ASYMPTOTIC ELASTICITY AND THE FULL ELASTICITY PROPERTY IN ATOMIC MONOIDS

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ABSTRACT. Let M be a commutative atomic monoid (i.e. every nonzero nonunit of M can be factored as a product of irreducible elements). Let $\rho(x)$ denote the elasticity of $x \in M$, $\mathcal{R}(M) = \{\rho(x) \mid x \in M\}$ the set of elasticities of elements in M, and $\rho(M) = \sup \mathcal{R}(M)$ the elasticity of M. We say M is *fully elastic* if $\mathcal{R}(M) = \mathbb{Q} \cap [1, \rho(M)]$. We examine the full elasticity property in the context of numerical semigroups and block monoids over finitely generated abelian groups. In particular, we show several large classes of block monoids are fully elastic. We also define $\bar{\rho}(x) = \lim_{n\to\infty} \rho(x^n)$ to be the *asymptotic elasticity* of x. We determine some basic properties of $\bar{\rho}$ and discuss the set of values attained by $\bar{\rho}$ in both the numerical semigroup and block monoid cases.

1. INTRODUCTION

Let M be a finitely generated commutative cancellative monoid with M^* the set of nonunits of M and $\mathcal{A}(M)$ the set of irreducibles (or atoms). We suppose M is *atomic* (i.e. every element of M^* is a sum of atoms). Such monoids have applications in combinatorics, algebraic geometry, commutative algebra, number theory, and computational algebra (see [10, pp. iii-iv]). Much recent literature has been devoted to the study of monoids in which elements fail to factor uniquely. In particular, a central topic of focus has been the *elasticity* of elements of M, which measures their failure to factor uniquely. While much is known about the supremum of the set of elasticities, we study here the complete set of elasticities in several important classes of monoids. We begin with some definitions and notations.

For $x \in M^*$, define

$$\mathcal{L}(x) = \{n \mid x = \alpha_1, \dots, \alpha_n \text{ with each } \alpha_i \in \mathcal{A}(M)\}$$

to be the set of lengths of factorizations of x into irreducibles. Define

$$L(x) = \sup \mathcal{L}(x) \qquad \text{and} \qquad l(x) = \inf \mathcal{L}(x),$$

and define

$$\rho(x) = \frac{L(x)}{l(x)}$$

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to be their quotient. $\rho(x)$ is called the *elasticity* of x.

We also define

$$\mathcal{R}(M) = \{\rho(x) \mid x \in M^*\}$$

to be the set of elasticities of nonunits in M, and

$$\rho(M) = \sup\{\rho(x) \mid x \in M^*\}$$

to be the supremum of this set. $\rho(M)$ is called the elasticity of M. The notion of elasticity was introduced by Valenza in [11] in the context of rings of integers of algebraic number fields.

We now state several basic facts about elasticity which we will use freely throughout the paper.

(i) $1 \leq \rho(x) \leq \rho(M) \leq \infty$ for all $x \in M^*$
(ii) $\rho(x)$ is rational for all $x \in M^*$
(iii) $\rho(M) = \frac{m}{n} \in \mathbb{Q}$ and $\alpha_1 \dots \alpha_m = \beta_1 \dots \beta_n$ for some irreducibles $\alpha_i, \beta_j \in \mathcal{A}(M)$.

(i) and (ii) are obvious, and (iii) is Theorem 7 in [1].

Following Zaks in [12], we say the atomic monoid M is a *half-factorial monoid* (HFM) if for all nonunits $x \in M$, every irreducible factorization of x has the same length. Thus, $\rho(M)$ is a measure of how far M deviates from being a HFM. In particular, $\rho(M) = 1$ if and only if M is a HFM. For more information on elasticity, we refer the reader to [2] and [5].

In section 2, we define the asymptotic elasticity of a monoid and give some basic properties and results. Section 3 introduces the notion of a fully elastic monoid, and explores it in the context of block monoids. We also characterize in this section the set of asymptotic elasticities for a block monoid. In section 4, we study the full elasticity property in terms of numerical monoids, and give some results concerning asymptotic elasticities.

2. Asymptotic elasticity

For all $x \in M^*$, define

$$\bar{L}(x) = \lim_{n\to\infty} \frac{L(x^n)}{n},$$

$$\bar{l}(x) = \lim_{n\to\infty} \frac{l(x^n)}{n}.$$

From [4], we know both these limits exists, although $\bar{L}(x)$ may be infinite.

