Three-Dimensional Jump Systems and Manhattan Polytopes

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August 7, 2002

Abstract

Jump systems are sets of lattice points of varying sizes, and which satisfy a "two-step" axiom. This paper explores the question of whether the union of two-dimensional bounded sets of integer points embedded in $\mathbf{Z}^3$ form a jump system, and other results such as varied operations on three-dimensional jump systems.

1 An Introduction to Jump Systems

The concept of jump systems, that is, nonempty sets of vectors satisfying a "two-step" axiom, was created in 1995 by Bouchet and Cunningham. Jump systems are primarily used to describe degree systems of graphs and in problems of matroid theory. We introduce the basic definitions and concepts needed to analyze jump systems in this introductory section. For more information, see other papers on jump systems. [1, 2, 3, 4]

First fix a finite set S. For elements of $\mathbb{Z}^S$, we use the "taxicab" norm $|x| = \sum_{i \in S} |x_i|$ and corresponding distance $d(x, y) = |x - y|$.

For convenience in our proofs, we introduce the concept of the *generalized interval* of integers a and b denoted [[a,b]] and defined as follows: $[[a, b]] =: \{x \in \mathbb{Z} : \min(a, b) \leq x \leq \max(a, b), \}$.

We also use the concept of a box, which is important in the proofs of this paper. For a,b $\in \mathbf{Z}^S$, define $\text{box}(a, b) =: \{x \in \mathbb{Z}^S : x_i \in [[a_i, b_i]], \forall i\}$ For $x, y \in \mathbb{Z}^S$, the box of x and y, denoted $\text{box}(x, y)$, is defined as follows:

$$\text{box } (x, y) := \{z \in \mathbb{Z}^S : z_i \in [[x_i, y_i]], i \in S\}$$

For $x, y \in \mathbb{Z}^S$, we call z a *step from x to y* or an *(x,y)-step* if $d(x, z) = 1$ and $z \in \text{box}(x, y)$. We say that a collection of integer points $J \subseteq \mathbb{Z}^S$ is a *jump system* if it satisfies Axiom 1.

Axiom 1 (Two-Step Axiom). *Given $x, y \in J$ and (x,y)-step z, either $z \in J$ or $\exists$ (z,y)-step z' such that $z' \in J$.*

The following operations on jump systems preserve Axiom 1.1 and greatly simplify proofs concerning the properties of jump systems:

Reflection: For given $i \in S$, the reflection of J in the i^{th} coordinate is the set obtained by negating x_i in each point x.

Translation: Given $v \in \mathbb{Z}^S$, adding v to each point in J translates J by v.

Intersection with a Box: Given box B such that $J \cap B \neq \emptyset$, then $J \cap B$ is a jump system. This is a special case of a result of *Lovász* [3].

We will proceed with a discussion of faces of jump systems and the relationship of these faces to our Manhattan polytopes. We will discuss Manhattan polytopes and their properties, and then present several theorems involving these polytopes, followed by an examination of some additional operations that can be performed on jump systems.

Understanding the concept of faces is necessary for the understanding of many theorems and proofs about jump systems. We give the definition of faces as follows:

Let $J \subseteq \mathbb{Z}^S$ be a jump system. Let $v \in \mathbb{R}^S$ $v \neq 0$, and let $m_v = \max\{v^T x, x \in J\}$. The set $f_v = \{x \in J : v^T x = m_v\}$ is called a *face* of J. The value m_v is called the *face value* of f_v. In short, the points on the "v-face," f_v, of J, are the points in J that lie furthest in the direction indicated by the vector v.

It is a useful fact that the only faces f_v of a jump system that can contain more than one point are the faces which have $v \in \{-1, 0, 1\}^S$ $(v \neq 0)$. These faces are sufficient for describing jump systems.

2 Three-Dimensional Jump Systems: Manhattan Polytopes

In this section, we present some theorems and definitions necessary for the introduction of Manhattan polytopes and following theorems and results.

2.1 Properties of MPs

We now define Manhattan polytope. Let $S \subseteq \mathbb{Z}^2 \times \{r\}, r \in \mathbb{Z}$. Let $v \in \{-1, 0, 1\}^2 \times \{0\}$ $(v \neq 0)$, and let m_v be the face value of the face f_v. We define the *Manhattan polytope (MP)*, P_S associated with S as follows:

$$P_S = \{x \in \mathbb{Z}^2 \times \{r\} : v^T x \leq m_v\}.$$

We say that a point x in S is in the MP if for all nonzero $v \in \{-1, 0, 1\}^2 \times \{0\}$, $v^T x \leq m_v$. Note that Manhattan polytopes are a special class of jump systems, namely those with no holes, or "gaps".

2.2 MP Translation

We now present a theorem that answers the question: Given J_1, J_2, MPs, and $J_1 = J_2 + v + e_3$ for vector of translation v and unit vector in the third coordinate e_3, when is $J_1 \cup J_2$ a jump system? This result sheds light on some necessary geometric conditions for three dimensional Manhattan polytopes, which are special-case jump systems.

We define corresponding points $a \in J_1$ and $b \in J_2$ as points that share the relationship $b = a + v$, where v is some vector of translation.

We also define *feasible points* to be points that are not in the jump system J.

Theorem 1 (Translation Theorem). *We have J_1, a Manhattan polytope (MP) embedded in $\mathbb{Z}^3$ via $\mathbb{Z}^2 \times \{p\}$. J_2, which is also a MP, equals J_1 translated by some vector v. We have that $J_2 \subseteq \mathbb{Z}^2 \times \{r\}$, and $|p - r| = 1$.*

For J_1, J_2, MPs, the following are equivalent:

1. *$J_1 \cup J_2 \subseteq \mathbb{Z}^3$ is a jump system.*
2. *For any two corresponding points, $a = (a_1, a_2, p)$ and $b = (b_1, b_2, r)$, Either*

   a) $a_1 = b_1$ and $|a_2 - b_2| \leq 1$, or

   b) $a_2 = b_2$ and $|a_1 - b_1| \leq 1$.

Proof. $(1 \Rightarrow 2)$

We look at corresponding corners a and b, such that they are the intersections of the North and Northeast faces of their respective MPs. For $a = (a_1, a_2, p)$ and $b = (b_1, b_2, r)$ corresponding corners where $|p - r| = 1$, we prove by way of contradiction.

There are four possibilities for the relationships of a_1 and b_1 and a_2 and b_2 if neither $a_1 = b_1$ nor $a_2 = b_2$. That is, when $a_1 > b_1$ and $a_2 > b_2$, when $a_1 < b_1$ and $a_2 < b_2$, when $a_1 < b_1$ and $a_2 > b_2$, and when $a_1 > b_1$ and $a_2 < b_2$. See Figure 1. We notice that by coordinate swapping and reflection, all of the above cases collapse to case 1, where $a_1 > b_1$ and $a_2 > b_2$. We examine this case, the result of which will lead to further restrictions on the variables.

Case 1: Assume $a_1 > b_1$ and $a_2 > b_2$. ¿From $a_1 > b_1$, we have $a_1 - b_1 > 0$, and thus $a_1 - b_1 \geq 1$. Similarly, from $a_2 > b_2$, we have $a_2 - b_2 > 0$, and thus $a_2 - b_2 \geq 1$. Taken together, these inequalities produce the joint inequality $(a_1 + a_2) - (b_1 + b_2) \geq 2$. We have $a, b \in J$, so, consider an (a,b)-step z s.t. $z = (a_1, a_2, r) \notin J$. By the jump system axiom, $\exists z'$, a (z,b)-step s.t. $z' \in J$. $z' = (a_1 - 1, a_2, r)$ or $z' = (a_1, a_2 - 1, r)$. Either case produces a $z' \in J$ because z' would give $((a_1 - 1) + a_2) - (b_1 + b_2) \geq 2 - 1 \geq 1$ or $(a_1 + (a_2 - 1)) - (b_1 + b_2) \geq 2 - 1 \geq 1$, separating z' and the nearest point in J, b, by at least one unit. So $z' \notin J$ and the two step axiom does not hold. We have a contradiction.

