Difference equations from discretization of a continuous epidemic model with immigration of infectives

Sophia Jang [1] and Saber Elaydi [2]

Abstract. A continuous-time epidemic model with immigration of infectives is introduced. Systems of difference equations obtained from the continuous-time model by using nonstandard discretization technique are presented. Comparisons between the continuous-time model and its discrete counterpart are made.

1 Introduction

The aim of this paper is to use the nonstandard discretization techniques, due to Mickens [9], to study the discrete analogue of the continuous *SIS* model of Brauer and Van den Driessche [3]. This work is a continuation of the work of Elaydi and collaborators [8] [2]. In these two papers, the authors used effectively nonstandard discretization techniques in competitive, cooperative and predator-prey Lotka Volterra equations. There has been a surge of activities in this area. In addition to a volume edited by Mickens [9], there will be two issues of the *Journal of Difference Equations and Applications* published in honor of Mickens (volumes 9:11, 9:12, 2003). Other notable contributions are the work of Jiang and Rogers [6], Krawcewicz and Rogers [7], Roeger and Allen [11], Roeger [12] [13], Ushiki [14] and the references therein.

Although the continuous-time logistic equation has only equilibrium dynamics, its discrete counterpart, the well known discrete logistic equation, exhibits period doubling bifurcation cascade to chaos [1] [4]. This discrete logistic equation can be obtained via a simple forward Euler approximation with step size $\Delta t = 1$. Since the step size $\Delta t = 1$ is large, the discrepancy between the ordinary differential equation and its difference approximation is inevitable.

[1]Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504
[2]Department of Mathematics, Trinity University, San Antonio, TX 78212

Another attempt is to consider the well-studied Lotka-Volterra ordinary differential equations of two populations. Ushiki [14] presented a forward Euler approximation with step size h. It was demonstrated again that the discrete model possesses period-doubling bifurcation route to chaos. Several other researchers used piecewise constant arguments to obtain a discrete analogue of the Lotka-Volterra equation. Jiang and Rogers [6] studied the competitive case and Krawcewicz and Rogers [7] discussed the cooperative case. Both studies showed dynamical inconsistency between the continuous and discrete models.

It is shown here that the existence criteria of the steady states in the continuous-time and discrete-time models are the same. Moreover, both continuous and discrete time models have the same equilibrium. However, unlike the continuous SIS model for which global asymptotic stability of the steady state can be easily established by using well known theory of Dulac criterion and Poincaré-Bendixson Theorem, the local stability of the interior steady state for the discrete model is not trivial. For the steady state on the boundary, global asymptotic stability of the discrete model can be obtained by using simple comparison arguments. For the interior equilibrium, we are only able to prove its local stability. It is demonstrated that the discrete models derived from Mickens' nonstandard discretization method have similar dynamics as the continuous model.

2 Model derivation

Let $S(t)$ and $I(t)$ denote the number of susceptibles and infectives of a population at time t. It is assumed that there is a constant flow of new members into the population, of which a fraction p ($0 \leq p \leq 1$) is infective. Let $d > 0$ be the per capita natural death rate of the population. The disease related death rate is denoted by $\alpha \geq 0$. In this model, a simple mass action βSI is used to model disease transmission, where β is a positive constant, and a fraction $\gamma \geq 0$ of these infectives recovers. We refer the reader to Brauer and van den Driessche [3] for more biological discussion about the continuous-time model.

The continuous SIS model studied by Brauer and van den Driessche [3] is given below.

$$\begin{aligned}\dot{S} &= (1-p)A - \beta SI - dS + \gamma I \\ \dot{I} &= pA + \beta SI - (d+\gamma+\alpha)I \\ & \quad S(0), I(0) \geq 0.\end{aligned} \tag{2.1}$$

The dynamics of model (2.1) are well understood. All solutions of (2.1) converge to the steady state when the system has only a single equilibrium. When there are two equilibriums, one is on the boundary and the other is in the interior, solutions with positive initial condition always asymptotically approach the interior steady state.