For $x \in M^*$, we define

$$\bar{\rho}(x) = \frac{\bar{L}(x)}{\bar{l}(x)} = \lim_{n\to\infty} \rho(x^n)$$

to be the *asymptotic elasticity* of x, which exists since $\bar{L}(x)$ and $\bar{l}(x)$ do. Moreover, it is shown in [1, Theorem 12] and [9, Theorem 2] that $\bar{L}(x)$ and

$\bar{l}(x)$ are rational for all elements $x \in M^*$, so that $\bar{\rho}(x)$ is rational as well. We also define

$$\bar{\mathcal{R}}(M) = \{\bar{\rho}(x) \mid x \in M^*\}$$

to be the set of asymptotic elasticities, and

$$\bar{\rho}(M) = \sup \bar{\mathcal{R}}(M)$$

to be the asymptotic elasticity of M. We begin with several basic properties of $\bar{\rho}$.

Lemma 1. *Let x be a nonunit of M. Then*

(i) $\bar{\rho}(x) \geq \rho(x^n) \geq \rho(x)$ *for all $n \in \mathbb{N}$*
(ii) $\bar{\rho}(M) = \rho(M)$.

Proof. (i) It is easy to verify that $L(x_1x_2) \geq L(x_1) + L(x_2)$ and $l(x_1x_2) \leq l(x_1) + l(x_2)$ for all $x_1, x_2 \in M^*$. It follows that $L(x^k) \geq kL(x)$ and $l(x^k) \leq kl(x)$ for all $k \in \mathbb{N}$, from which

$$\rho(x^k) = \frac{L(x^k)}{l(x^k)} \geq \frac{kL(x)}{kl(x)} = \rho(x).$$

Thus for all $x \in M^*$, $\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) \geq \rho(x)$. For all $n \in \mathbb{N}$, $x^n \in M^*$ so

$$\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) = \lim_{k\to\infty} \rho(x^{kn}) = \bar{\rho}(x^n) \geq \rho(x^n) \geq \rho(x),$$

which completes the proof of (i).

For (ii), it follows from (i) that

$$\bar{\rho}(M) = \sup\{\bar{\rho}(x) \mid x \in M^*\} \geq \sup\{\rho(x) \mid x \in M^*\} = \rho(M).$$

Suppose $\bar{\rho}(M) > \rho(M)$. Then there exists $x \in M^*$ such that $\bar{\rho}(x) > \rho(M)$. Let $\varepsilon = \bar{\rho}(x) - \rho(M) > 0$. Since $\lim_{n\to\infty} \rho(x^n) = \bar{\rho}(x)$, there exists $N \in \mathbb{N}$ such that $n > N \implies |\bar{\rho}(x) - \rho(x^n)| < \varepsilon \implies \rho(x^n) > \rho(M)$, which contradicts the maximality of $\rho(M)$. Hence, $\bar{\rho}(M) \leq \rho(M)$, and the result follows. ◻

Theorem 2. *Let $x \in M^*$. Then the following are equivalent:*

(i) $\bar{\rho}(x) = \rho(x)$
(ii) $\rho(x^n) = \rho(x)$ *for all $n \in \mathbb{N}$*
(iii) *There is an integer $m \geq 2$ such that $\rho(x^{m^n}) = \rho(x)$ for all $n \in \mathbb{N}$.*

Proof. From Lemma 1 (i),

$$\bar{\rho}(x) \geq \rho(x^n) \geq \rho(x) \quad \text{for all } n \in \mathbb{N}.$$

It follows that if $\bar{\rho}(x) = \rho(x)$ then $\rho(x^n) = \rho(x)$ for all $n \in \mathbb{N}$, so (i) implies (ii). That (ii) implies (iii) is obvious. Finally, if (iii) holds then

$$\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) = \lim_{k\to\infty} \rho(x^{m^k}) = \rho(x),$$

so (iii) implies (i). ◻

The following lemma will allow us to relate the set of elasticities of a monoid to its set of asymptotic elasticities.

Lemma 3. *Let $x \in M^*$. There is an integer $\alpha \in \mathbb{N}$ depending on x such that $\bar{\rho}(x) = \rho(x^\alpha)$.*

Proof. By [1, Theorem 12], there are integers $m, n \geq 1$ such that $l(x^{km}) = km\bar{l}(x)$ and $L(x^{kn}) = kn\bar{L}(x)$ for all $k \in \mathbb{N}$. Taking $\alpha = mn$, we have

$$\rho(x^\alpha) = \rho(x^{mn}) = \frac{L(x^{mn})}{l(x^{mn})} = \frac{mn\bar{L}(x)}{mn\bar{l}(x)} = \bar{\rho}(x).$$

□

We conclude this section by relating $\bar{\mathcal{R}}(M)$ to $\mathcal{R}(M)$ and its set of limit points. If $S \subseteq \mathbb{R}$ then $x \in \mathbb{R}$ is called a *limit point* of S if $U - \{x\}$ intersects S for any neighborhood U of x in $\mathbb{R}$.