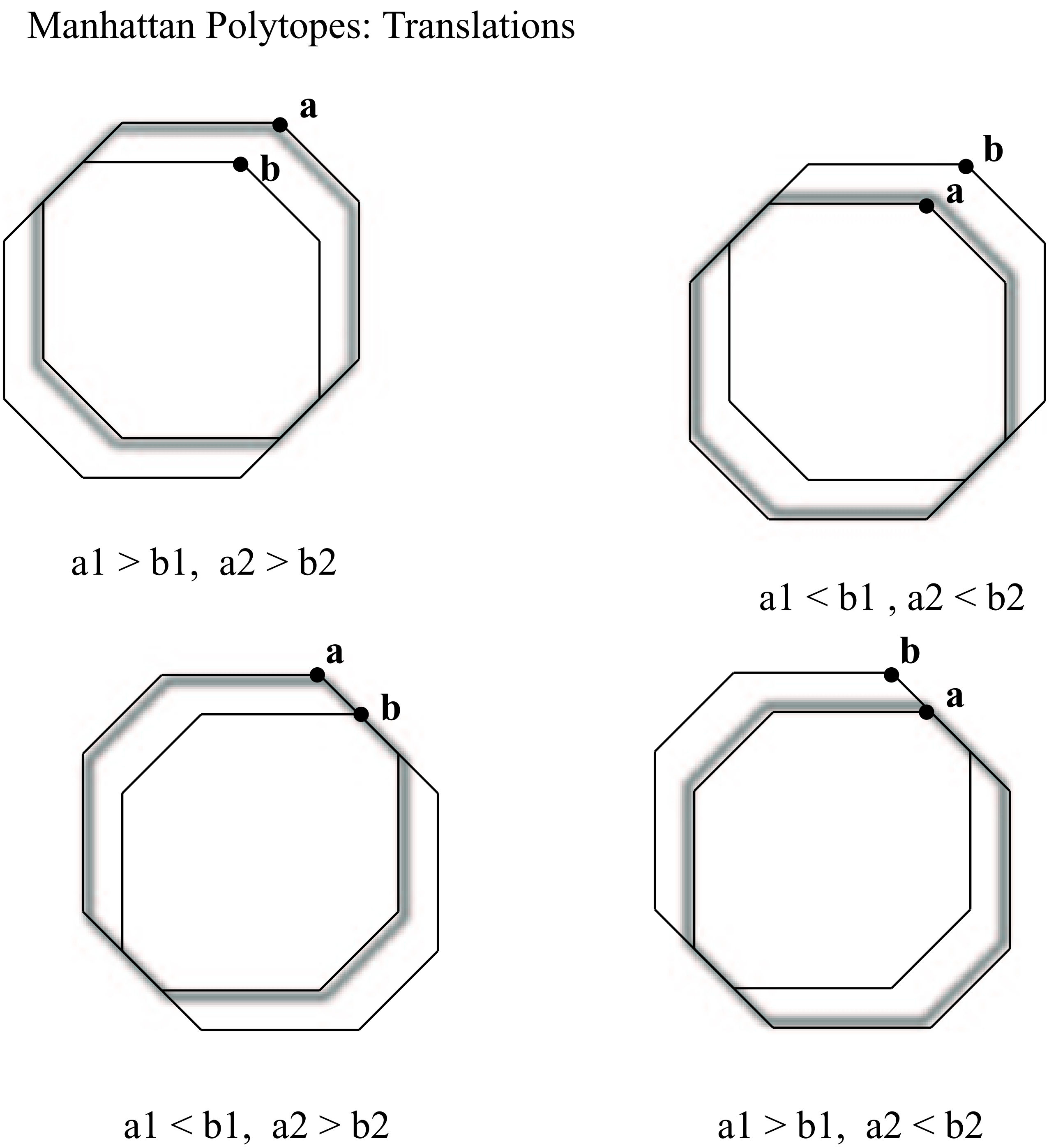


Figure 1: Manhattan Polytope Translation

Case 2: Since we have determined that cases in which neither $a_1 = b_1$ nor $a_2 = b_2$ are impossible, we can conclude that either $a_1 = b_1$ or $a_2 = b_2$ or both.

Subcase 1: If both $a_1 = b_1$ and $a_2 = b_2$, we satisfy (2), because J_1 and J_2 line up perfectly.

Subcase 2: Given $a_1 = b_1$, we determine the restrictions on a_2 and b_2. By way of contradiction, assume $|b_2 - a_2| > 1$. Given corresponding corners $a = (a_1, a_2, p)$ and $b = (b_1, b_2, r)$ We take step $z = (a_1, a_2, r) = (b_1, a_2, r)$, and we know that $z \notin J$ because b is on the North face of J_2. By the Jump System axiom, $z' = (b_1, a_2 - 1, r) \in J$, but without loss of generality from $|b_2 - a_2| > 1$ we get $a_2 - b_2 \geq 2$, or $a_2 - b_2 > 1$, which yields $a_2 - 1 > b_2$, a contradiction because if that were true, $z' \notin J$. We conclude, then, that when $a_1 = b_1$, $|b_2 - a_2| \leq 1$.

Subcase 3: Given $a_2 = b_2$, we determine the restrictions on a_1 and b_1. By way of contradiction, assume $|b_1 - a_1| > 1$. Given corresponding corners $a = (a_1, a_2, p)$ and $b = (b_1, b_2, r)$ We take step $z = (a_1, a_2, r) = (a_1, b_2, r)$, and we know that $z \notin J$ because b is on the Northeast face of J_2. By the Jump System axiom, $z' = (a_1 - 1, b_2 - 1, r) \in J$, but without loss of generality from $|b_1 - a_1| > 1$ we get $a_1 - b_1 \geq 2$, or $a_1 - b_1 > 1$, which yields $a_1 - 1 > b_1$, a contradiction because if that were true, $z' \notin J$. We conclude, then, that when $a_2 = b_2$, $|b_1 - a_1| \leq 1$.

Thus, (2) holds.

$(2 \Rightarrow 1)$

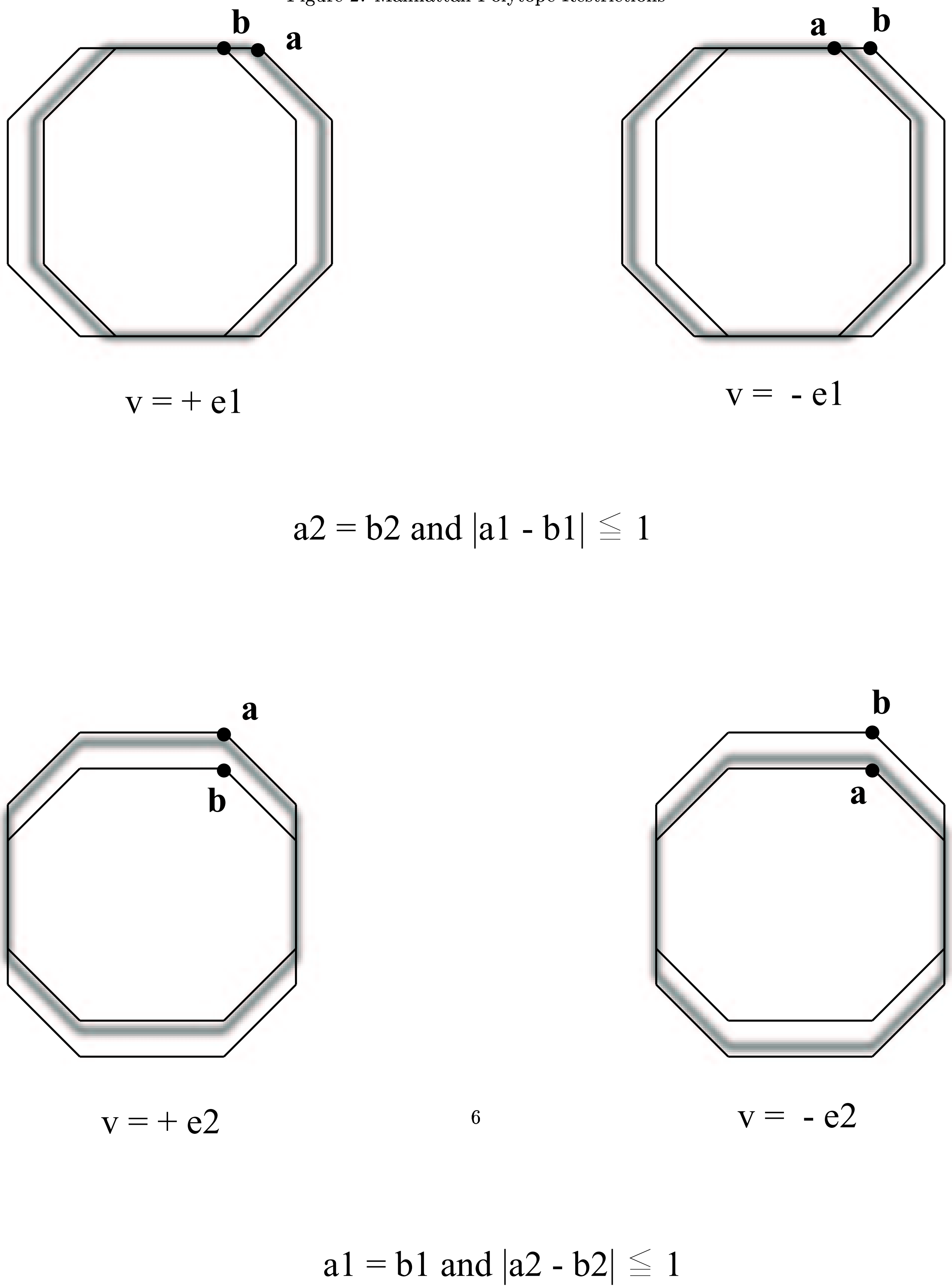
¿From our criteria, we note that J_1 can only be vertically or horizontally translated by a unit vector ($v = 0$, horizontal translation $v = \pm e_1$, or vertical translation $v = \pm e_2$). See Figure 2. Let x, y be arbitrary points in J, and let z be an (x,y)-step.

Case 1: If $x, y \in J_1$ or $x, y \in J_2$, we know that since J_1 and J_2 are solid MPs, the two-step axiom holds for any two points in either J_1 or J_2, and we're done.

Case 2: If $x \in J_1$, $y \in J_2$, and (x,y)-step $z \in J$, we're done. So assume $z \notin J$. We now look for a (z,y)-step z' s.t. $z' \in J$. We notice that $v \neq 0$ because any (x,y)-step z we could take would be in J.

Subcase 1: Notice that by reflection and coordinate swapping, cases $v = \pm e_1$ and $v = \pm e_2$ collapse to the single case for $v = +e_1$, without loss of generality. So for $v = +e_1$, we note that the only interesting case is when x is on a face of J_1. This is because if x was not on a face, any (x,y)-step z would be in J. We are interested in all possible (x,y)-steps z s.t. $z \notin J$. We look at these steps for arbitrary x on each face of J_1. Before examining each face individually, we will note that the case for x on the NE face can yield the case for x on the SE face by reflecting J_1 in the second coordinate and shifting J_1 back to its original position in the first quadrant. Thus without loss of generality we need

Figure 2: Manhattan Polytope Restrictions



only examine the case for x on the NE face rather than both NE and SE. The same reflection can be applied to the case for x on the NW face, so we need not examine the case for x on the SW face. We lastly note that the case for x on the W face is similar to the NW and SW cases, so we handle the West case in a small argument following the NW case. We therefore focus mainly on the cases when x is on E, NE, and NW faces. We also ignore the cases where x is on either the N or S faces. In these cases, all possible (x,y)-steps z are either in J, or or outside J and not in the direction of y.

