Global stability of periodic orbits of non-autonomous difference equations and population biology

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Abstract

Elaydi and Yakubu showed that a globally asymptotically stable(GAS) periodic orbit in an autonomous difference equation must in fact be a fixed point whenever the phase space is connected. In this paper we extend this result to periodic nonautonomous difference equations via the concept of skew-product dynamical systems. We show that for a k-periodic difference equation, if a periodic orbit of period r is GAS, then r must be a divisor of k. In particular sub-harmonic, or long periodic, oscillations cannot occur. Moreover, if r divides k we construct a non-autonomous dynamical system having minimum period k and which has a GAS periodic orbit with minimum period r. Our methods are then applied to prove a conjecture by J. Cushing and S. Henson concerning a non-autonomous Beverton-Holt equation which arises in the study of the response of a population to a periodically fluctuating environmental force such as seasonal fluctuations in carrying capacity or demographic parameters like birth or death rates.

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1 Introduction

A continuous map $F : X \to X$ on a metric space generates an autonomous difference equation

$$x_{n+1} = F(x_n),\ \ n \in \mathbb{Z}^+ \tag{1.1}$$

where for each $x_0 \in X$, $x_n = F^n(x_0)$, $F^0 = id$, and $\mathbb{Z}^+$ is the set of nonnegative integers. Another useful way of looking at equation (1.1) is to consider the discrete semi-dynamical system, or semi-flow π on X, (X, π) where $\pi : X \times \mathbb{Z}^+ \to X$ is such that

$$\begin{aligned} &1. \quad \pi \text{ is continuous,} \\ &2. \quad \pi(x_0, 0) = x_0 \text{ for } x_0 \in X, \\ &3. \quad \pi(\pi(x_0, r), s) = \pi(x_0, s + r) \text{ with } \pi(x_0, n) = F^n(x_0). \end{aligned} \tag{1.2}$$

where $F^n = F \circ F \circ \cdots \circ F$ is the n^{th} composition of F. The orbit $\mathcal{O}(x_0)$ of $x_0 \in X$ is defined as $\mathcal{O}(x_0) = \{F^n(x_0) : n \in \mathbb{Z}^+\}$. A point $\bar{x}$ is k-periodic with a minimal period k if $F^k(\bar{x}) = \bar{x}$, $F^r(\bar{x}) \neq \bar{x}$ for all $0 < r < k$. The orbit $\mathcal{O}(\bar{x})$ of a k-periodic point is denoted by c_k and is called a k-cycle, $c_k = \{\bar{x}_1 = \bar{x}, \bar{x}_2 = F(\bar{x}_1), \ldots, \bar{x}_k = F^{k-1}(\bar{x}_1)\}$. If $c_k = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ is a k-cycle, then we consider each point in c_k as a fixed point of the map $g = F^k$.

Elaydi and Yakubu [5] proved the following fundamental result.

Theorem 1.1 *Let $F : X \to X$ be a continuous map on a connected metric space. If a k-cycle c_k is globally asymptotically stable, then c_k must be a fixed point.*

One of our main objectives in this paper is to extend this theorem to nonautonomous difference equations of the form $x_{n+1} = F(n, x_n)$. These time-varying equations are a better fit to model species that are sensitive to seasonal or environmental variations. The main difficulty here is that such equations fail to define discrete dynamical systems on the same phase space X since the semi-group property in (1.2) is lost. However, one may define a discrete dynamical system, called a skew-product dynamical system, on an enlarged space which takes into account that the function F on the right side is evolving in (discrete) time along with the solution. The concept of the skew-product dynamical system comes from Sell [10], 1971. It was developed in Sacker and Sell [8], 1973; Sacker and Sell [9], 1977. See also Sacker [7], 2003 for further discussion.

Since all our dynamical systems in this paper are discrete, we will drop the adjective discrete in our terminology.

2 Skew-product dynamical system

Definition 2.1 *Let X and Y be two topological spaces. A dynamical system π on a product space $X \times Y$ is said to be a skew-product dynamical system if there exists continuous mappings $\pi : X \times Y \times \mathbb{Z} \to X \times Y$ and $\sigma : Y \times \mathbb{Z} \to Y$ such that*

$$\pi(x, y, n) = (\varphi(x, y, n), \sigma(y, n)).$$

where σ is a semi-dynamical system on Y.

If $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$, then π is called a skew-product semi-dynamical system.