Theorem 4. *Let $\mathcal{R}(M)'$ denote the set of limit points of $\mathcal{R}(M)$. Then*

$$\bar{\mathcal{R}}(M) \subseteq \mathcal{R}(M)' \cup \{\rho(x) \mid \rho(x) = \bar{\rho}(x)\}.$$

Also, $\bar{\mathcal{R}}(M) \subseteq \mathcal{R}(M)$.

Proof. Let $r = \bar{\rho}(z) \in \bar{\mathcal{R}}(M)$ and suppose r is not a limit point of $\mathcal{R}(M)$. Then there is an open interval I of radius $\varepsilon > 0$ around r such that $I - \{r\}$ is disjoint from $\mathcal{R}(M)$. Thus the sequence $\{\rho(z^k)\}_{k\in\mathbb{N}}$ is disjoint from $I - \{r\}$. Since this sequence has limit r, there exists $N \in \mathbb{N}$ such that $\rho(z^k) \in I$ for all $k > N$. Since $\rho(z^k) \notin I - \{r\}$, it follows that $\rho(z^k) = \bar{\rho}(z^k) = r$ for all $k > N$. Thus, $r \in \{\rho(x) \mid \rho(x) = \bar{\rho}(x)\}$, which proves the first statement. The second statement follows immediately from Lemma 3. □

3. The full elasticity property and $\mathcal{B}(G)$

As a basis for studying the entire set of elasticities $\mathcal{R}(M)$ of a monoid M, we propose the following

Definition. For all $x \in M^*$, $\rho(x) \in \mathbb{Q}$ and $1 \leq \rho(x) \leq \rho(M)$. Thus, for every monoid M, ρ defines a function $\rho\colon M^* \to \mathbb{Q} \cap [1, \rho(M)]$. If for a given monoid M the function ρ is surjective, we say M is *fully elastic.* Equivalently, a monoid M is fully elastic if $\mathcal{R}(M) = \mathbb{Q} \cap [1, \rho(M)]$.

We note that this definition is valid when $\rho(M) = \infty$ if we understand $[1, \rho(M)]$ to mean $[1, \infty)$. We also remark that if the monoid M is a HFM then $\rho(M) = 1$ and M is trivially fully elastic. We first study the full elasticity property in the context of block monoids. In section 4, we explore it in terms of numerical monoids.

Throughout this section G will denote a nontrivial finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, we can write $G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, for some integers $r \geq 0$ and $1 < n_1 | \ldots | n_k$. We will use this result freely throughout this section. We also note that if G is finite (i.e. $r = 0$) then k is known as the *rank* of G.

A *zero-sequence* of G is a nonempty sequence $\{g_1, \dots, g_t\}$ of (not necessarily distinct) elements of G such that $\sum_{i=1}^{t} g_i = 0$. A zero-sequence is called *minimal* if it contains no proper zero-subsequence. Zero-sequences are also called *blocks*. The *length* of the zero-sequence $B = \{g_1, \dots, g_t\}$, denoted by $|B|$, is defined to be t.

Let $\mathcal{Z}(G)$ denote the set of zero-sequences of G. If $B_1 = \{g_1, \dots, g_n\}$ and $B_2 = \{h_1, \dots, h_m\}$ are zero-sequences in $\mathcal{Z}(G)$, we define $B_1 \sim B_2$ if $n = m$ and there's a permutation $\sigma \in S_n$ such that $g_i = h_{\sigma(i)}$ for all $1 \le i \le n$. We note that $\sim$ is an equivalence relation on $\mathcal{Z}(G)$. Define $\mathcal{B}(G)$ to be the set $\mathcal{Z}(G)/\sim$ under the operation

$$\{g_1, \dots, g_n\} \cdot \{h_1, \dots, h_m\} = \{g_1, \dots, g_n, h_1, \dots, h_m\}.$$

$\mathcal{B}(G)$ is a commutative atomic monoid called the *block monoid* of G. Let $\mathcal{U}(G)$ denote the subset of $\mathcal{B}(G)$ consisting of minimal zero-sequences of G. Then the elements of $\mathcal{U}(G)$ are precisely the irreducibles of $\mathcal{B}(G)$. We also note that the empty block acts as the identity in $\mathcal{B}(G)$. In what follows, we will often write blocks in the form $g_1^{x_1} \dots g_t^{x_t}$, where the x_i are nonnegative integers and $g_1, \dots, g_t$ are distinct group elements.