A) For $x = (x_1, x_2, p)$ on the E face of J_1, possible (x,y)-steps z are: $z = (x_1 + 1, x_2, p), z = (x_1, x_2 + 1, p), z = (x_1 - 1, x_2, p), z = (x_1, x_2 - 1, p), z = (x_1, x_2, r)$. Since x is on the E face, $z = (x_1, x_2 + 1, p), z = (x_1 - 1, x_2, p)$ and $z = (x_1, x_2 - 1, p)$ are all in J. We disregard these cases. We also see that while $z = (x_1 + 1, x_2, p) \notin J$, it will never be in the direction of y since x is on the E face and all points have first coordinates less than or equal to x_1. We examine the only feasible point, $z = (x_1, x_2, r)$. Consider $z' = (x_1 - 1, x_2, r)$. Observe that $d(z, z') = 1$. We note that the E face is the (1,0) face where x_1 is maximized, and we get the inequality $x_1 \geq y_1 + 1 > y_1$. Since x_1 is the first coordinate of z, it is clear that $z' \in \mathrm{box}(z, y)$. If x_1 is maximized on the E face of J_1, then $x_1 - 1$ is maximized on the E face of J_2 because $v = +e_1$. So if z' has first coordinate less than or equal to $x_1 - 1$, which it does, then z' is behind or on the E face of J_2. We also see that x, which is on the E face of J_1, has second coordinate x_2. Since z' also has second coordinate x_2, it must be on the corresponding E face of J_2. Therefore, $z' \in J_2$ and $z' \in J$.

B) For $x = (x_1, x_2, p)$ on the NE face of J_1, possible (x,y)-steps z are: $z = (x_1 + 1, x_2, p), z = (x_1, x_2 + 1, p), z = (x_1 - 1, x_2, p), z = (x_1, x_2 - 1, p), z = (x_1, x_2, r)$. Since x is on the NE face, $z = (x_1 - 1, x_2, p)$ and $z = (x_1, x_2 - 1, p)$ are both in J. We disregard these cases. We examine the feasible points $z = (x_1 + 1, x_2, p), z = (x_1, x_2 + 1, p)$, or $z = (x_1, x_2, r)$.

a) For $z = (x_1, x_2 + 1, p)$, $z \notin J$ and could be in the direction of y. Consider $z' = (x_1 - 1, x_2 + 1, p)$. Observe $d(z, z') = 1$. Recall that the North and South faces yield only trivial (x,y)-steps, and thus any points on those faces are not discussed. We therefore conclude that $z' \in J$ because x is not on the North face. We have only to prove now that $z' \in \mathrm{box}(x, y)$. Since we know x is on NE face, $x_1 + x_2 \geq y_1 + y_2 + 1$ and since z takes us to $(x_1, x_2 + 1, p)$, we know that $y_2 \geq x_2 + 1$. Substituting, we get $x_1 + x_2 \geq y_1 + y_2 + 1 \geq y_1 + x_2 + 2$, which yields $x_1 \geq y_1 + 2$. So from $x_1 \geq y_1 + 2$ and $x_1 \geq x_1 - 1$, and from $y_2 \geq x_2 + 1$, which translates to $x_2 \leq y_2 - 1$, and $x_2 \leq x_2 + 1$, we clearly see that $z' \in \mathrm{box}(z, y)$.

b) For $z = (x_1 + 1, x_2, p)$, $z \notin J$ and could be in the direction of y. Consider $z' = (x_1 + 1, x_2 - 1, p)$. Observe $d(z, z') = 1$. Recall section B under subcase 1, and the cases dealing with x on the East face. If x were on the East face, we would follow the given procedure. We therefore know that $z' \in J$ because x is not on E face. We have only to prove now that $z' \in \mathrm{box}(x, y)$. Since we know x is on NE face, $x_1 + x_2 \geq y_1 + y_2 + 1$ and since z takes us to $(x_1 + 1, x_2, p)$, we know that $y_1 \geq x_1 + 1$. Substituting, we get $x_1 + x_2 \geq y_1 + y_2 + 1 \geq x_1 + y_2 + 2$, which yields $x_2 \geq y_2 + 2$. So from $x_2 \geq y_2 + 2$ and $x_2 \geq x_2 - 1$, and from $y_1 \geq x_1 + 1$, which translates to $x_1 \leq y_1 - 1$, and $x_1 \leq x_1 + 1$, we clearly see that $z' \in \text{box}(z, y)$.

c) For $z = (x_1, x_2, r)$, $z \notin J$ and is in the direction of y. For this z, there are two possible choices for z'. Both z' are one step from z, and both $z' \in J$. We can step to $z'_1 = (x_1 - 1, x_2, r)$ or $z'_2 = (x_1, x_2 + 1, r)$. Which step to take depends on where y is. We must prove that we can always step one way or the other, and that at least one situation can happen. We have x on the NE face, so we have the inequality $x_1 + x_2 \geq y_1 + y_2 + 1$. Assume that neither $z'_1 \in \text{box}(z, y)$ nor $z'_2 \in \text{box}(z, y)$ and $y_1 \geq x_1$ and $y_2 \geq x_2$. Combining the inequalities, we get $y_1 + y_2 \geq x_1 + x_2$. Combining this inequality with the first, we get $y_1 + y_2 \geq x_1 + x_2 \geq y_1 + y_2 + 1$, which gives a contradiction. Now that we know that either one situation or the other must happen, we can proceed to show that both z' steps are in J. Both steps clearly have $d(z, z') = 1$. Depending on which z' step was taken, z' was determined to be in the direction of y. We have left to prove that both z' are in J. We know that x is on the NE face of J_1, so $x_1 + x_2$ is maximized on J_1. Any point on J_2 with the sum of its first two coordinates less than or equal to $x_1 + x_2 - 1$, will be behind or on the NE face of J_2. Both of our z' satisfy this requirement and have a sum of $x_1 + x_2 - 1$ for their first two coordinates, so both z' are on the NE face of J_2. We noted before that x was not on the N face of J_1, so since neither z' has increased second coordinate, we know that both z' are below the N face of J_2. Both z' are safely within J_2, so we can say that both $z' \in J$.

C) For $x = (x_1, x_2, p)$ on the NW face of J_1, possible (x,y)-steps z are: $z = (x_1 + 1, x_2, p), z = (x_1, x_2 + 1, p), z = (x_1 - 1, x_2, p), z = (x_1, x_2 - 1, p), z = (x_1, x_2, r)$. Since x is on the NW face, $z = (x_1 + 1, x_2, p), z = (x_1, x_2 - 1, p)$ and $z = (x_1, x_2, r)$ are all in J. We disregard these cases. We examine feasible points $z = (x_1, x_2 + 1, p)$ and $z = (x_1 - 1, x_2, p)$.

For both possible z, we can easily see that either z' can be a step down from its corresponding z. For $z = (x_1, x_2 + 1, p)$, $z' = (x_1, x_2 + 1, r)$, and for $z = (x_1 - 1, x_2, p)$, $z' = (x_1 - 1, x_2, r)$. For both of these z', it is clear that $d(z, z') = 1$. We also know that both z' are in the direction of y since they both step down to third coordinate r, where y lives. We have only left to show that both z' are in J. Because of $v = +e_1$, if x is on any part of J_1, the following is true: $-y_1 + y_2 \geq -x_1 + x_2 + 1$. Now, if x is on any western face of J_1, the inequality becomes an equality. ¿From the equality we know that before either of these (x,y)-steps z were taken, any point on a western face of J_1 was one step away from its corresponding point on the same western face of J_2. So once that step z is taken, we are still within the boundary of J_2's faces, allowing us to step down from either step z to make our way towards y, and leaving either step down (both z') in J_2 and thus in J. So both possible z' are in J.

D) We note that the case for x on the West face of J_1 is similar to that for x on the NW face, except that instead of the two feasible points listed for the

NW case, the West case has only one feasible point, namely, $z = (x_1 - 1, x_2, p)$. The argument remains the same.

We conclude that $2 \Rightarrow 1$. □

2.3 Degenerate MPs: The Box Theorem

In this section, we present a result for a special class of three-dimensional jump systems: those consisting of two"layers" (or cross sections) such that each layer is a box. First, however, we define some terms that will be used in the statement of the theorem and its proof:

For points a and b in $\mathbb{Z}^2$, let $B = \text{box}(a, b)$ be given. A point $k \in B$ is called a *corner of* B if and only if there exists some point $j \in \mathbb{Z}^2$ such that $\text{box}(k, j) = B$. The two points k and j are called *opposite corners* of B.

Given a box B, it is clear that, for nonzero $v \in \{1, -1\} \times \{0\}$, f_v corresponds to a corner of B. Given two boxes, B and B', let f_v be a face of B and let f'_v be a face of B'. Then f_v and f'_v are *corresponding faces* of B and B'. Particularly, for $a \in B$ and $a' \in B'$, we call a and a' *corresponding corners* if for some nonzero $v \in \{1, -1\} \times \{0\}$, $a \in f_v$ and $a' \in f'_v$.

Theorem 2. *Given $B \subseteq \mathbb{Z}^2 \times R$ and $B' \subseteq \mathbb{Z}^2 \times P$, for r and p in $\mathbb{Z}$ where $|r - p| = 1$, the following are equivalent:*

1. $d(q, q') \leq 2$ where q and q' are any two corresponding corners in B and B' respectively.

2. Either: $k'_1 = k_1$ and $|k_2 - k'_2| \leq 1$ for all corresponding corners, $k \in B$ and $k' \in B'$, or $k'_2 = k_2$ and $|k_1 - k'_1| \leq 1$ for all corresponding corners, $k \in B$ and $k' \in B'$.