Now let us construct a skew-product semi-dynamical system from the nonautonomous difference equation

$$x_{n+1} = F(n, x_n) \tag{2.1}$$

We first define σ. Let $C = C(\mathbb{Z} \times X, X)$ be the space of continuous functions from $\mathbb{Z} \times X$ to X equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z} \times X$. Now for $G \in C$, we define $\sigma(G, n) = G_n$ as the shift map, $G_n(t, x) = G(t + n, x)$ and the *Hull* of G, $\mathcal{H}(G) \doteq Cl\{G_n : n \in \mathbb{Z}\}$ and set

$$Y = \mathcal{H}(F)$$

where F is given in (2.1). In the general theory of skew-product dynamical systems it is assumed that σ is defined (or extendible) to all of $\mathbb{Z}$. In the present (periodic) case this is immediate. It is also assumed that Y is compact. By [10] compactness follows if and only if $F(n, x)$ is bounded on every set of the form $\mathbb{Z} \times K \subset \mathbb{Z} \times X$ with K compact. Again, periodicity guarantees the compactness. For $G \in Y$ the map φ is defined as

$$\varphi(x_0, G, n) \doteq \Phi(n, G)x_0, \tag{2.2}$$

Hence, letting $G_0 = G$,

$$\pi((x, G_i), n) = (\Phi(n, G_i)x, G_{i+n}).$$

The operator Φ (nonlinear, in general) in (2.2) is then given as follows: For $G \in Y$ set $G_0 = G$. Then

$$\begin{aligned} \Phi(0, G_0) &= id \\ \Phi(n, G_0) &= \Phi(1, G_{n-1})\Phi(n-1, G_0) \\ &= \Phi(1, G_{n-1})\Phi(1, G_{n-2}) \ldots \Phi(1, G_0) \end{aligned} \tag{2.3}$$

Remark 2.2 *An* <u>*autonomous*</u> *equation,*

$$x_{n+1} = H(x_n), \tag{2.4}$$

in this context is one such that $\sigma(n, H) = H$, *for all* n, *i.e. a stationary point of* σ. *For example, if* F *in (2.1) is k-periodic*

$$F(n+k, x) = F(n, x), \qquad \text{for all } n > 0,$$

then defining $H(x) \doteq \Phi(k, F)x$ *produces an autonomous equation (2.4).*

Since we are dealing with maps (2.1) which for fixed n are not necessarily invertible we will refer to the positive semi-orbit as simply the "orbit" of x_0 given by

$$\mathcal{O}(x_0) = \{\Phi(n, F)x_0 : n \in \mathbb{Z}^+\}.$$

In the case $X = \mathbb{R}^m$, Φ is in general a nonlinear operator acting on X. In case (2.1) is linear, $F(n, x) \doteq A(n)x$, then we can suppress the x-variable and $\Phi(n, A)$ is just the fundamental matrix of the system

$$x_{n+1} = A(n)x_n, \qquad \Phi(0, A) = id$$

If p is the projection map, then a skew-product dynamical system can be illustrated by the following commuting diagram .

$$\begin{array}{ccc} X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\ \Big\downarrow{\scriptstyle p\times id} & & \Big\downarrow{\scriptstyle p} \\ Y \times \mathbb{Z}^+ & \xrightarrow{\sigma} & Y \end{array}$$

For $y \in Y$ the *fiber over* y is defined to be $\mathcal{F}_y \doteq p^{-1}(y)$. Thus the above diagram shows that π is "fiber preserving": For every $(y, n) \in Y \times \mathbb{Z}^+$, $\pi((p^{-1} \times id)(y, n)) \subset p^{-1}(\sigma(y, n))$.

The following example is useful in understanding convergence in the function space C. Recall that $G : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ is in the omega-limit set ω_F if for each $n \in \mathbb{Z}^+$

$$|F_\tau(n, x) - G(n, x)| \to 0$$

uniformly for x in compact subsets of $\mathbb{R}$ as $\tau \to \infty$ along some subsequence τ_G.

Example 2.3 *Define*

$$F(n, x) = (-1)^n(1 + \frac{1}{n})x$$

There are two elements in ω_F:

$$\omega_F = \{G_+, G_-\}$$

where $G_+(n,x) = (-1)^n x$ and $G_- = -G_+$ since for each fixed $n \in \mathbb{Z}^+$

$$F_\tau(n,x) - G_+(n,x) = (-1)^{n+\tau}x - (-1)^n x + \frac{(-1)^{n+\tau}}{n+\tau}x = \frac{(-1)^{n+\tau}}{n+\tau}x \to 0$$

as $\tau \to \infty$ along the subsequence $\tau = 2k$. A similar argument holds for G_-.