We now describe our discretization procedure. Let $\phi_i(h) = h + O(h^2)$, $0 < \phi_i(h) < 1$ for $i = 1, 2$. Then we replace $\dot{S}$ by $\dfrac{S_{n+1} - S_n}{\phi_1(h)}$, $\dot{I}$ by $\dfrac{I_{n+1} - I_n}{\phi_2(h)}$, S by S_{n+1}, I by I_{n+1}, SI by $S_{n+1}I_n$ in $\dot{S}$ and by S_nI_n in $\dot{I}$. Then substituting in (2.1) yields

$$\begin{aligned}\frac{S_{n+1} - S_n}{\phi_1(h)} &= (1-p)A - \beta S_{n+1}I_n - dS_{n+1} + \gamma I_n \\ \frac{I_{n+1} - I_n}{\phi_2(h)} &= pA + \beta S_nI_n - (d+\gamma+\alpha)I_{n+1}\end{aligned} \tag{2.2}$$

For simplicity, we write ϕ_i instead of $\phi_i(h)$ for $i = 1, 2$. Simplifying (2.2) we obtain the following system of difference equations.

$$\begin{aligned}S_{n+1} &= \frac{S_n + (1-p)\phi_1 A + \gamma\phi_1 I_n}{1 + \beta\phi_1 I_n + d\phi_1} \\ I_{n+1} &= \frac{I_n + \phi_2 pA + \beta\phi_2 S_n I_n}{1 + (d+\gamma+\alpha)\phi_2} \\ & \quad S_0, I_0 \geq 0.\end{aligned} \tag{2.3}$$

When $\beta = 0$, system (2.3) becomes

$$\begin{aligned}S_{n+1} &= \frac{S_n + (1-p)\phi_1 A + \gamma\phi_1 I_n}{1 + d\phi_1} \\ I_{n+1} &= \frac{I_n + \phi_2 pA}{1 + (d+\gamma+\alpha)\phi_2} \\ & \quad S_0, I_0 \geq 0.\end{aligned} \tag{2.4}$$

System (2.3) has a unique steady state $E_0 = (S_0^*, I_0^*)$, where $S_0^* = \dfrac{(1-p)A}{d} + \frac{\gamma}{d} I_0^*$ and $I_0^* = \dfrac{pA}{d+\gamma+\alpha}$. Since the equation for I_n can be decoupled from S_n, the global dynamics of (2.3) can be easily understood.

Theorem 2.1. *Every solution of* (2.3) *converges to* E_0.

Proof. Let $\delta = \dfrac{1}{1+(d+\gamma+\alpha)\phi_2}$. Then $0 < \delta < 1$. Using formula (1.2.8) in [5] we obtain

$$I_n = \delta^n I_0 + \left(\frac{1-\delta^n}{1-\delta}\right) \delta\phi_2 pA. \tag{2.5}$$

Hence

$$\lim_{n\to\infty} I_n = \frac{\delta\phi_2 pA}{1-\delta} = \frac{pA}{(d+\gamma+\alpha)} = I^*.$$

If we let $\epsilon = \dfrac{1}{1+d\phi_1}$, then substituting (2.5) in the first equation in (2.4) yields

$$S_{n+1} = \epsilon S_n + \epsilon(1-p)\phi_1 A + \epsilon\gamma\phi \left[\delta^n I_0 + \frac{(1-\delta^n)}{1-\delta}\delta\phi_2 pA\right].$$

Using formula (1.2.6) in [5] we obtain

$$S_n = \epsilon^n S_0 + \sum_{k=0}^{n-1} \epsilon^{n-k-1} \left[\epsilon(1-p)\phi_1 A + \epsilon\gamma\phi_1 \left\{\delta^k I_0 + \frac{(1-\delta^k)}{1-\delta}\delta\phi_2 pA\right\}\right].$$

Since $\epsilon^n S_0 \to 0$ and $\sum\limits_{k=0}^{n-1} \epsilon^{n-k-1}\delta^k I_0 \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \epsilon(1-p)\phi_1 A \lim_{n\to\infty} \sum_{k=0}^{n-1} \epsilon^{n-k-1} \\
&\quad + pA\delta\epsilon\gamma\phi_1\phi_2 \lim_{n\to\infty} \sum_{k=0}^{n-1} \epsilon^{n-k-1} \left(\frac{1-\delta^k}{1-\delta}\right) \\
&= \frac{(1-p)A}{d} + \frac{\gamma pA}{d(d+\gamma+\alpha)} \\
&= S^*
\end{aligned}$$