Proof. Let $B = \text{box}(a, b)$ for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ where $a_3 = b_3 = r$ and $a_i \leq b_i$. Let $B' = \text{box}(a', b')$ for $a' = (a'_1, a'_2, a'_3)$ and $b' = (b'_1, b'_2, b'_3)$ where $a'_3 = b'_3 = p$ and $a'_i \leq b'_i$. Call the other two corners of B, c and d. Particularly, let $c = (c_1, c_2, c_3) = (a_1, b_2, r)$ and $d = (d_1, d_2, d_3) = (b_1, a_2, r)$. Call the other two corners of B', c' and d'. Particularly, let $c' = (c'_1, c'_2, c'_3) = (a'_1, b'_2, p)$ and $d' = (d'_1, d'_2, d'_3) = (b'_1, a'_2, p)$.

(1⇒2) Assume $\forall q \in \{a, b, c, d\}$, $d(q, q') \leq 2$. Then, particularly, for $q = a$, $d(a, a') = |a_1 - a'_1| + |a_2 - a'_2| + |r - p| = |a_1 - a'_1| + |a_2 - a'_2| + 1 \leq 2$. So $|a_1 - a'_1| + |a_2 - a'_2| \leq 1$.Thus, either $|a_1 - a'_1| + |a_2 - a'_2| = 0$ implying that $a'_1 = a_1$ and $a'_2 = a_2$, or $|a_1 - a'_1| + |a_2 - a'_2| = 1$. If the latter is true, then either $|a_1 - a'_1| = 1$ and $|a_2 - a'_2| = 0$, i.e. $a'_1 = a_1 \pm 1$ and $a'_2 = a_2$, or $|a_2 - a'_2| = 1$ and $|a_1 - a'_1| = 0$ i.e. $a'_2 = a_2 \pm 1$ and $a'_1 = a_1$.

Case 1: ($a'_1 = a_1, a'_2 = a_2$) Since $a'_1 = a_1, a'_2 = a_2$, $c'_1 = a'_1 = a_1 = c_1$, i.e. $c'_1 = c_1$. Either $|c_2 - c'_2| \leq 1$ or $|c_2 - c'_2| > 1$. Notice that $|c_2 - c'_2| > 1 \Rightarrow |c_2 - c'_2| \geq 2$ so, $d(c, c') = |c_1 - c'_1| + |c_2 - c'_2| + |c_3 - c'_3| = |c_2 - c'_2| + 1 \geq 3$ contradicting the hypothesis that $d(q, q') \leq 2$ $\forall q \in \{a, b, c, d\}$. Thus $|c_2 - c'_2| \leq 1$, i.e. either $|c_2 - c'_2| = 0$ or $|c_2 - c'_2| = 1$.

Subcase 1: ($|c_2 - c_2'| = 1 \leq 1$) We will show that for all corresponding corners k and k', $k_1' = k_1$ and $|k_2 - k_2'| \leq 1$. We consider all pairs of corresponding corners in turn.

First, consider b and b'. Since, $c_2 = b_2$ and $c_2' = b_2'$, $|b_2 - b_2'| = |c_2 - c_2'| = 1$ i.e. $|b_2 - b_2'| \leq 1$. Either $b_1' = b_1$ or $b_1' \neq b_1$. Notice that $b_1' \neq b_1 \Rightarrow |b_1 - b_1'| > 0 \Rightarrow |b_1 - b_1'| = 1 \geq 1$, so $\mathrm{d}(b, b') = |b_1 - b_1'| + |b_2 - b_2'| + |b_3 - b_3'| = |b_1 - b_1'| + 1 + 1 \geq 3$ contradicting the hypothesis that $\mathrm{d}(q, q') \leq 2 \ \forall q \in \{a, b, c, d\}$. Thus $b_1' = b_1$.

Now, consider d, d'. Since $b_1' = b_1$, $d_1' = b_1' = b_1 = d_1$ i.e. $d_1' = d_1$. Also, $d_2' = a_2' = a_2 = d_2$ so $|d_2 - d_2'| = |a_2 - a_2'| = 0 \leq 1$.

Summarizing: $a_1' = a_1, a_2' = a_2$ i.e. $|a_2 - a_2'| = 0 \leq 1$, $c_1' = c_1, |c_2 - c_2'| = 1 \leq 1$, $b_1' = b_1, |b_2 - b_2'| = 1 \leq 1$, $d_1' = d_1, |d_2 - d_2'| = 0 \leq 1$.

Hence, for all corresponding corners k and k', $k_1' = k_1$ and $|k_2 - k_2'| \leq 1$.

Subcase 2: ($|c_2 - c_2'| = 0$) We will show that for all corresponding corners k and k', $k_2' = k_2$ and $|k_1 - k_1'| \leq 1$. Consider c and c'. Recall that $c_1' = c_1$ so $|c_1 - c_1'| = 0 \leq 1$. Since $|c_2 - c_2'| = 0$,$c_2 = c_2'$.

Consider b and b'. Since $c_2' = c_2$, $b_2' = c_2' = c_2 = b_2$ i.e. $b_2' = b_2$. Now, either $|b_1 - b_1'| \leq 1$ or $|b_1 - b_1'| > 1$. Notice that $|b_1 - b_1'| > 1 \Rightarrow |b_1 - b_1'| \geq 2$, so $\mathrm{d}(b, b') = |b_1 - b_1'| + |b_2 - b_2'| + |b_3 - b_3'| = |b_1 - b_1'| + 0 + 1 \geq 3$ contradicting the hypothesis. Thus $|b_1 - b_1'| \leq 1$.

Now consider, d and d'. Since $d_1 = b_1$ and $d_1' = b_1'$, $|d_1 - d_1'| = |b_1 - b_1'| \leq 1$. Also $d_2' = a_2' = a_2 = d_2$ i.e. $d_2' = d_2$.

Summarizing: $a_2' = a_2, a_1' = a_1$ so $|a_1 - a_1'| = 0 \leq 1$, $c_2' = c_2, c_1' = c_1$ so $|c_1 - c_1'| = 0 \leq 1$, $b_2' = b_2, |b_1 - b_1'| \leq 1$, $d_2' = d_2, |d_1 - d_1'| \leq 1$.

Hence, for all corresponding corners k, k', $k_2' = k_2$ and $|k_1 - k_1'| \leq 1$.

Case 2: ($a_2' = a_2, a_1' = a_1 \pm 1$)' We will show for all corresponding corners k and k', $k_2' = k_2$ and $|k_1 - k_1'| \leq 1$. First, consider c and c'. Since $a_1' = a_1 \pm 1$, $c_1' = a_1' = a_1 \pm 1 = c_1 \pm 1$ i.e. $c_1' = c_1 \pm 1$. Thus $|c_1 - c_1'| = |c_1 - (c_1 \pm 1)| = 1 \leq 1$. Now, either $c_2' = c_2$ or $c_2' \neq c_2$. Notice that $c_2' \neq c_2 \Rightarrow |c_2 - c_2'| > 0 \Rightarrow |c_2 - c_2'| \geq 1$. So $\mathrm{d}(c, c') = |c_1 - c_1'| + |c_2 - c_2'| + |c_3 - c_3'| = |c_1 - (c_1 \pm 1)| + |c_2 - c_2'| + |r - p| = 1 + |c_2 - c_2'| + 1 \geq 3$ contradicting the hypothesis. Thus $c_2' = c_2$.

Now, consider b and b'. Since $c_2' = c_2$, and $b_2' = c_2' = c_2 = b_2$i.e. $b_2' = b_2$. Either, $|b_1 - b_1'| \leq 1$ or $|b_1 - b_1'| > 1$. But notice that $|b_1 - b_1'| > 1 \Rightarrow |b_1 - b_1'| \geq 2$ so $\mathrm{d}(b, b') = |b_1 - b_1'| + |b_2 - b_2'| + |b_3 - b_3'| = |b_1 - b_1'| + 0 + 1 \geq 3$ contradicting the hypothesis. Thus $|b_1 - b_1'| \leq 1$.

Consider d and d'. Since $a_2' = a_2$, $d_2' = a_2' = a_2 = d_2$ i.e. $d_2' = d_2$. Also since $d_1' = b_1'$ and $d_1 = b_1$, $|d_1 - d_1'| = |b_1 - b_1'| \leq 1$.

Summarizing: $a_2' = a_2, |a_1 - a_1'| = 1 \leq 1$, $c_2' = c_2, |c_1 - c_1'| = 1 \leq 1$, $b_2' = b_2, |b_1 - b_1'| \leq 1$, $d_2' = d_2, |d_1 - d_1'| \leq 1$.

Hence, for all corresponding corners k, k', $k_2' = k_2$ and $|k_1 - k_1'| \leq 1$.

Case 3: ($a_1' = a_1, a_2' = a_2 \pm 1$) We will show for all corresponding corners k and k', $k_1' = k_1$ and $|k_2 - k_2'| \leq 1$. Again, we consider all points in turn.

First consider d and d'. Since $a_2' = a_2 \pm 1$, $d_2' = a_2' = a_2 \pm 1 = d_2 \pm 1$. Thus $|d_2 - d_2'| = |d_2 - (d_2 \pm 1)| = 1 \leq 1$. Now, either $d_1' = d_1$ or $d_1' \neq d_1$. But notice that $d_1' \neq d_1 \Rightarrow |d_1 - d_1'| > 0 \Rightarrow |d_1 - d_1'| \geq 1$, so $\mathrm{d}(d, d') =$

$|d_1 - d'_1| + |d_2 - d'_2| + |d_3 - d'_3| = |d_1 - d'_1| + 1 + |r - p| = |d_1 - d'_1| + 1 + 1 \geq 3$ contradicting the hypothesis. Thus $d'_1 = d_1$.

Now consider b and b'. Since $d'_1 = d_1$, $b'_1 = d'_1 = d_1 = b_1$ i.e $b'_1 = b_1$. Either $|b_2 - b'_2| \leq 1$ or $|b_2 - b'_2| > 1$. But $|b_2 - b'_2| > 1 \Rightarrow |b_2 - b'_2| \geq 2$, so $\mathrm{d}(b, b') = |b_1 - b'_1| + |b_2 - b'_2| + |b_3 - b'_3| = |b_2 - b'_2| + |r - p| = |b_2 - b'_2| + 1 \geq 3$ contradicting the hypothesis. Thus $|b_2 - b'_2| \leq 1$.

