deg(\tilde{p}_k) = k.$$

(Notice that the description of the action of T_ω on x, whose form depends on whether $x \leq 1/2$ or $x \geq 1/2$, is identical to the description of the given operation R, only now the dependence is on $\deg(p_k)$.)

The operation R just introduced partitions each of the sets $\left\{\frac{\hat{p}_k(\omega)}{\omega^k-1}\right\}$ and $\left\{\frac{\tilde{p}_k(\omega)}{\omega^k+1}\right\}$ into *cycles* (or orbits.) We will say that one of these cycles in $\left\{\frac{\hat{p}_k(\omega)}{\omega^k-1}\right\}$ (resp. in $\left\{\frac{\tilde{p}_k(\omega)}{\omega^k+1}\right\}$) satisfies the $1/2$-*condition for* ω if

$$\deg(\hat{p}_k) < k \text{ if } \frac{\hat{p}_k(\omega)}{\omega^k - 1} \leq 1/2 \text{ and } \deg(\hat{p}_k) = k \text{ if } \frac{\hat{p}_k(\omega)}{\omega^k - 1} \geq 1/2$$

$$\left(\text{resp. } \deg(\tilde{p}_k) < k \text{ if } \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \leq 1/2 \text{ and } \deg(\tilde{p}_k) = k \text{ if } \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \geq 1/2\right)$$

for all the elements of the cycle. (The ambiguity concerning the value $1/2$ is irrelevant.) We can now state the following important result.

Theorem 3. *Given ω and k, if $x = \frac{\tilde{p}_k(\omega)}{\omega^k+1}$ or $x = \frac{\hat{p}_k(\omega)}{\omega^k-1}$ and the R-cycle of the rational expression satisfies the 1/2-condition then $T^k_\omega(x) = x$.*

Proof. Suppose $x = \frac{\tilde{p}_k(\omega)}{\omega^k+1}$. We study the orbit of x under T_ω by studying the orbit of the rational expression under the operation R; this is possible since both actions are identical given that the orbit of the rational expression satisfies the 1/2-condition. To simplify notation, we consider only the coefficients of the polynomial $\tilde{p}_k(\omega)$, which we present as an element of S^k, where $S = \{-1, 0, 1\}$. Thus, if $\vec{v} = (v_1, v_2, \ldots, v_k)$ is the element of S^k containing the coefficients of $\tilde{p}_k(\omega)$ then the action $R : S^k \to S^k$ is given by

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 0 \text{ if } v_k = 0,$$

and

$$R(\vec{v})_i = -v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 1 \text{ if } v_k \neq 0.$$

That is,

$$R(\vec{v}) = (|v_k|, (-1)^{|v_k|}v_1, (-1)^{|v_k|}v_2, \ldots, (1)^{|v_k|}v_{k-1})$$

and, continuing the same way

$$R^k(\vec{v}) = (|v_1|, (-1)^{|v_1|}|v_2|, (-1)^{|v_1|+|v_2|}|v_3|, \ldots, (1)^{|v_1|+\ldots|v_{k-1}|}|v_k|).$$

Now, for any polynomial of the form $\tilde{p}_k$, $v_i = (-1)^X|v_i|$, where $X = \sum_{j=1}^{i-1}|v_j|$ because for every term with coefficient 1 there is an even number of terms of smaller degree in the polynomial, while for every term with coefficient -1 there is an odd number of terms of smaller degree in the polynomial. Therefore, since $R^k(\vec{v}) = \vec{v}$ we must have $T^k_\omega(x) = x$.