The following notation is sometimes convenient, especially in the construction of examples. Define $f_n(x) = F(n,x)$ so that the equation (2.1) takes the form

$$x_{n+1} = f_n(x_n) \tag{2.5}$$

It is however tempting to believe that subsequential limits of the sequence f_n give rise to autonomous equations (2.5). The previous example should dispel this notion.

Remark 2.4 *By definition, a point (x_0, y) is a periodic point with period k if and only if for all $n \in \mathbb{Z}^+$*

$$\pi((x_0,y), k+n) = \pi((x_0,y), n) \tag{2.6}$$

By the semi-group property (1.2) it suffices to verify

$$\pi((x_0,y), k) = (x_0, y). \tag{2.7}$$

, i.e. if a point returns to itself after "time" k, then so do all iterates of the point.

We wish to make precise a notion of periodicity *in the state space X* where this latter statement fails to hold.

Note that in particular, periodicity implies (letting $y = G_0$) that $G_{n+k} \doteq \sigma(G_n, k) = G_n$. In the simplified notation, with $g_n(x) = G(n,x)$ one has $g_{n+k} = g_n$ for all $n \in \mathbb{Z}$.

Definition 2.5 (Geometric r-cycle) *Let f_n be periodic with period k and let $r > 0$ be an integer, $r \le k$. By a geometric r-cycle we mean an ordered set of points*

$$\mathcal{C} = \{c_0, c_1, \ldots c_{r-1}\}, \qquad c_i \in X \tag{2.8}$$

with the property that for $i = 0, 1, \ldots, r-1$

$$f_{(i+nr) \bmod k}(c_i) = c_{i+1 \bmod r} \qquad \textit{for all } n \in \mathbb{Z}$$

We have the following lemma which describes the "layout" in $X \times Y$ of the orbit of (c_0, f_0) under the action of the skew-product flow π.

Lemma 2.6 *Let $\mathcal{C}$ in (2.8) be a geometric r-cycle and let $s = [r, k]$, the least common multiple. Then the π-orbit of (c_0, f_0) intersects each fiber $\mathcal{F}_i, i = 1, \ldots, k$ in exactly $\ell = s/k$ points and each of these points is periodic under the skew-product flow with period s.*

PROOF. *We observe that in the skew-product flow the orbit of (c_0, f_0) is periodic with minimum period s and therefore the set*

$$S = \{\pi((c_0, f_0), n) | n \in \mathbb{Z}^+\} \subset X \times Y$$

is minimal, invariant under π and consists of s distinct *points. Therefore for each i, $0 \leq i < r$ the mapping*

$$f_i : S \cap \mathcal{F}_i \rightarrow S \cap \mathcal{F}_{i+1 \bmod k}$$

is onto. To see that it is one-to-one, observe that the cardinality

$$\mathcal{C}_i \doteq crd(S \cap \mathcal{F}_i)$$

is a non-increasing integer valued function and therefore stabilizes at some fixed value from which it follows that $\mathcal{C}_i$ is constant. Thus each $S \cap \mathcal{F}_i$ contains the same number of points, namely s/k. $\square$

We next give two examples. The first is a 6-periodic difference equation having a geometric 4-cycle consisting of 2 distinct points. So that the reader is not misled into thinking that the number of distinct points in a geometric cycle must coincide with the "ℓ" of Lemma 2.6 we then construct another example of a 6-periodic difference equation having a geometric 4-cycle consisting of 4 distinct points.

A simple modification will then be made to cause these cycles to become *locally* asymptotically stable.

Example 2.7 *Define*

$$\mathcal{C} = \{0, 1, 1, 0\}$$

and

$$
\begin{aligned}
f_0(x) &= 1 - x \\
f_2(x) &= f_0(x) \ \textit{for} \ x \leq 1 \ \textit{and} \ = 0 \ \textit{otherwise} \\
f_4(x) &= f_0(x) \ \textit{for} \ x \geq 0 \ \textit{and} \ = 1 \ \textit{otherwise} \\
f_1(x) &= x \\
f_3(x) &= f_1(x) \ \textit{for} \ x \leq 1 \ \textit{and} \ = 1 \ \textit{otherwise} \\
f_5(x) &= f_1(x) \ \textit{for} \ x \geq 0 \ \textit{and} \ = 0 \ \textit{otherwise} \\
f_n &= f_{n \bmod 6}, \quad n \geq 6
\end{aligned}
\tag{2.9}
$$