□

Suppose now $\beta > 0$ and $p = 0$. Then system (2.3) takes the following form.

$$\begin{aligned} S_{n+1} &= \frac{S_n + \phi_1 A + \gamma\phi_1 I_n}{1 + \beta\phi_1 I_n + d\phi_1} \\ I_{n+1} &= \frac{I_n + \beta\phi_2 S_n I_n}{1 + (d+\gamma+\alpha)\phi_2} \\ & \quad S_0, I_0 \geq 0. \end{aligned} \tag{2.6}$$

System (2.6) may be rewritten as $(S_{n+1}, I_{n+1}) = H(S_n, I_n)$, where $S_{n+1} = F(S_n, I_n), I_{n+1} = G(S_n, I_n)$ and $H = (F, G)$. A steady state (S, I) of (2.6) must satisfy

$$(d+\gamma+\alpha)I = \beta I \frac{A + \gamma I}{\beta I + d}.$$

One solution is $I = 0$ and another solution is

$$I = \frac{\beta A - d(d+\gamma+\alpha)}{(d+\alpha)\beta}. \tag{2.7}$$

As in the continuous model, we let

$$\sigma = \beta A - d(d+\gamma+\alpha).$$

Then $\left(\frac{A}{d}, 0\right)$ is the only feasible steady state of (2.6) if $\sigma < 0$. However, if $\sigma > 0$, then a nontrivial steady state (S^*, I^*) exists where I^* is given by (2.7) and $S^* = \dfrac{A + \gamma I^*}{\beta I^* + d} = \frac{d+\gamma+\alpha}{\beta}$.

The following lemma is trivial but it can be used to study system (2.6).

Lemma 2.2. *Let* $a = \max\left\{\frac{\gamma}{\beta}, \frac{A}{d}\right\}$. *Then* $\dfrac{\phi_1 A + \gamma\phi_1 x}{1 + \beta\phi_1 x + d\phi_1} \leq a$ *for all* $x \geq 0$.

If $\sigma < 0$, then (2.6) has only boundary steady state $\left(\frac{A}{d}, 0\right)$, which can be shown to be globally asymptotically stable.

Theorem 2.3. *If* $\beta > 0$, $p = 0$ *and* $\sigma < 0$, *then every solution of* (2.3) *converges to* $\left(\frac{A}{d}, 0\right)$.

Proof. If $I_0 = 0$, then $I_n = 0$ for $n \geq 0$ and thus $\lim_{n\to\infty} S_n = \frac{A}{d}$. We may assume $I_0 > 0$, then $I_n > 0$ for $n \geq 0$. If there exists $k = 0, 1, \cdots$ such that

$$S_k \leq \frac{d+\gamma+\alpha}{\beta}$$

then

$$S_{k+1} \leq \frac{\frac{d+\gamma+\alpha}{\beta} + \phi_1 A + \gamma\phi_1 I_k}{1 + \beta\phi_1 I_k + d\phi_1} < \frac{d+\gamma+\alpha}{\beta} \quad \text{as} \quad \sigma < 0.$$

Therefore $S_n < \frac{d+\gamma+\alpha}{\beta}$ for $n > k$. As a result,

$$I_{n+1} = \frac{(1+\beta\phi_2 S_n)I_n}{1+(d+\gamma+\alpha)\phi_2} < I_n$$

for $n > k$, which implies $\lim_{n\to\infty} I_n = \bar{I} \geq 0$ exits. Notice that if $\bar{I} = 0$, then for any $\epsilon > 0$ there exists $n' > 0$ such that $-\epsilon < I_n < \epsilon$ for $n \geq n'$. Thus

$$S_{n+1} \geq \frac{S_n + \phi_1 A}{1 + \beta\phi_1\epsilon + d\phi_1} \quad \text{for} \quad n \geq n',$$

and we have $\liminf_{n\to\infty} S_n \geq \frac{A}{d}$. Similarly it can be proven that $\limsup_{n\to\infty} S_n \leq \frac{A}{d}$. Hence $\lim_{n\to\infty} S_n = \frac{A}{d}$ and the assertion is shown. Suppose now $\bar{I} > 0$. Then