The proof for the case when $x = \frac{\hat{p}_k(\omega)}{\omega^k-1}$ is very similar. We just use the appropriate form for R, namely

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 0 \text{ if } v_k = 0,$$

and

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = -1 \text{ if } v_k \neq 0,$$

and the specific properties of the polynomials $\hat{p}_k$. □

Suppose now that k is an odd number ≥ 5, and consider the polynomial $s_1(\omega) = \omega^{k-1} - \omega^{k-2} + \omega^{k-3} - \ldots - \omega$. This polynomial is of the form $\hat{p}_k(\omega)$. Also consider the R-cycle of $\frac{s_1(\omega)}{\omega^k-1}$, consisting, in addition to that rational expression itself, of $\frac{s_2(\omega)}{\omega^k-1}$, $\frac{s_3(\omega)}{\omega^k-1}$, $\ldots$, $\frac{s_i(\omega)}{\omega^k-1}$, $\ldots$ $\frac{s_{k-1}(\omega)}{\omega^k-1}$, and $\frac{s_k(\omega)}{\omega^k-1}$. Here

$s_2(\omega) = \omega^k - \omega^{k-1} + \omega^{k-2} \ldots - \omega^2$,
$s_3(\omega) = \omega^k - \omega^{k-1} + \ldots + \omega^3 - \omega$,
$\vdots$

$s_i(\omega) = \omega^k - \omega^{k-1} + \ldots + (-1)^{(i+1)}\omega^i + (-1)^i\omega^{i-2} + \ldots - \omega$,
$\vdots$

$s_{k-1}(\omega) = \omega^k - \omega^{k-1} + \omega^{k-3} - \ldots - \omega,$
$s_k(\omega) = \omega^k - \omega^{k-2} + \omega^{k-3} - \ldots - \omega.$

Lemma 3. *If $\frac{s_{k-1}(\omega)}{\omega^k - 1} \geq 1/2$ for some $\omega > 1$ then the R-cycle above satisfies the 1/2-condition for that ω.*

Proof. First consider those s_i having degree k. If $2 \leq i \leq k-2$, we would like that $s_{k-1}(\omega) \leq s_i(\omega)$. That this is so follows from the fact that $s_i(\omega) - s_{k-1}(\omega) = (\omega - 1)(\omega^{k-3} - \omega^{k-4} + \ldots + (-1)^{(i+1)}\omega^{i+1}) > 0$ for all $\omega > 1$. Thus, $\frac{s_{k-1}(\omega)}{\omega^k-1} \geq 1/2$ forces $\frac{s_i(\omega)}{\omega^k-1} > 1/2$. Likewise, $s_k(\omega) - s_{k-1}(\omega) = \omega^{k-1} - \omega^{k-2} > 0$ for $\omega > 1$ so $\frac{s_k(\omega)}{\omega^k-1} > 1/2$ as well.

Finally, we need to show that $\frac{s_{k-1}(\omega)}{\omega^k-1} \geq 1/2$ implies that $\frac{s_1(\omega)}{\omega^k-1} \leq 1/2$. By cross-multiplying and subtracting in the last inequality, we reformulate our desideratum as $\omega^k - 2\omega^{k-1} + 2\omega^{k-2} - \ldots + 2\omega - 1 \geq 0$. But, again cross-multiplying and subtracting, the first inequality can be written as $\omega^k - 2\omega^{k-1} + 2\omega^{k-3} - 2\omega^{k-4} + \ldots - 2\omega + 1 > 0$ and this inequality is easily seen to imply the previous one when $\omega > 1$. □