For the flow in the base $Y = \{f_0, \ldots, f_5\}$ define $\sigma(f_n, 1) = f_{n+1 \bmod 6}$. It is easily seen that the functions with the even subscripts interchange " 0" and " 1" while those with odd subscripts leave those points fixed. Thus $\mathcal{C}$ is a geometric 4*-cycle.*

Example 2.8 *Define*

$$\mathcal{C} = \{0, 1, 2, 3\}$$

and

$$
\begin{aligned}
f_0(x) &= x + 1 \\
f_2(x) &= f_0(x) \ \textit{for} \ x \leq 2 \ \textit{and} \ = 3 \ \textit{otherwise} \\
f_4(x) &= f_0(x) \ \textit{for} \ x \geq 0 \ \textit{and} \ = 1 \ \textit{otherwise} \\
f_1(x) &= 3 - x \\
f_3(x) &= f_1(x) \ \textit{for} \ x \leq 3 \ \textit{and} \ = 0 \ \textit{otherwise} \\
f_5(x) &= f_1(x) \ \textit{for} \ x \geq 0 \ \textit{and} \ = 3 \ \textit{otherwise} \\
f_n &= f_{n \bmod 6}, \quad n \geq 6
\end{aligned}
\tag{2.10}
$$

with the flow in the base defined as before. Then $f_i(0) = 1$ and $f_i(2) = 3$ for i even, while $f_i(1) = 2$ and $f_i(3) = 0$ for i odd proving that $\mathcal{C}$ is a geometric 4*-cycle.*

To get the local asymptotic stability in the second example take one function, f_0 say, and undefine it in the interval $-\frac{1}{2} < x < \frac{1}{2}$. Then redefine it in the interval $-\frac{1}{4} < x < \frac{1}{4}$ to be $1 + \frac{x}{2}$ and finally on the remaining two intervals join the end points with line segments. All the functions are continuous and piecewise linear and therefore Lipschitz continuous. Letting

$$g = f_5 \circ f_4 \circ \cdots \circ f_0$$

it is easily seen, looking at difference quotients, that $g \circ g$ has Lipschitz constant $\frac{1}{2}$ at any of the fixed points giving rise to the periodic orbit described above. The first example is treated similarly.

3 Periodic difference equation

In this section we consider a k-periodic nonautonomous difference equation

$$x_{n+1} = F(n, x_n) \tag{3.1}$$

Our aim here is to extend Elaydi and Yakubu Theorem (Theorem 1.1) to equation (3.1). The following result is an extension of Theorem 1.1(for autonomous equations) to nonconnected metric spaces and will be needed to treat the non-autonomous case. Recall that the basin of attraction $W^s(x^*)$ [3],[4] of a fixed point x^* is defined as

$$W^s(x^*) = \{x : \lim_{n\to\infty} F^n(x) = x^*\}.$$

Theorem 3.1 *Let $h : \Omega \to \Omega$ be a continuous map on a metric space Ω where Ω is the union of k components. If h has a globally asymptotically stable r-cycle, then $r \leq k$.*

PROOF. Let $g = h^r$. Then g has r fixed points $c_r = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\}$ each of which has a basin of attraction $W^s(\bar{x}_i)$ that is open and invariant under g [3]. Moreover, by assumption $\Omega = \bigcup\limits_{i=1}^{r} W^s(\bar{x}_i)$. Let M be one of the components of Ω, and let $\{\bar{x}_{n_1}, \bar{x}_{n_2}, \ldots, \bar{x}_{n_l}\}$ be all the points in c_r that belong to M. Define $G_j = W^s(x_{n_j}) \cap M$. Then $G_j \neq \phi$ and relatively open in M. Hence, if $l > 1$, $M = \bigcup\limits_{j=1}^{l} G_j$ is a separation of M which contradicts the connectedness of M. This implies that M contains at most one point in the r-cycle c_r. Since there are only k components, $r \leq k$. $\square$

Even if X is connected, Eq.(3.1) may have globally asymptotically stable cycles of period $k > 1$ in the non-autonomous case. The following example demonstrates this new phenomenon.