$$1 = \lim_{n\to\infty} \frac{I_{n+1}}{I_n} = \lim_{n\to\infty} \frac{1+\beta\phi_2 S_n}{1+(d+\gamma+\alpha)\phi_2}$$

implies $\lim_{n\to\infty} S_n = \frac{d+\gamma+\alpha}{\beta}$. Consequently, solutions of (2.6) converge to $(\frac{d+\gamma+\alpha}{\beta}, \bar{I})$, a fixed point of H. Since $(\frac{A}{d}, 0)$ is the only fixed point of H, we obtain a contradiction and the result follows.

Suppose on the other hand that $S_n > \frac{d+\gamma+\alpha}{\beta}$ for $n = 0, 1, \cdots$. Then $I_{n+1} > I_n$ for $n = 0, 1, \cdots$ and thus $\lim_{n\to\infty} I_n > 0$ exists (maybe ∞). Notice that $S_{n+1} \le \dfrac{S_n}{1+d\phi_1} + a$ for $n \ge 0$ by Lemma 2.2, and hence $\limsup\limits_{n\to\infty} S_n \le \dfrac{a(1+d\phi_1)}{d\phi_1}$. Consequently, if $\lim\limits_{n\to\infty} I_n = \infty$, then from the first equation of (2.6), we have $\lim\limits_{n\to\infty} S_{n+1} = \dfrac{\gamma}{\beta} < \dfrac{d+\gamma+\alpha}{\beta} \le \liminf\limits_{n\to\infty} S_n$ and obtain a contradiction. Therefore $\lim_{n\to\infty} I_n = \hat{I}$ is a positive real number. As a consequence, $\lim\limits_{n\to\infty} S_n = \dfrac{d+\gamma+\alpha}{\beta}$. But since $\sigma < 0$,

$$\begin{aligned}
\frac{d+\gamma+\alpha}{\beta} &= \lim_{n\to\infty} S_{n+1} \\
&= \frac{\dfrac{d+\gamma+\alpha}{\beta} + \phi_1 A + \gamma\phi_1\hat{I}}{1+\beta\phi_1\hat{I}+d\phi_1} \\
&< \frac{(d+\gamma+\alpha)(1+d\phi_1+\beta\phi_1\hat{I})}{\beta(1+\beta\phi_1\hat{I}+d\phi_1)} \\
&= \frac{d+\gamma+\alpha}{\beta}.
\end{aligned}$$

We thus arrive at a contradiction. Therefore, there must exist $k_0 \ge 0$ such that $S_{k_0} \le \dfrac{d+\gamma+\alpha}{\beta}$ and as a result $\left(\dfrac{A}{d}, 0\right)$ is globally asymptotically stable. $\square$

If $\beta > 0$, $p = 0$ and $\sigma > 0$, then system (2.3) has two steady states $\left(\frac{A}{d}, 0\right)$ and (S^*, I^*). Their stability are summarized below.

Theorem 2.4. *If $\beta > 0$, $p = 0$ and $\sigma > 0$, then $\left(\dfrac{A}{d}, 0\right)$ is unstable and (S^*, I^*) is locally asymptotically stable. Moreover,*

$$\frac{\gamma}{\beta} \le \liminf_{n\to\infty} S_n \le \limsup_{n\to\infty} S_n \le \frac{A}{d}$$

for any solution (S_n, I_n) of (2.3) with $S_0, I_0 > 0$.