It is now a consequence of Theorem 3 and Lemma 3 that if for some $\omega > 1$, $x = \frac{s_{k-1}(\omega)}{\omega^k-1} \geq 1/2$, then $x = T_\omega^k(x)$. One such ω is, of course, the largest real root of $f_k(x) = x^k - 2x^{k-1} + 2x^{k-3} - \ldots - 2x + 1$ which will henceforth be denoted by $\overline{\eta}_k$. Thus, $x = \frac{s_1(\overline{\eta}_k)}{\overline{\eta}_k^k - 1}$ satisfies $x = T_{\overline{\eta}_k}^k(x)$. We have not excluded the a priori possibility that $x = T_{\overline{\eta}_k}^n$ for some $1 \leq n < k$, i.e. that x have (prime) period $< k$. If this indeed happened then for some $2 \leq i \leq k$ we would have $s_1(\overline{\eta}_k) = s_i(\overline{\eta}_k)$. However, these two polynomials (and in fact any pair s_j, s_l, $j \neq l$) can have the same value at only finitely many points. Therefore we can select η_k arbitrarily close to, but larger than $\overline{\eta}_k$, i.e. in $[\overline{\eta}_k, \overline{\eta}_k + \delta)$ for arbitrarily small δ, so that $f_k(\eta_k) \geq 1/2$ (f_k is nondecreasing in a neighborhood of its largest real root), and so that $s_i(\eta_k) \neq s_j(\eta_l)$, $j \neq l$. In this case $x = \frac{s_1(\eta_k)}{\eta_k^k - 1}$ is T_{η_k}-periodic of (prime) period k. From this the inequality $\overline{\eta}_k \geq \overline{\omega}_k$ follows. This inequality is useful because we will be able to show that $\lim_{n\to\infty} \overline{\eta}_{2n+1} = \sqrt{2}$.

Lemma 4. *(i) $f_k(\sqrt{2}) = 2\sqrt{2} - 3$ for all odd k; and (2) $\lim_{k\to\infty} f_k(x) = \infty$ for $x > \sqrt{2}$.*

Proof. (i) $f_k(x) = x^k - 2x^{k-1} + 1 + (1 - \frac{1}{x})(x^{k-3} + x^{k-5} + \ldots + x^2)$. Thus $f_k(x) = x^k - 2x^{k-1} + 1 + \frac{2x^{k-2}}{x+1} - \frac{2x}{x+1} = \frac{x^{k-2}}{x+1}(x^3 - x^2 - 2x + 2) + \frac{1-x}{1+x}$. Now we compute $f_k(\sqrt{2}) = 0 + \frac{1-\sqrt{2}}{\sqrt{2}+1} = 2\sqrt{2} - 3$. (ii) When $x > \sqrt{2}$ a simple calculation shows that $x^3 - x^2 - 2x + 2 > 0$. Thus, $f_k(x) = \alpha\frac{x^{k-2}}{x+1} - \beta$ for some $\alpha > 0$, implying that $\lim_{n\to\infty} f_{2n+1}(x) = \infty$. □

Theorem 4. $\lim_{n\to\infty} \overline{\eta}_{2n+1} = \sqrt{2}$.

Proof. By the preceding lemma, given any $\epsilon > 0$, once n is large enough we will have $f_{2n+1}(\sqrt{2}+\epsilon) > 1$. Since $f_{2n+1}(\sqrt{2}) = 2\sqrt{2} - 3 < 0$ for such n, we must have $\sqrt{2} < \overline{\eta}_{2n+1} < (\sqrt{2}+\epsilon)$. Now let ϵ decrease to 0. □

Corollary 4. $\lim_{n\to\infty} \overline{\omega}_{2n+1} = \sqrt{2}$.

Proof. From Proposition 1(i) and the comment preceding Lemma 4 we have $\sqrt{2} \le \overline{\omega}_{2n+1} \le \overline{\eta}_{2n+1}$. Now use Theorem 4. □

Thus, $T_{\sqrt{2}}$ has, besides the fixed point, periodic points of all even periods and none of odd period. Also,

Corollary 5. *(i)* $\lim_{n\to\infty} \overline{\omega}_{(2n+1)2^m} = 2^{2^{-m-1}}$*; (ii)* $\lim_{m\to\infty} \overline{\omega}_{(2n+1)2^m} = 1$.

Corollary 6. *For all* m, $\overline{\omega}_{2^m} = 1$.

Proof. This follows from part (ii) of the Corollary 5 and from the previously mentioned fact that $\overline{\omega} : (\mathbb{N}, \prec) \to (\mathbb{R}, \le)$ is order-reversing. □

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