Example 3.2 *Consider Eq.(3.1) with $X = \mathbb{R}$, where*

$$\begin{aligned} f_0(x) &= 1 + 0.5x, & f_1(x) &= -0.5 + 0.5x, \\ f_{2n} &= f_0, & f_{2n+1} &= f_1. \end{aligned}$$

Notice that for any $x_0 \in \mathbb{R}$, x_{2n} converges to 0 and x_{2n+1} converges to 1. Thus, the geometric 2-cycle $\{0, 1\}$ is globally asymptotically stable.

The following example, which motivates the next theorem, shows that the relationship between the periods of the equation and its solution is very special.

Example 3.3 *Let α, $\beta \in (0,1)$ with $\alpha \neq \beta$ and define*

$$\begin{aligned} f_0(x) &= 1+\alpha x, & f_1(x) &= -\alpha+\alpha x, \\ f_2(x) &= 1+\beta x, & f_3(x) &= -\beta+\beta x, \\ f_n &= f_{n \bmod 4}, & n &> 3 \end{aligned}$$

Then

$$\begin{aligned} f_3 \circ f_2 \circ f_1 \circ f_0 = f_1 \circ f_0 \circ f_3 \circ f_2 &= \alpha^2\beta^2 x, \\ f_0 \circ f_3 \circ f_2 \circ f_1 = f_2 \circ f_1 \circ f_0 \circ f_3 &= 1-\alpha^2\beta^2(1-x). \end{aligned} \tag{3.2}$$

Thus, again we have a globally asymptotically stable geometric 2-cycle $\{0,1\}$ in $\mathbb{R}$ while the base $Y = \{f_0, f_1, f_2, f_3\}$ is a 4-cycle.

In the sequel, we will assume that the maps $\{f_n\}$ of equation (3.1) constitute a k-periodic sequence, i.e. $f_{n+k} = f_n$ for all $n \in \mathbb{Z}^+$. Hence the space $Y = \{f_0, f_1, \ldots, f_{k-1}\}$ (as defined in Section 2). We now consider the skew-product semi-dynamical system π on $X \times Y$ defined by

$$\pi((x, f_i), n) = (\Phi(n, f_i)x, f_{n+i}).$$

We are now ready to state our main result.

Theorem 3.4 *Assume that X is a connected metric space and each $f_i \in Y$ is a continuous map on X. Let $c_r = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\}$ be a geometric r-cycle of equation (3.1). If c_r is globally asymptotically stable then $r|k$, i.e. r divides k.*

PROOF. Let $c_r = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r\}$ be an r-cycle of equation (3.1) and set $s = [r,k]$, the least common multiple of r and k. Then by Lemma 2.6 each fiber $p^{-1}(f_i)$, $0 \leq i \leq k-1$, contains $l = \frac{s}{k}$ points $(x_{n_1}, f_i), (x_{n_2}, f_i), \ldots, (x_{n_l}, f_i)$ and each one of these points has period s in the map

$$h : \Omega \doteq X \times \{f_1, \ldots, f_k\} \to \Omega$$

$$h(x, f_i) \doteq (\Phi(1, f_i)x, f_{i+1})$$

But X connected implies Ω has k components so by Theorem 3.1, $s \leq k$. But $s = [r,k] \geq k$ so $s = k$. Finally, $[r,k] = k$ gives us $r|k$. $\square$

The next theorem shows how to construct such a dynamical system given any two positive integers r and k with $r|k$.

Theorem 3.5 *Given any two positive integers r and k with $r|k$ then there exists a non-autonomous dynamical system having minimum period k and which has a globally asymptotically stable geometric r-cycle with minimum period r.*

PROOF. The proof is constructive. Define $p = k/r$ and choose $\alpha_i \in (0,1),\ i = 0, 1, \ldots, p-1$ such that the α_i are independent over the rationals $\mathbb{Q}$, i.e.

$$\sum_{i=0}^{p-1} q_i \alpha_i \in \mathbb{Q} \text{ and } q_i \in \mathbb{Q} \text{ imply } q_i = 0 \quad \text{for all } i$$