Proof. It is straightforward to verify that (S^*, I^*) is locally asymptotically stable. Indeed, the linearization of system (2.6) about the steady state yields the following Jacobian matrix

$$J = \begin{pmatrix} \dfrac{1}{1+\beta\phi_1 I^* + d\phi_1} & \dfrac{\phi_1(\gamma + d\gamma\phi_1 - S^*\beta - A\beta\phi_1)}{(1+\beta\phi_1 I^* + d\phi_1)^2} \\ \dfrac{\beta\phi_2 I^*}{1+(d+\gamma+\alpha)\phi_2} & 1 \end{pmatrix}.$$

Note that

$$\gamma + d\gamma\phi_1 - \beta S^* - A\beta\phi_1 \quad < \quad 0 \tag{2.8}$$

implies

$$detJ = \frac{1}{1+\beta\phi_1 I^* + d\phi_1} - \frac{(\beta\phi_1\phi_2 I^*)(\gamma + d\gamma\phi_1 - S^*\beta - A\beta\phi_1)}{[1+(d+\gamma+\alpha)\phi_2](1+\beta\phi_1 I^* + d\phi_1)^2} > 0,$$

and also $trJ = 1 + \dfrac{1}{1+\beta\phi_1 I^* + d\phi_1} > 0$. Applying the Jury conditions [1], we have eigenvalues λ of J satisfying $|\lambda| < 1$ if and only if $detJ < 1$ and $trJ < 1 + detJ$. Clearly $trJ < 1 + detJ$ by (2.8), and $detJ < 1$ if and only if

$$\frac{-\beta\phi_2 I^*(\gamma + d\gamma\phi_1 - \beta S^* - A\beta\phi_1)}{[1+(d+\gamma+\alpha)\phi_2][1+\beta\phi_1 I^* + d\phi_1]} < \beta I^* + d.$$

Since $A\beta = d^2 + d\gamma + d\alpha + (\alpha + d)\beta I^*$, a straightforward calculation shows that the latter inequality is true. Hence the steady state (S^*, I^*) is locally asymptotically stable.

Furthermore, the linearization of system (2.6) about the steady state $\left(\dfrac{A}{d}, 0\right)$ has the following Jacobian matrix

$$J_0 = \begin{pmatrix} \dfrac{1}{1+d\phi_1} & \dfrac{\phi_1(\gamma + \phi_1 d\gamma - \frac{A}{d}\beta - A\beta\phi_1)}{(1+d\phi_1)^2} \\ 0 & \dfrac{1+\phi_2\beta\frac{A}{d}}{1+(d+\gamma+\alpha)\phi_2} \end{pmatrix}.$$

Clearly the eigenvalues of J_0 are $\lambda_1 = \dfrac{1}{1+d\phi_1}$ and $\lambda_2 = \dfrac{1+\beta\frac{A}{d}\phi_2}{1+(d+\gamma+\alpha)\phi_2}$. Since $\sigma > 0$, we have $\lambda_2 > 1$ and $\left(\dfrac{A}{d}, 0\right)$ is unstable.

We proceed to prove the rest of the assertions. Observe that $\frac{\gamma}{\beta} < \frac{d+\gamma+\alpha}{\beta} < \frac{A}{d}$. If $S_n < \frac{\gamma}{\beta}$ for $n = 0, 1, \cdots$, then $I_{n+1} < I_n$ for $n \geq 0$ and thus $\lim_{n\to\infty} I_n = \hat{I} \geq 0$ exists. If $\hat{I} = 0$, then we have $\liminf_{n\to\infty} S_n \geq \frac{A}{d}$ and obtain a contradiction. If $\hat{I} > 0$, then by using a similar argument as in the proof of the previous theorem, we have $\lim_{n\to\infty} S_n = \frac{d+\gamma+\alpha}{\beta}$. But this again is impossible as $S_n < \frac{\gamma}{\beta}$ for $n \geq 0$. We therefore conclude that $S_{n^*} \geq \frac{\gamma}{\beta}$ for some $n^* \geq 0$. It is then straightforward to show that $S_{n^*+1} > \frac{\gamma}{\beta}$. As a consequence $S_n \geq \frac{\gamma}{\beta}$ for all n large and $\liminf_{n\to\infty} S_n \geq \frac{\gamma}{\beta}$.