Finally, consider c and c'. Since $a'_1 = a_1$, $c'_1 = a'_1 = a_1 = c_1$ i.e. $c'_1 = c_1$. Also, since $c'_2 = b'_2$ and $c_2 = b_2$, we have that $|c_2 - c'_2| = |b_2 - b'_2| \leq 1$.

Summarizing $a'_1 = a_1, |a_2 - a'_2| = 1 \leq 1$, $d'_1 = d_1, |d_2 - d'_2| = 1 \leq 1$, $b'_1 = b_1, |b_2 - b'_2| \leq 1$, $c'_1 = c_1, |c_2 - c'_2| \leq 1$.

Hence, for all corresponding corners k and k', $k'_1 = k_1$ and $|k_2 - k'_2| \leq 1$.

(2⇒ 1) Assume that , either: $k'_1 = k_1$ and $|k_2 - k'_2| \leq 1$ for all corresponding corners, k and k' in B and B', or $k'_2 = k_2$ and $|k_1 - k'_1| \leq 1$ for all corresponding corners, k and k' in B and B'. Let q be any corner. Then $\mathrm{d}(q, q') = |q_1 - q'_1| + |q_2 - q'_2| + |r - p| = |q_1 - q'_1| + |q_2 - q'_2| + 1 \leq 2$.

□

Theorem 3. *Given boxes $B \subseteq \mathbb{Z}^2 \times \{r\}$ and $B' \subseteq \mathbb{Z}^2 \times \{p\}$ where r and p are in $\mathbb{Z}$ and $|r - p| = 1$, let $J = B \cup B'$. Then the following are equivalent:*

1. *J is a jump system.*
2. *For all corresponding corners q and q' in B and B' respectively, $d(q, q') \leq$* 2.

Proof. Let the points a and b be opposite corners in B such that $a_i \leq b_i$, and let the points a' and b' be opposite corners in B' such that $a'_i \leq b'_i$.

(1⇒2) Let J be a jump system. Assume that ∃ a pair of corners k and k' such that $\mathrm{d}(k, k') > 2$. We seek a contradiction. Specifically, by reflection and translation, assume $k = a$ and $k' = a'$. Then $\mathrm{d}(k, k') = \mathrm{d}(a, a') = |a_1 - a'_1| + |a_2 - a'_2| + |a_3 - a'_3| = |a_1 - a'_1| + |a_2 - a'_2| + |r - p| = |a_1 - a'_1| + |a_2 - a'_2| + 1 > 2$. So $|a_1 - a'_1| + |a_2 - a'_2| > 1 \Rightarrow |a_1 - a'_1| + |a_2 - a'_2| \geq 2$. Thus, one of the following must hold:

1. $|a_1 - a'_1| \geq 1$, $|a_2 - a'_2| \geq 1$
2. $|a_1 - a'_1| \geq 2$, $|a_2 - a'_2| = 0$
3. $|a_1 - a'_1| = 0$, $|a_2 - a'_2| \geq 2$

By coordinate-swapping, case 3 is similar to case 2, so we consider cases 1 and 2 without loss of generality.

Case 1: ($|a_1 - a'_1| \geq 1$, $|a_2 - a'_2| \geq 1$) Clearly $a_1 \neq a'_1$ and $a_2 \neq a'_2$. Without loss of generality, let $a'_1 > a_1$. (If $a_1 > a'_1$, rename B as B' and vice versa.) Thus, $a'_1 \geq a_1 + 1$. We now consider two possibilities for a'_2: $a'_2 > a_2$ i.e. $a'_2 \geq a_2 + 1$ and $a'_2 < a_2$ i.e. $a'_2 \leq a_2 - 1$.

Subcase 1: ($a'_2 \geq a_2 + 1$) Let $x = a = (a_1, a_2, r)$,$y = a' = (a'_1, a'_2, p)$, and let $z = (x_1, x_2, y_3) = (a_1, a_2, p)$. Then, clearly $z \in$ box $(x, y) = [[x_1, y_1]] \times [[x_2, y_2]] \times [[r, p]]$, and since $|r - p| = 1$, $\mathrm{d}(x, z) = 1$. Thus, z is an (x, y)-step.

But, $z \notin B$ since $z_3 = p \neq r$ and, since $z_1 = a_1 \leq a_1' - 1 < a_1' \leq b_1'$ i.e. $z_1 \notin [a_1', b_1']$, $z \notin B'$. Thus $z \notin J$. Now, since J is a jump system, $\exists$ a (z, y)-step $w \in J$. Thus $w \in \text{box}(z, y) = [a_1, a_1'] \times [a_2, a_2'] \times [p, p]$. Particularly, $w_3 \in [p, p]$ so $w_3 = p$. Thus, with the requirement that d(z, w)=1, the only possibilities for w are: $w^{(1)} = (a_1 + 1, a_2, p)$ and $w^{(2)} = (a_1, a_2 + 1, p)$. But since $w_2^{(1)} = a_2 < a_2' \leq b_2'$, $w_2^{(1)} \notin [a_2', b_2']$, so $w^{(1)} \notin B' = [a_1', b_1'] \times [a_2', b_2'] \times [p, p]$. Also, since $w_1^{(2)} = a_1 < a_1' \leq b_1'$, $w_1^{(2)} \notin [a_1', b_1']$ so $w^{(2)} \notin B'$. Clearly both $w^{(1)}$ and $w^{(2)}$ are not in B since $w_3^{(1)} = w_3^{(2)} = p \neq r$. Thus, both possibilities for w are not in J, so $\nexists (z, y)$-step in J, contradicting the hypothesis that J is a jump system. Thus either J is not a jump system or d$(a, a') \leq 2$.

Subcase 2: *($a_2' \leq a_2 - 1$)* Again, let $x = a = (a_1, a_2, r)$, $y = a' = (a_1', a_2', p)$, and let $z = (x_1, x_2 - 1, x_3) = (a_1, a_2 - 1, r)$. Clearly, d$(x, z) = 1$. Also, since, $y_2 = a_2' \leq a_2 - 1 = z_2 < a_2 = x_2$, $z_2 \in [y_2, x_2]$so, clearly $z \in$ box $(x, y) = [[x_1, y_1]] \times [y_2, x_2] \times [[r, p]]$. Thus, z is an (x, y)-step. But $z \notin B'$ since $z_3 = r \neq p$ and $z \notin B$ since $z_2 = a_2 - 1 < a_2 \leq b_2$. Thus $z \notin J$. Since J is a jump system, $\exists$ a (z, y)-step $w \in J$. Thus $w \in \text{box}(z, y) \cap J$.

Let $t \in \text{box}(z, y)$. Then $t_2 \in [a_2', a_2 - 1]$, so $t_2 \leq a_2 - 1 < a_2 \leq b_2$ so $t_2 \notin [a_2, b_2]$. Thus $t \notin B$. Hence $\forall t \in \text{box}(z, y)$, $t \notin B$ i.e. $\text{box}(z, y) \cap B = \emptyset$, so $\text{box}(z, y) \cap J = \text{box}(z, y) \cap B'$. Now, let $t \in \text{box}(z, y) \cap J = \text{box}(z, y) \cap B' = \text{box}(z, y) \cap \text{box}(a', b')$. Then: $t_1 \in [[z_1, y_1]] \cap [a_1', b_1'] = [a_1, a_1'] \cap [a_1', b_1']$, so $t_1 = a_1'; t_3 \in [[z_3, y_3]] \cap [p, p] = [[r, p]] \cap [p, p]$ so $t_3 = p$. Thus t has the form (a_1', t_2, p) where $t_2 \in [a_2', a_2 - 1] \cap [a_2', b_2']$. Since $w \in \text{box}(z, y) \cap J, w$ has form (a_1', w_2, p) where $w_2 \in [a_2', a_2 - 1] \cap [a_2', b_2']$. Recall $|a_1 - a_1'| \geq 1$, so d$(z, w) = |z_1 - w_1| + |z_2 - w_2| + |r - p| = |a_1 - a_1'| + |(a_2 - 1) - w_2| + 1 \geq 2$. But since w is a (z, y)-step, d(z, w)=1, a contradiction. Hence, no such (z, y)-step, w can exist, contradicting the hypothesis that J is a jump system. So either J is not a jump system or d$(a, a') \leq 2$.

Case 2: *($|a_1 - a_1'| \geq 2$, $|a_2 - a_2'| = 0$)* Again, without loss of generality, assume $a_1' > a_1$. Then $a_1' \geq a_1 + 2$. Also, clearly $a_2' = a_2$. Let $x = a = (a_1, a_2, r)$, $y = a' = (a_1', a_2', p)$, and let $z = (x_1, x_2, y_3) = (a_1, a_2, p)$. Then clearly, d$(x, z) = 1$ and $z \in \text{box}(x, y)$, so z is an (x, y)-step. But since $z_3 = p \neq r$, $z \notin B$, and since $z_1 = a_1 < a_1' \leq b_1'$, $z_1 \notin [a_1', b_1']$ so $z \notin B'$. Thus $z \notin J$.

Since J is a jump system $\exists$ a (z, y)-step $w \in J$. Thus $w \in\text{box}(z, y) = [[z_1, y_1]] \times [[z_2, y_2]] \times [p, p] = [a_1, a_1'] \times [a_2, a_2] \times [p, p]$. Particularly $w_3 = p$. Thus $w \notin B$. So, with the requirement that d$(z, w) = 1$, the only possibility for w is $w = (a_1 + 1, a_2, p) = (z_1 + 1, z_2, p)$. But then $w_1 = a_1 + 1 < a_1 + 2 \leq a_1' \leq b_1'$ so $w \notin [a_1', b_1']$. i.e. $w \notin B'$. Thus $w \notin J$ and so $\nexists$ a (z, y)-step in J, contradicting the fact that J is a jump system. Thus, either J is not a jump system or d$(a, a') \leq 2$.