We next define function classes, $L_0, L_1, \ldots, L_{p-1}$. For $r \geq 2$ set $\nu = \frac{1}{r-1}$ and define

$$L_i = \{f_{i,0}, f_{i,1}, \ldots, f_{i,p-1}\}, \quad f_{ij} : \mathbb{R} \to \mathbb{R}$$

where

$$f_{i,0}(x) = \nu + \alpha_i x, \quad f_{i,r-1}(x) = -\alpha_i + \alpha_i x \tag{3.3}$$

and for $k = 1, 2, \ldots, r-2$,

$$f_{i,k}(x) = (1+\alpha_i)\nu + (1 - \frac{\alpha_i}{k})x. \tag{3.4}$$

We now define the base space of our skew-product system to be

$$Y = \bigcup_{j=1}^{p-1} L_j$$

Claim: Y consists of $pr = k$ distinct points. To prove this we first look at two functions, one from (3.4), the other from the second definition in (3.3). If they agree then for some integer m and for some $\gamma, \beta \in \{\alpha_0, \ldots, \alpha_{p-1}\}$

$$-\gamma + \gamma x = (1+\beta)\nu + (1 - \frac{\beta}{m})x, \qquad \text{for all } x$$

Equating the constant part we obtain

$$\nu\beta + \gamma = -\nu$$

which contradicts the assumption that the α's are independent over the rationals. The remaining arguments are similar and this proves the Claim.

We now define the flow on Y as follows:

$$\begin{aligned} f_{0,0} \to f_{0,1} \to \cdots \to f_{0,r-1} \to f_{1,0} \to \\ \cdots \to f_{1,r-1} \to \cdots \to f_{r-1,r-1} \to f_{0,0}\ , \end{aligned} \tag{3.5}$$

i.e. "add and carry base r".

We now compute the orbit of the initial point $x_0 = 0$ starting in the fiber over $f_{i,0}$ for any i.

$$\begin{aligned} x_0 &= 0 \\ x_1 &= f_{i,0}(0) = \nu \\ x_2 &= f_{i,1}(x_1) = 2\nu \\ &\vdots \\ x_{r-1} &= f_{i,r-2}(x_{r-2}) = (r-1)\nu = 1 \\ x_r &= f_{i,r-1}(x_{r-1}) = 0 \end{aligned} \tag{3.6}$$

and this is *independent of* i ! Thus, the first (or $\mathbb{R}$) coordinate of the orbit in the product space through $(x_0, f_{0,0})$ say, takes exactly r steps to return to its initial starting value $x_0 = 0$ and similarly for other points on the orbit. The global attraction follows from the linearity of the mappings and the fact that each one is a strict contraction. We have thus constructed the geometric r-cycle

$$\{0, \nu, 2\nu, \ldots, (r-2)\nu, 1\}.$$

For $r = 1$ choose

$$f_0 = \frac{1}{2}x, \quad f_1 = \frac{1}{3}x, \cdots, f_{k-1} = \frac{1}{k+1}x$$

with the sequential flow

$$f_0 \to f_1 \to \cdots \to f_{k-1} \to f_0$$

Then $x_0 = 0$ is a globally attracting fixed point. $\square$

4 The Beverton-Holt Equation

The Beverton-Holt equation studied extensively by Cushing and Henson [1,2] is

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \qquad x_0 \geq 0 \tag{4.1}$$

It is known that for $\mu > 1$, $K > 0$, all non-zero solutions converge to the positive equilibrium point $x^* = K$, and for $\mu \leq 1$, $K > 0$, all solutions converge to the equilibrium point $x^* = 0$.

A modification of this equation that arises in the study of populations living in a periodically (seasonally) fluctuating environment replaces the constant carrying capacity K with a periodic sequence $\{K_n\}$ of positive carrying capacities. Thus we have a periodic Beverton-Holt equation

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n} \tag{4.2}$$

where $K_{n+p} = K_n > 0$ for all $n \in \mathbb{Z}^+$ and a minimal period $p \geq 2$, and $\mu > 1$. Equation (4.2) can be written in the form (3.1)

$$\begin{aligned} x_{n+1} &= F(n, x_n), \quad \text{where } F(i,x) = \frac{\mu K_i x}{K_i + (\mu-1)x} \quad \text{with} \\ F_{n+p} &= F_n \quad \text{ for all } n \in \mathbb{Z}^+, \quad \text{where } F_j(i,x) = F(i+j,x). \end{aligned} \tag{4.3}$$