On the other hand if $S_n > \frac{A}{d}$ for $n = 0, 1, \cdots$, then $I_{n+1} > I_n$ for $n \geq 0$ and thus $\lim_{n\to\infty} I_n = \bar{I} > 0$ exists. By using similar arguments as in the proof of Theorem 2.3, we have $\lim_{n\to\infty} S_{n+1} = \frac{\gamma}{\beta}$ if $\bar{I} = \infty$ and $\lim_{n\to\infty} S_n = \frac{d+\gamma+\alpha}{\beta}$ if $\bar{I}$ is a real number. In any case we obtain a contradiction. Hence $S_{k'} \leq \frac{A}{d}$ for some $k' \geq 0$ and

$$S_{k'+1} \leq \frac{A + dA\phi_1 + \gamma d\phi_1 I_{k'}}{d(1 + \beta\phi_1 I_{k'} + d\phi_1)} < \frac{A + dA\phi_1 + \beta A\phi_1 I_{k'}}{d(1 + \beta\phi_1 I_{k'} + d\phi_1)} = \frac{A}{d}.$$

Therefore $S_n \leq \frac{A}{d}$ for all n large and $\limsup_{n\to\infty} S_n \leq \frac{A}{d}$ is shown. □

Suppose now $\beta > 0$ and $p > 0$. Then steady state (S, I) of (2.3) must satisfy

$$\beta(d+\alpha)I^2 - \sigma I - dpA = 0,$$

where $\sigma = \beta A - d(d+\gamma+\alpha)$. One root is negative and the other root is

$$\bar{I} = \frac{\sigma + \sqrt{\sigma^2 + 4\beta dpA(d+\alpha)}}{2\beta(d+\alpha)} > 0.$$

Consequently,

$$\bar{S} = \frac{A + \gamma\bar{I}}{\beta\bar{I} + d}$$

and $(\bar{S},\bar{I})$ is the only feasible steady state of system (2.3). Similar to the continuous model, it can be shown that $\bar{I} > I_0^*$, where I_0^* is the I-component of the steady state of (2.3) when $\beta = 0$. The linearization of (2.3) at $(\bar{S},\bar{I})$ yields the following Jacobian matrix

$$J = \begin{pmatrix} \dfrac{1}{1+\beta\phi_1\bar{I}+d\phi_1} & \dfrac{\phi_1(\gamma+d\gamma\phi_1-\beta\bar{S}-(1-p)A\beta\phi_1)}{(1+\beta\phi_1\bar{I}+d\phi_1)^2} \\ \dfrac{\beta\phi_2\bar{I}}{1+(d+\gamma+\alpha)\phi_2} & \dfrac{1+\beta\phi_2\bar{S}}{1+(d+\gamma+\alpha)\phi_2} \end{pmatrix}.$$

Unlike the continuous model for which global asymptotic stability of the positive steady state can be easily proved by using the Dulac criterion and the Poincaré-Bendixson Theorem, the local stability of the steady state for the discrete model is not trivial and requires a lot of computations. However, these computations are straightforward. We will use the Jury condition to show $(\bar{S},\bar{I})$ is locally asymptotically stable for (2.3) if $\phi_1 < \dfrac{d+\alpha}{d\gamma - A\beta}$ when $d\gamma > A\beta$ is true. When $d\gamma \leq A\beta$, then the inequality is unnecessary.

Proposition 2.5. *If $d\gamma \leq A\beta$, then $(\bar{S},\bar{I})$ is locally asymptotically stable for model (2.3). If $d\gamma > A\beta$, then $(\bar{S},\bar{I})$ is locally asymptotically stable if $\phi_1 < \dfrac{d+\alpha}{d\gamma - A\beta}$.*

Proof. Since $trJ > 0$, the Jury condition states that $(\bar{S},\bar{I})$ is locally asymptotically stable if and only if the Jacobian matrix J satisfying $trJ < 1 + detJ < 2$. We first claim that $detJ < 1$, where $detJ$ is

$$\frac{1+\beta\phi_2\bar{S}}{(1+\beta\phi_1\bar{I}+d\phi_1)[1+(d+\gamma+\alpha)\phi_2]} - \frac{\beta\phi_1\phi_2\bar{I}[\gamma+d\gamma\phi_1-\beta\bar{S}-(1-p)A\beta\phi_1]}{[1+(d+\gamma+\alpha)\phi_2](1+\beta\phi_1\bar{I}+d\phi_1)^2}.$$