(2$\Rightarrow$ 1)

Assume that $\forall q \in \{a, b, c, d\}$, d$(q, q') \leq 2$. By the preceding theorem, either of the following holds:

1. $k_1' = k_1$ and $|k_2 - k_2'| \leq 1$ for all corresponding corners, $k \in B$ and $k' \in B'$,
2. $k_2' = k_2$ and $|k_1 - k_1'| \leq 1$ for all corresponding corners, $k \in B$ and $k' \in B'$.

By coordinate-swapping, without loss of generality, consider case 2. We will show that J is a jump system. Since $c_1 = a_1$, $d_1 = b_1$, $c_1' = a_1'$, and $d_1' = b_1'$, clearly $k_1 \in \{a_1, b_1\}$ and $k_1' \in \{a_1', b_1'\}$. Thus, for $k_1 = a_1$, $|k_1 - k_1'| = |a_1 - a_1'| \leq 1 \Rightarrow |a_1 - a_1'| = 0$ or $|a_1 - a_1'| = 1$. By identical reasoning, for $k_1 = b_1$, we have that $|b_1 - b_1'| = 0$ or $|b_1 - b_1'| = 1$.

Thus we obtain the following cases:

1.$|a_1 - a_1'| = 0$, $|b_1 - b_1'| = 0$

2.$|a_1 - a_1'| = 1$, $|b_1 - b_1'| = 0$

3.$|a_1 - a_1'| = 0$, $|b_1 - b_1'| = 1$

4.$|a_1 - a_1'| = 1$, $|b_1 - b_1'| = 1$

Case 1: ($|a_1 - a_1'| = 0$, $|b_1 - b_1'| = 0$) Since $\forall k \in \{a, b, c, d\}$,$k_1 \in \{a_1, b_1\}$, then $\forall k \in \{a, b, c, d\}$, $|k_1 - k_1'| = 0$, so $k_1' = k_1$ and recall $\forall k \in \{a, b, c, d\}$, $k_2' = k_2$. Notice that $J = B \cup B' = \text{box}(a, b) \cup \text{box}(a', b') = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1', b_1'] \times [a_2', b_2'] \times [p, p]) = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \times ([a_1, b_1] \times [a_2, b_2] \times [p, p]) = [a_1, b_1] \times [a_2, b_2] \times [[r, p]]) = \text{box}[(a_1, a_2, r), (b_1, b_2, p)] = \text{box}[a, (b_1', b_2', p)] = \text{box}(a, b')$. Thus, J is a box, so J is a jump system.

For the rest of the cases, let x and y be in J, and let z be an (x, y)-step not in J. Assume that x and y are not both in B or both in B' since the two-step axiom will clearly hold. We seek a (z, y)-step in J.

Case 2: ($|a_1 - a_1'| = 1$, $|b_1 - b_1'| = 0$) Clearly $a_1' \neq a_1$. Without loss of generality, let $a_1' > a_1$. (If $a_1' < a_1$, rename B' as B and vice versa.) Thus $a_1' = a_1 + 1$. Clearly, $b_1' = b_1$, and since $b_2' = b_2$, we have that $b_i' = b_i$ for $i \in [1, 2]$.

We first show that x,y, and z are all in $\text{box}(a, b')$ by showing that $J \subseteq \text{box}(a, b')$. Notice $J = B \cup B' = \text{box}(a, b) \cup \text{box}(a', b') = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1', b_1'] \times [a_2', b_2'] \times [p, p]) = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1 + 1, b_1] \times [a_2, b_2] \times [p, p])$. But $\text{box}(a, b') = [[a_1, b_1']] \times [[a_2, b_2']] \times [[a_3, b_3']] = [[a_1, b_1]] \times [[a_2, b_2]] \times [[r, p]] = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1, b_1] \times [a_2, b_2] \times [p, p])$. Clearly $[a_1 + 1, b_1] \subseteq [a_1, b_1]$ so $J \subseteq \text{box}(a, b')$.

Since x and y are in J, x and y are in $\text{box}(a, b')$. Notice $\text{box}(x, y) = [[x_1, y_1]] \times [[x_2, y_2]] \times [[x_3, y_3]]$. But $\forall i \in [1, 3]$, $x_i \in [[a_i, b_i']]$ and $y_i \in [[a_i, b_i']]$, so $[[x_i, y_i]] \subseteq [[a_i, b_i']]$. Thus $\text{box}(x, y) \subseteq \text{box}(a, b')$. Since $z \in \text{box}(x, y)$, $z \in \text{box}(a, b')$.

We now seek a specification of z. Since $z \in \text{box}(a, b') = [a_1, b_1] \times [a_2, b_2] \times [[r, p]]$, either $z_3 = r$ or $z_3 = p$. If $z_3 = r$, then clearly $z \in \text{box}(a, b) \subseteq J$ so it must be that $z_3 = p \neq r$. Now, if $z_1 > a_1 = a_1' - 1$, then $z_1 \geq a_1'$ so $a_1' \leq z_1 \leq b_1' = b_1$, so recalling that $k_2' = k_2$ $\forall k \in \{a, b, c, d\}$, $z \in [a_1', b_1'] \times [a_2', b_2'] \times [p, p] = \text{box}(a', b') = B' \subseteq J$ contradicting the hypothesis that $z \notin J$. Thus $z_1 \not> a_1$ i.e. $z_1 = a_1$, so we have that $z = (z_1, z_2, z_3) = (a_1, z_2, p)$ where $z_2 \in [a_2, b_2]$.

We now seek a (z, y)-step in J. We consider two possibilities for x: $x \in B$ and $x \in B'$.

Subcase 1: ($x \in B$) If $x \in B$, then, by hypothesis, $y \in B' = \text{box}(a', b')$. We seek a (z, y)-step $w \in J$. Let $w = (z_1 + 1, z_2, z_3)$. Clearly, $\text{d}(z, w) = 1$. Also $w_1 = z_1 + 1 = a_1 + 1 = a_1'$ and since $y_1 \in [a_1', b_1']$, $a_1' \leq y_1$. Thus $z_1 < w_1 = a_1' \leq y_1$ so $w_1 \in [z_1, y_1]$ and $w \in \text{box}(z, y)$. Hence w is a (z, y)-step.

Notice $w = (a_1 + 1, z_2, p)$ where $z_2 \in [a_2, b_2]$. But $B' = \text{box}(a', b') = [a_1 + 1, b_1] \times [a_2, b_2] \times [p, p]$. Thus, $w \in B'$. Hence, w is a (z, y)-step in J as desired.

Subcase 2: ($x \in B'$) By hypothesis, $y \in B = \text{box}(a, b)$. Particularly, $y_3 = r$. Let $w = (z_1, z_2, r)$ where $z_2 \in [a_2, b_2]$. Since $z = (z_1, z_2, p)$and $|r - p| = 1$, clearly $\text{d}(z, w) = 1$. Also, since $w_3 = r = y_3$, clearly, $w \in \text{box}(z, y)$. Thus w is a (z, y)-step. Since $w = (z_1, z_2, r) = (a_1, z_2, r)$ and $B = [a_1, b_1] \times [a_2, b_2] \times [r, r]$,$w \in B$. Hence w is a (z, y)-step in J as desired.

Case 3: ($|a_1 - a'_1| = 0$, $|b_1 - b'_1| = 1$) By reflection and translation, this case is identical to Case 2.

Case 4: ($|a_1 - a'_1| = 1$, $|b_1 - b'_1| = 1$) Clearly, $a'_1 \neq a_1$. Without loss of generality, let $a'_1 > a_1$. (If $a'_1 < a_1$, rename B' as B and vice versa.)Thus $a'_1 = a_1 + 1$. Then either $b'_1 = b_1 + 1$ or $b'_1 = b_1 - 1$.

Subcase 1: $b'_1 = b_1 + 1$ We first show that x,y, and z are all in $\text{box}(a, b')$ by showing that $J \subseteq \text{box}(a, b')$. Notice $J = B \cup B' = \text{box}(a, b) \cup \text{box}(a', b') = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a'_1, b'_1] \times [a'_2, b'_2] \times [p, p]) = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1 + 1, b_1 + 1] \times [a_2, b_2] \times [p, p])$. But $\text{box}(a, b') = [[a_1, b'_1]] \times [[a_2, b'_2]] \times [[a_3, b'_3]] = [a_1, b_1 + 1] \times [a_2, b_2] \times [[r, p]] = ([a_1, b_1 + 1] \times [a_2, b_2] \times [r, r]) \cup ([a_1, b_1 + 1] \times [a_2, b_2] \times [p, p])$. Clearly $[a_1 + 1, b_1 + 1] \subseteq [a_1, b_1 + 1]$ so $J \subseteq \text{box}(a, b')$. Since x and y are in J, x and y are in $\text{box}(a, b')$. Note that $\text{box}(x, y) = [[x_1, y_1]] \times [[x_2, y_2]] \times [[x_3, y_3]]$. But $\forall i \in [1, 3], x_i \in [[a_i, b'_i]]$ and $y_i \in [[a_i, b'_i]]$, so $[[x_i, y_i]] \subseteq [[a_i, b'_i]]$. Thus $\text{box}(x, y) \subseteq \text{box}(a, b')$. Since $z \in \text{box}(x, y)$, $z \in \text{box}(a, b')$.

We now seek a specification of z. Since $z \in \text{box}(a, b') = [a_1, b_1 + 1] \times [a_2, b_2] \times [[r, p]]$, and $|r - p| = 1$, either $z_3 = r$ or $z_3 = p$.