From Remark 2.2 we know that (2.4) with $H(x) = \Phi(p, F)x$ is autonomous. We next calculate $\Phi(p, f)x$. An easy calculation shows

$$\Phi(2, f)x = \frac{\mu^2 K_1 K_0 x}{K_1 K_0 + (\mu - 1)M_1 x}.$$

One may show inductively that for $x \in \mathbb{R}^+$,

$$\Phi(p, f)x = \frac{\mu^p K_{p-1} K_{p-2} \cdots K_0 x_0}{K_{p-1} K_{p-2} \cdots K_0 + (\mu - 1)M_{p-1} x} \tag{4.4}$$

where M_n satisfies the difference equation:

$$M_{n+1} = K_{n+1} M_n + \mu^{n+1} K_n K_{n-1} \ldots K_0, \quad M_0 = 1. \tag{4.5}$$

Thus

$$M_{p-1} = \prod_{j=0}^{p-2} K_{j+1} + \sum_{m=0}^{p-2} \left(\prod_{i=m+1}^{p-2} K_{i+1} \right) \mu^{m+1} \cdot K_m K_{m-1} \cdots K_0.$$

Letting $L_{p-1} = K_{p-1} K_{p-2} \ldots K_0$ in equation (4.5), we obtain

$$H(x) \doteq \Phi(p, f)x = \frac{\mu^p L_{p-1} x_0}{L_{p-1} + (\mu - 1)M_{p-1} x_0} \tag{4.6}$$

The mapping $x_{n+1} = H(x_n)$ thus has the unique positive fixed point

$$\bar{x} = \frac{\mu^p - 1}{\mu - 1} \frac{L_{p-1}}{M_{p-1}} \tag{4.7}$$

which is globally asymptotically stable either by [1] or the general treatment in the next section.

By Theorem 3.4, either $\bar{x}$ is of minimal period p or of minimal period r where $r|p$. However, $\bar{x}$ cannot be a fixed point for each component map, i.e. $\bar{x} = F(i,\bar{x})$ for $i = 0, 1, \dots p-1$. For if so, then $K_0 = K_1 = \cdots = K_{p-1}$, which contradicts $p \geq 2$ being the minimal period.

Next we investigate conditions under which an r-periodic point $\bar{x}$ with a minimal period $r|p$ exists, i.e. a geometric r-cycle. Once an r-periodic point is found, then there are no other periodic orbits since $\bar{x}$ is globally asymptotically stable.

Notice that if the periodic point in (4.7) has a minimal period r, then

$$\bar{x} = \frac{\mu^p - 1}{\mu - 1} \, \frac{L_{p-1}}{M_{p-1}} = \frac{\mu^r - 1}{\mu - 1} \, \frac{L_{r-1}}{M_{r-1}}. \tag{4.8}$$

Hence

$$M_{p-1} = \left(\frac{\mu^p - 1}{\mu^r - 1} \right) K_p \cdots K_r M_{r-1} \tag{4.9}$$

Theorem 4.1 *Suppose $\mu > 1$ and $K_i > 0, 0 \leq i \leq p-1$ and $r|p$. Then equation (4.3) has an r-periodic point which is globally asymptotically stable if and only if condition (4.9) holds.*

Remark 4.2 *Clearly by choosing r to be the smallest integer satisfying both (4.8) and $r|p$, we obtain a periodic solution of minimum period r.*

5 Extension and Generalizations

The results obtained for the autonomous Beverton-Holt equation can be extended to monotone equations satisfying certain conditions which guarantee the existence of a positive globally asymptotically stable fixed point.

(i) Thus we consider the more general Ricatti equation

$$x_{n+1} = f(x_n) \qquad f(x) \doteq \frac{ax + b}{cx + d} \tag{5.1}$$

where we assume the following conditions

$$\begin{aligned} &1.\ a, c, d > 0,\ b \geq 0 \\ &2.\ ad - bc \neq 0 \\ &3.\ bc > 0 \quad \text{or} \quad a > d \end{aligned} \tag{5.2}$$

The first condition in (5.2) guarantees that $f : \mathbb{R}^+ \to \mathbb{R}^+$ while the second conditon says the mapping is not a constant map. The last condition guarantees a positive fixed point and has an alternative which allows application to the Beverton-Holt equation ($b = 0$). We now investigate the behavior of these maps under composition. Thus let

$$g(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$$

An easy computation yields

$$g \circ f(x) = \frac{(a\alpha + c\beta)x + (b\alpha + d\beta)}{(a\gamma + c\delta)x + (b\gamma + d\delta)}$$

from which the conditions (5.2) are easily verified.