Thus $detJ < 1$ if and only if the following inequality is true

$$\begin{aligned} & \beta\phi_2\bar{S} + 2\beta^2\phi_1\phi_2\bar{S}\bar{I} + \alpha\beta\phi_1\phi_2\bar{S} - \beta\gamma\phi_1\phi_2\bar{I} - \beta d\gamma\phi_1^2\phi_2\bar{I} + (1-p)A\beta^2\phi_1^2\phi_2\bar{I} \\ < \; & (d+\gamma+\alpha)\phi_2 + \beta\phi_1\bar{I} + 2\beta(d+\gamma+\alpha)\phi_1\phi_2\bar{I} + \beta^2\phi_1^2\bar{I}^2 + d\phi_1 + 2d(d+\gamma+\alpha)\phi_1\phi_2 \\ + \; & \beta^2(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I}^2 + 2d\beta\phi_1^2\bar{I} + 2d\beta(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I} + d^2\phi_1^2 + d^2(d+\gamma+\alpha)\phi_1^2\phi_2 \end{aligned} \tag{2.9}$$

Note by (2.3) we have

$$\beta\bar{S}\bar{I} = (d+\gamma+\alpha)\bar{I} - pA < (d+\gamma+\alpha)\bar{I}. \tag{2.10}$$

By using (2.10) and after some simplifications, inequality (2.9) becomes

$$\begin{aligned} & -2\beta\phi_1\phi_2 pA - \beta\gamma\phi_1\phi_2\bar{I} - \beta d\gamma\phi_1^2\phi_2\bar{I} + (1-p)A\beta^2\phi_1^2\phi_2\bar{I} \qquad (2.11) \\ < \; & \beta\phi_1\bar{I} + \beta^2\phi_1^2\bar{I}^2 + \beta^2(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I}^2 + 2d\beta\phi_1^2\bar{I} + 2d\beta(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I} \\ + \; & d^2\phi_1^2 + d^2(d+\gamma+\alpha)\phi_1^2\phi_2 \end{aligned}$$

The only positive term left on the left hand side of (2.11) is $(1-p)A\beta^2\phi_1^2\phi_2\bar{I}$. Since (2.10) is true and

$$(1-p)A = \beta\bar{S}\bar{I} + d\bar{S} - \gamma\bar{I},$$

we can conclude from (2.11) that $detJ < 1$.

To show $trJ < 1 + detJ$, it is equivalent to show the inequality

$$\begin{aligned} & \beta^2\phi_1\phi_2\bar{S}\bar{I} + \beta^3\phi_1^2\phi_2\bar{S}\bar{I}^2 + d\beta\phi_1\phi_2\bar{S} + 2d\beta^2\phi_1^2\phi_2\bar{S}\bar{I} + \beta d^2\phi_1^2\phi_2\bar{S} \\ & < \beta(d+\gamma+\alpha)\phi_1\phi_2\bar{I} + \beta^2(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I}^2 + d(d+\gamma+\alpha)\phi_1\phi_2 \qquad (2.12) \\ & + 2d\beta(d+\gamma+\alpha)\phi_1^2\phi_2\bar{I} + d^2(d+\gamma+\alpha)\phi_1^2\phi_2 - \beta\gamma\phi_1\phi_2\bar{I} - d\gamma\beta\phi_1^2\phi_2\bar{I} \\ & + (1-p)A\beta^2\phi_1^2\phi_2\bar{I}. \end{aligned}$$

Applying (2.10), (2.12) becomes

$$\beta d\gamma\phi_1^2\phi_2\bar{I} - 2d\beta pA\phi_1^2\phi_2 < \beta(d+\alpha)\phi_1\phi_2\bar{I} + A\beta^2\phi_1^2\phi_2\bar{I}.$$

If $d\gamma \leq A\beta$, then the inequality is trivially true. If $d\gamma > A\beta$, then a sufficient condition for the inequality to be true is by requiring $\phi_1 < \dfrac{d+\alpha}{d\gamma - A\beta}$. □

Numerical simulations of system (2.3) and (2.6) showed that solutions converge to the interior steady state. Moreover, global asymptotic stability of the interior equilibrium was also proved for the continuous-time model (2.1). We conjecture that the interior steady state is globally asymptotic stable for the discrete models. However, since the system is neither competitive nor cooperative, one may construct a Liapunov function to show the conjecture.

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