Assume $z_3 = r$. Since $z \notin J$, z is not in $B = \text{box}(a, b)$ so $z_1 \notin [a_1, b_1]$. But $z_1 \in [a_1, b_1 + 1]$, so $z_1 = b_1 + 1$. Thus $z = (b_1 + 1, z_2, r)$ where $z_2 \in [a_2, b_2]$. Now, either $x \in B$ or $x \in B'$. If $x \in B$, let $w = (z_1, z_2, p)$. Since $|z_3 - w_3| = |r - p| = 1$, clearly $\text{d}(z, w) = 1$. By hypothesis $y \in B'$, so $y_3 = p$. Thus, $w \in \text{box}(z, y)$. Notice, $w = (z_1, z_2, p) = (b_1 + 1, z_2, p)$ where $z_2 \in [a_2, b_2]$ and recall $B' = \text{box}(a', b') = [a_1 + 1, b_1 + 1] \times [a_2, b_2] \times [p, p]$, so $w \in B'$. Thus, w is a (z, y)-step in J as desired.

If $x \in B'$, let $w = (z_1 - 1, z_2, r)$. Clearly, $\text{d}(z, w) = 1$. Also $w_1 = z_1 - 1 = b_1$. By hypothesis $y \in B$, so $a_1 \leq y_1 \leq b_1 = w_1$. Thus $y_1 \leq b_1 = w_1 = z_1 - 1 < z_1$ i.e. $w_1 \in [y_1, z_1]$, and $w \in \text{box}(z, y)$. Therefore, w is a (z, y)-step. Recall $B = [a_1, b_1] \times [a_2, b_2] \times [r, r]$. Clearly, $w \in B$. Thus, w is a (z, y)-step in J as desired.

Now suppose $\exists$ x,y and z that violate the two-step axiom. If $z_3 \neq r$, reflect and translate until $z_3 = r$. Then we have a contradiction, since, as just shown, for $z_3 = r$, the axiom holds. Thus $\nexists$ x,y and z that violate the two-step axiom. In other words, J is a jump system.

Subcase 2: $b'_1 = b_1 - 1$ We first show that x,y,and z lie in the box $K = [a_1, b_1] \times [a_2, b_2] \times [[r, p]]$. Notice $J = \text{box}(a, b) \cup \text{box}(a', b') = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a'_1, b'_1] \times [a'_2, b'_2] \times [p, p]) = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1 + 1, b_1 - 1] \times [a_2, b_2] \times [p, p])$. Also notice $K = ([a_1, b_1] \times [a_2, b_2] \times [r, r]) \cup ([a_1, b_1] \times [a_2, b_2] \times [p, p])$. Clearly $J \subseteq K$ since $[a_1 + 1, b_1 - 1] \subset [a_1, b_1]$, so x and y are in K. Thus, by previous arguments, $\text{box}(x, y) \subset K$. Since $z \in \text{box}(x, y)$,$z \in K$.

We seek a specification of z. Since $z \in K$, $z_3 \in [[r,p]]$ so $z_3 = r$ or $z_3 = p$. If $z_3 = r$, then, trivially, $z_3 \in [r,r]$ and since $z_1 \in [a_1, b_1]$, $z_2 \in [a_2, b_2]$, $z \in \text{box}(a,b) = B$, contradicting the hypothesis that $z \notin J$. Thus $z_3 = p \neq r$. Since $z \notin J$, $z \notin B'$ so $z_1 \notin [a_1+1, b_1-1]$ but $z_1 \in [a_1, b_1]$. Thus, $z_1 = a_1$ or $z_1 = b_1$. Assume $z_1 = a_1$. Then $z = (a_1, z_2, p)$ where $z_2 \in [a_2, b_2]$. Now either $x \in B$ or $x \in B'$. If $x \in B$, then let $w = (z_1+1, z_2, p)$. Clearly $\text{d}(z,w)$=1. Also $w_1 = z_1 + 1 = a_1 + 1$. By hypothesis $y \in B$, so $y_1 \in [a_1+1, b_1-1]$. Thus $z_1 < w_1 = z_1 + 1 = a_1 + 1 \leq y_1$, so $w_1 \in [z_1, y_1]$, so clearly, $w \in \text{box}(z,y)$. Therefore, w is a (z,y)-step. Notice $w = (z_1+1, z_2, p) = (a_1+1, z_2, p)$ where $z_2 \in [a_2, b_2]$, and recall $B' = [a_1+1, b_1-1] \times [a_2, b_2] \times [p,p]$. Clearly, $w \in B'$ so w is a (z,y)-step in J as desired.

If $x \in B'$, then let $w = (z_1, z_2, r) = (a_1, z_2, r)$. Since $|r-p| = 1$, clearly $\text{d}(z,w)$=1. By hypothesis $y \in B$, so $y_3 = r$. Thus $w \in \text{box}(z,y)$, so w is a (z,y)-step. Since $z_2 \in [a_2, b_2]$, it is clear that $w \in \text{box}(a,b) = B$. Thus w is a (z,y)-step in J as desired.

Now, suppose $\exists x$,y,and z that violate the two-step axiom. If $z_1 \neq a_1$, then reflect and translate until $z_1 = a_1$. Then we have $z_1 = a_1$ and points x,y and z that violate the two-step axiom, which is a contradiction since, as has just been shown, the axiom holds for $z_1 = a_1$. Thus $\nexists x$,y,and z that violate the two-step axiom. In other words, J is a jump system.

□

3 Operations on Jump Systems

In the following sections we present results concerning two interesting operations on jump systems. The first, defined as "squashing" a jump system is an operation that, like reflection and translation, is performed on just one jump system. The second operation, however, that of forming the Cartesian product of jump systems, involves two jump systems.

3.1 The "Squashing" Operation

Given a set of integer points in three-space,"squashing" is defined as taking the projection of all two-dimensional cross sections of the set onto a chosen horizontal plane. The xy-grid is preserved but the z coordinate is eliminated. See Figure 3 for an illustration of the Squash Theorem.

Theorem 4 (The Squash Theorem). *Let J be a jump system. If* $J^* = \{(a,b,0) : (a,b,c) \in J\}$*, then* J^* *is a jump system.*

Proof. Let $x, y \in J^*$ and let z be an (x, y)-step. Given a point $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$, define $p' = (p_1, p_2, 0)$. Then $\exists$ points $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in J such that $x = u'$ and $y = v'$.

Case 1: Assume $x = u'$ and $y = v'$ are two distinct points in J^* where u and v lie in the same horizontal cross section, J_r, of J - i.e. $v_3 = u_3$. Since J is

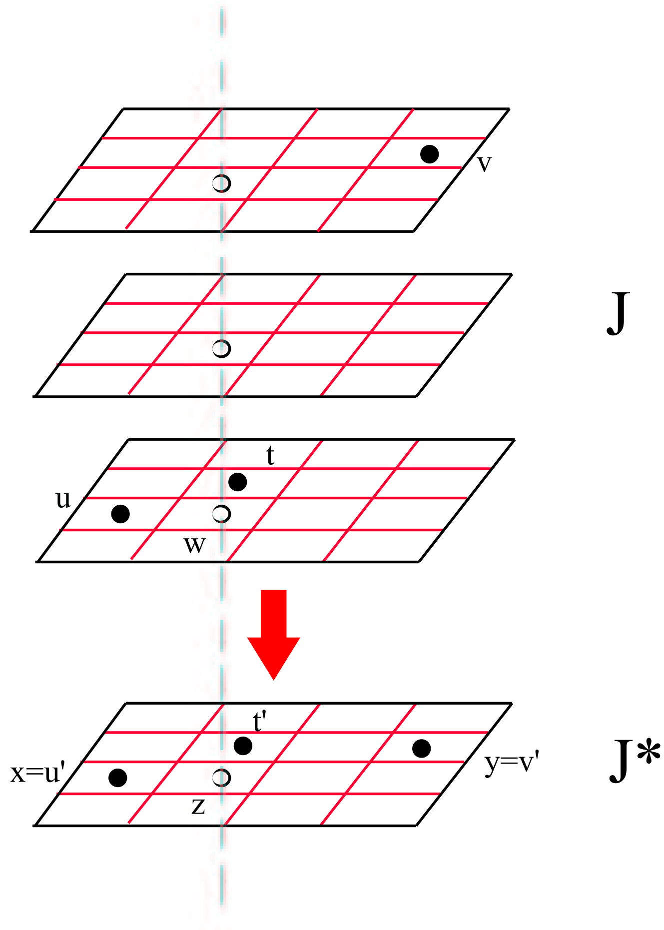


Figure 3: The Illustrated Squash Theorem

a jump system, the jump system axioms hold for any three points a, b and c, where c is an (a,b)-step, that lie in $J_r \subset J$. Thus, J_r is a jump system. But since translation preserves the axioms, the axioms also hold for the translation of J_r, ${J_r}^* = \{(a,b,c) + (0,0,-c) : (a,b,c) \in J_r\} \subset J^*$. Since for u and v in J_r, $x = u' = (u_1, u_2, 0) \in {J_r}^*$ and $y = v' = (v_1, v_2, 0) \in {J_r}^*$, the axioms hold for x, y, and any (x,y)-step, z, in ${J_r}^*$.

Here is an alternate argument: Note that the horizontal cross section of J, J_r, in which u and v lie is the intersection of J with the region $(-\infty, +\infty) \times (-\infty, +\infty) \times [u_3, u_3]$ which is a box. Since, for any jump system J, and box B, $J \cap B$ is a jump system, J_r is a jump system.

Case 2: Now, assume $x = u' = (u_1, u_2, 0)$ and $y = v' = (v_1, v_2, 0)$ are two distinct points in J^*, where u and v lie in different horizontal cross sections of J, i.e. $v_3 \neq u_3$. Given an (x,y)-step z, either $z \in J^*$ and we're done, or $z \notin J^*$. So, assume $z \notin J^*$. We seek a (z,y)-step in J^*.

Because z is an (x,y)-step, $z \in box(x,y)$, and $d(x,z) = 1$. Since $z \in$ box(x,y), $z_i \in [[x_i, y_i]], \forall\ i \in [1,3]$. Particularly, $z_3 \in [0,0]$ so $z_3 = 0$. Thus $z = (z_1, z_2, 0)$ where $z_1 \in [[x_1, y_1]] = [[u_1, v_1]]$, $z_2 \in [[x_2, y_2] = [[u_2, v_2]]$.