The following change of variables [4, p.87]

$$cx_n + d = \frac{y_{n+1}}{y_n}$$

reduces (5.1) to

$$y_{n+2} - py_{n+1} - qy_n = 0 \tag{5.3}$$

where $p = a + d$ and $q = bc - ad$. The general solution of (5.3) has the form

$$y_n = c_1 \lambda_{max}^n + c_2 \lambda_{min}^n$$

where λ_{max} and λ_{min} are the roots of the characterstic equation $\lambda^2 - p\lambda - q = 0$,

$$\lambda_{max} = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}, \quad \lambda_{min} = \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2}$$

One can show that if $c_1 = 0$ then $x_0 \leq 0$, i.e. x_0 is either at the unstable fixed point "0" (in the case $b = 0$) or $x_0 < 0$ which is outside our domain of consideration. Thus $c_1 \neq 0$ from which it follows that

$$\frac{y_{n+1}}{y_n} \to \lambda_{max}.$$

From this it follows that

$$x_n \to x^* \doteq \frac{\lambda_{max} - d}{c},$$

and therefore x^* is a globally asymptotically stable fixed point in $\mathbb{R}^+$.

We now consider the periodic Ricatti equation

$$x_{n+1} = f_n(x_n) \doteq \frac{a_n x_n + b_n}{c_n x_n + d_n} \tag{5.4}$$

where the coefficients satisfy (5.2) and the coefficients have period $k > 0$ and k is the smallest such integer. Referring to Remark 2.2 we compute the function H in (2.4) to be

$$H(x) = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0$$

which by our previous remarks has the same form (5.1) and satisfies (5.2). Thus, applying theorem (3.4) we conclude that (5.4) has a globally asymptotically stable geometric r-cycle and $r|k$.

(ii) As our final extension we consider a class of functions $\mathcal{K}$ which generalizes the class "A1" of Cushing and Henson [1]. Recall that a *concave* function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is one such that

$$h(\alpha x + \beta y) \geq \alpha h(x) + \beta h(y) \qquad \text{for all } x, y \in \mathbb{R}^+$$

where $\alpha, \beta \geq 0, \alpha + \beta = 1$. The following property is easily verified: *If f, g are concave and f is increasing then $f \circ g$ is concave.* We next define the class $\mathcal{K}$ to be all functions which satisfy
(1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous
(2) f is increasing and concave
(3) There exist x_1 and x_2 such that $f(x_1) > x_1$ and $f(x_2) < x_2$

We now list some properties of $\mathcal{K}$.
(a) $\mathcal{K}$ is closed under the operation of composition, i.e. $f, g \in \mathcal{K}$ implies $f \circ g \in \mathcal{K}$. Thus $\mathcal{K}$ is a semi-group under composition.
(b) Each f has a unique globally asymptotically stable fixed point $x_f > 0$
(c) If $f, g \in \mathcal{K}$ with $x_f < x_g$ then $x_f < x_{f \circ g} < x_g$ and $x_f < x_{g \circ f} < x_g$

Regarding property (a), conditions (1) and (2) are immediate. To establish (3) take $f, g \in \mathcal{K}$ and assume, without loss of generality $x_f \leq x_g$. If $x_f = x_g$ then $f(x) > x$ and $g(x) > x$ for $x < x_f$ and therefore $f \circ g(x) > x$ and similarly for $x > x_f$.

If $x_f < x_g$ then $g \circ f(x_f) = g(x_f) > x_f$ and $f \circ g(x_f) > f(x_f) = x_f$. The second statement of (3) follows from a similar agrument using x_g in place of x_f.

Regarding property (b), the existence and uniqueness follows immediately from stated properties of $\mathcal{K}$ and the globally asymptotic stability follows from the fact that for $x \in \mathbb{R}^+$ with $x \neq 0, x_f$,

$$\frac{f(x) - f(x_f)}{x - x_f} < 1$$

Finally, property (c) follows by applying the intermediate value theorem to the composite functions and arguing as in part (a).

Thus, for the k-periodic difference equation

$$x_{n+1} = F(n, x_n), \qquad x \in \mathbb{R} \tag{5.5}$$

if for all $n, f_n \in \mathcal{K}$, where $f_n(x) = F(n, x)$ then

$$g(x) \doteq f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0 \in \mathcal{K}$$

and therefore represents an autonomous equation

$$x_{n+1} = g(x_n)$$

having a unique globally asymptotically stable fixed point. Therefore (5.5) has a globally asymptotically stable geometric r-cycle and by Theorem 3.4, $r|k$.

This completely answers, in the affirmative, the first of the two conjectures of Cushing and Henson [2].

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