Since $z = (z_1, z_2, 0) \notin J^*$ and $J^* = \{(a,b,0) : (a,b,c) \in J\}$, then it must be that $\forall\ s \in \mathbb{Z}, (z_1, z_2, s) \notin J$. Otherwise , if $(z_1, z_2, s) \in J$, then $z = (z_1, z_2, 0) \in J^*$ contradicting the hypothesis that $z \notin J$.

Thus, the point $w = (w_1, w_2, w_3) = (z_1, z_2, u_3) \notin J$. Notice, $w \in$ box(u,v), since: $w_1 = z_1 \in [[u_1, v_1]], w_2 = z_2 \in [[u_2, v_2]]$, and $w_3 = u_3 \in [[u_3, v_3]]$. Also notice that $u = (u_1, u_2, u_3) = (u_1, u_2, 0) + (0,0,u_3) = x + (0,0,u_3)$, and $w = (z_1, z_2, u_3) = (z_1, z_2, 0) + (0,0,u_3) = z + (0,0,u_3)$. Thus, u and w are the translation of x and z up u_3 units in 3-space. Since $d(x,z) = 1$ and distances are preserved under translation, we know that $d(u,w) = 1$. But notice: $d(u,w) = 1$ and $w \in [[u,v]]$, so w is a (u,v)-step.

Since w is a (u,v)-step that is not in J, then, because J is a jump system, $\exists$ a (w,v)-step t that is in J. Clearly, t cannot be $(z_1, z_2, u_3 \pm 1)$ since then, t would be of the form $(z_1, z_2, s) \notin J$. Since t cannot be directly above or below w, t must lie in the same plane as w i.e. $t_3 = u_3$. Thus, $d(w,t) = 1$ and $t = (t_1, t_2, t_3) = (t_1, t_2, u_3) \in$ box(w,v) so $t_1 \in [[w_1, v_1]] = [[z_1, v_1]], t_2 \in [[w_2, v_2]] = [[z_2, v_2]]$.

Since $t \in J$, then, by the definition of $J^*, t' = (t_1, t_2, 0) \in J^*$. But notice that $t' \in$ box(z,y) since: $t'_1 = t_1 \in [[z_1, v_1]] = [[z_1, y_1]], t'_2 = t_2 \in [[z_2, v_2]] = [[z_2, y_2]]$ and, trivially, $t'_3 = 0 \in [0,0] = [[z_3, y_3]]$. Also notice that $t' = (t_1, t_2, 0) = (t_1, t_2, u_3) + (0,0,-u_3) = t + (0,0,-u_3)$ and $z = (z_1, z_2, 0) = (z_1, z_2, u_3) + (0,0,-u_3) = w + (0,0,-u_3)$. Thus z and t' are the translation of w and t down u_3 units in space. Since $d(w,t) = 1$, and distances are preserved under translation, $d(z,t') = 1$. But notice: $d(z,t') = 1$ and $t' \in$ box(z,y). Thus, t' is a (z,y)-step that is in J^* as desired.

Therefore, J^* is a jump system. □

3.2 The Cartesian Product

Theorem 5. *(The Cartesian Product Theorem) Let R and P be finite sets such that $R \cap P = \emptyset$. Let $J_R \subseteq \mathbb{Z}^R$ and $J_P \subseteq \mathbb{Z}^P$ be jump systems. Then $J_R \times J_P$ is a jump system $\Leftrightarrow$ J_R and J_P are jump systems.*

Proof. Note that $J_R \times J_P \subseteq \mathbb{Z}^{R\cup P}$. For this proof, we define $w \in \mathbb{Z}^{R\cup P}$ as $w = (w_R, w_P)$ where $w_R \in \mathbb{Z}^R$, and $w_P \in \mathbb{Z}^P$.

($\Rightarrow$) Assume $J_R \times J_P$ is a jump system. We will show that J_R is a jump system. (The argument for J_P is similar.) Let x_R and y_R be in J_R, and let z_R be an (x_R, y_R)-step $\notin J_R$. We seek an (x_R, z_R)-step in J_R.

Let $t_P \in J_P$. Then $x = (x_R, t_P)$ and (y_R, t_P) are in $J_R \times J_P$, while $z = (z_R, t_P) \notin J$ since $z_R \notin J_R$. Notice that $\mathrm{d}(x, z) = \sum_{i\in R\cup P} |x_i - z_i| = \sum_{i\in R} |x_i - z_i| + \sum_{i\in P} |x_i - z_i| = \mathrm{d}(x_R, z_R) + \mathrm{d}(t_P, t_P) = 1$. Also since z is an (x_R, y_R)-step, $z_R \in \mathrm{box}(x_R, y_R)$ so $\forall i \in R$, $z_i \in [[x_i, y_i]]$. Since $\forall i \in P$, $z_i = t_i \in [t_i, t_i] = [x_i, y_i]$, $\forall i \in P$ we also have that $z_i \in [x_i, y_i]$. Thus $z \in \mathrm{box}(x, y)$. So z is an (x, y)-step not in $J_R \times J_P$.

Since $J_R \times J_P$ is a jump system, $\exists$ a (z, y)-step $z' = (z'_R, z'_P) \in J_R \times J_P$. Thus $z' \in \mathrm{box}(z, y)$. Particularly, $\forall i \in P$ $z'_i \in [[z_i, y_i]] = [t_i, t_i] = [x_i, y_i]$, so $z'_i = t_i$ $\forall i \in P$. Thus $z'_P = t_P$, so $z' = (z'_R, t_P)$. Also $\mathrm{d}(z, z') = 1$ so $1 = \sum_{i\in R\cup P} |z_i - z'_i| = \sum_{i\in R} |z_i - z'_i| + \sum_{i\in P} |z_i - z'_i| = \mathrm{d}(z_R, z'_R) + \mathrm{d}(z_P, z'_P) = \mathrm{d}(z_R, z'_R) + \mathrm{d}(t_P, t_P) = \mathrm{d}(z_R, z'_R)$, i.e. $\mathrm{d}(z_R, z'_R) = 1$. Since $z' \in \mathrm{box}(z, y)$, $\forall i \in R$, $z'_i \in [[z_i, y_i]]$, i.e. $z'_R \in \mathrm{box}(z_R, y_R)$. Also, recall that $z'_R \in J_R$ so z'_R is a (z_R, y_R)-step in J_R as desired.

($\Leftarrow$) Assume J_R and J_P are jump systems. Let $x, y \in J_R \times J_P$, and let z be an (x, y)-step. If $z \in J_R \times J_P$ then we're done. So assume $z \notin J_R \times J_P$. We seek a (z,y)-step in $J_R \times J_P$.

Since z is an (x, y)-step, $d(x, z) = \sum_{i\in R\cup P} |x_i - z_i| = 1$. So, for some $k \in R \cup P$, $|x_k - z_k| = 1$, while $\forall i \in R \cup P$ where $i \neq k$, $|x_i - z_i| = 0$ i.e. $z_i = x_i$.

We consider two possibilities for $k : k \in R$ and $k \in P$.

Case 1: ($k \in R$) Assume $k \in R$. Then, $\forall i \in P$, clearly $i \neq k$ so $z_i = x_i$ i.e. $z_P = x_P \in J_P$. Since, $z = (z_R, z_P) \notin J_R \times J_P$, it must be that $z_R \notin J_R$. Notice that: $d(x_R, z_R) = \sum_{i\in R} |x_i - z_i| = |x_k - z_k| + \sum_{i\in R, i\neq k} |x_i - z_i| = 1 + 0 = 1$. Also, since z is an (x, y)-step, $\forall i \in R \cup P$ and thus $\forall i \in R, z_i \in [[x_i, y_i]]$ so $z_R \in [[x_R, y_R]]$. Thus z_R is an (x_R, y_R)-step. Since $z_R \notin J_R$ and J_R is a jump system, $\exists$ a (z_R, y_R)-step, $s_R \in J_R$. Now, let $z' \in \mathbf{Z}^{R\cup P}$ such that $z' = (z'_R, z'_P) = (s_R, z_P)$. Since $s_R \in J_R$ and $z_P \in J_P, z' \in J_R \times J_P$. Note:

$d(z, z') = \sum_{i\in R\cup P} |z_i - z'_i| = \sum_{i\in R} |z_i - z'_i| + \sum_{i\in P} |z_i - z'_i| = d(z_R, z'_R) + d(z_P, z'_P) = d(z_R, s_R) + d(z_P, z_P) = 1 + 0 = 1.$

Also, since s_R is (z_R, y_R)-step, $\forall i \in R$, $z'_i = s_i \in [[z_i, y_i]]$ and trivially, $\forall i \in P, z'_i = z_i \in [[z_i, y_i]]$. Thus $\forall i \in R \cup P, z'_i \in [[z_i, y_i]]$, so $z' \in \mathrm{box}(z, y)$.

Therefore, z' is (z, y)-step in $J_R \times J_P$ as desired.

Case 2: ($k \in P$) The argument for Case 2 is similar.

Therefore, $J_R \times J_P$ is a jump system. $\square$

An interesting result arises when we apply the Cartesian Product Theorem to the case where R is a two-element set and P is a one-element set. The following theorem states that, given any two-dimensional jump system, any set of n copies of that system in three-space is a jump system.

Theorem 6. *(The Translation-Iteration Theorem) Given a jump system, $J \subseteq \mathbb{Z}^2$, for $k \in \mathbb{Z}$, define $J_k = J \times \{k\}$. Then, for $[q,r] \subseteq \mathbb{Z}$, $J^* = \cup_{i=q}^{r} J_i$ is a jump system.*

Proof. Define $J' = [q,r]$. Clearly J' is a jump system. Thus, by the Cartesian Product Theorem, $J \times J'$ is a jump system. Notice that $J \times J' = J \times [q,r] = J \times \{t : t \in [q,r]\} = \cup_{i=q}^{r} J_i = J^*$. Hence J^* is a jump system. $\square$

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