If S is a nonempty subset of G, then define

$$\mathcal{B}(G, S) = \{\{g_1, \dots, g_t\} \in \mathcal{B}(G) \mid g_i \in S \text{ for all } 1 \le i \le t\}.$$

Then $\mathcal{B}(G, S)$ is a submonoid of $\mathcal{B}(G)$ with atoms $\mathcal{A}(\mathcal{B}(G, S)) = \mathcal{B}(G, S) \cap \mathcal{U}(G)$. For more information about block monoids, see [8].

Assume that G is a finite abelian group. For $B = \{g_1, \dots, g_t\} \in \mathcal{B}(G)$, the *cross number* of B is defined as

$$\mathbf{k}(B) = \sum_{i=1}^{t} \frac{1}{|g_i|},$$

where $|g_i|$ denotes the order of g_i in G. The *Davenport constant* of G, denoted by $D(G)$, is defined to be the maximum length of an irreducible in $\mathcal{B}(G)$. It is easy to argue that $D(G) \le |G|$ and if $G \cong \mathbb{Z}_n$ then $D(G) = n$. If $G \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$ with $1 < n_1 | \dots | n_k$, we define

$$M(G) = 1 + \sum_{i=1}^{k} (n_i - 1).$$

In general, we have $D(G) \ge M(G)$, and $D(G) = M(G)$ if G is a p-group or a group of rank less than 3, or if $|G| < 96$. It is also known that

$$\rho(\mathcal{B}(G)) = \frac{D(G)}{2}.$$

For a survey of known results concerning the Davenport constant and the cross number, and their relation to factorization theory, consult [5].

Let S be a nonempty subset of $G - \{0\}$. Following [3], we define $D_S(G)$ to be the maximum length of an irreducible in $\mathcal{B}(G, S)$. It is easy to verify that $D_S(G) \le D(G)$ and $D_G(G) = D(G)$.

Finally, we will use the following useful result in subsequent sections. We omit the simple proof.

Lemma 5. *Let G be an abelian group, H a subgroup of G, and S a nonempty subset of H. Then $\mathcal{R}(\mathcal{B}(H,S)) \subseteq \mathcal{R}(\mathcal{B}(G))$.*

3.1. **Finite abelian groups.** We first consider block monoids $\mathcal{B}(G)$ where G is a finite abelian group. The results in this section will allow us to characterize a large class of these block monoids as fully elastic.

Lemma 6. *Let $n \geq 2$ be a positive integer and S a nonempty subset of $\mathbb{Z}_n$. If there is an element $g \in S$ of order n such that $g^{-1} \in S$ then $\mathcal{B}(\mathbb{Z}_n, S)$ is fully elastic. In particular, $\mathcal{B}(\mathbb{Z}_n)$ is fully elastic.*

Proof. First, if $n = 2$ then $\mathcal{B}(\mathbb{Z}_n, S)$ is a HFM since $\mathcal{B}(\mathbb{Z}_n)$ is, hence it's fully elastic. Now suppose $n > 2$. By [3, Proposition 3] and the remarks preceding it, $\rho(\mathcal{B}(\mathbb{Z}_n, S)) = \frac{D_S(G)}{2} = \frac{n}{2}$. Let $x \in \mathbb{Q} \cap [1, \frac{n}{2}]$. Then it suffices to show that $\rho(B) = x$ for some $B \in \mathcal{B}(\mathbb{Z}_n, S)$. Suppose $u \leq v$ and consider the block $B = (g^n)^u((g^{-1})^n)^v$. The only possible irreducible divisors of B are g^n, $(g^{-1})^n$, and $g \cdot (g^{-1})$, which have lengths n or 2. Since the given factorization of B contains only maximal length irreducibles, it has minimal length and $l(B) = u + v$. Since the only irreducible of length 2 contains the element g, of which there are nu total in B, it follows that the factorization $B = (g \cdot (g^{-1}))^{nu}((g^{-1})^n)^{v-u}$ has maximal length and $L(B) = (n-1)u + v$. Now let $\frac{p}{q} = x$ for $p, q \in \mathbb{N}$. Take $u = p - q$ and $v = (n-1)q - p$. That $u \leq v$ follows from $\frac{p}{q} \leq \frac{n}{2}$. With these choices of u and v we have

$$\rho(B) = \frac{(n-1)u + v}{u + v} = \frac{(n-1)(p-q) + ((n-1)q - p)}{(p-q) + ((n-1)q - p)} = \frac{p}{q} = x,$$

so $\mathcal{B}(\mathbb{Z}_n, S)$ is fully elastic.

To see that $\mathcal{B}(\mathbb{Z}_n)$ is fully elastic, take $S = \{1, n-1\}$. By the above result, $\mathcal{B}(\mathbb{Z}_n, S)$ is fully elastic, and by Lemma 5, $\mathcal{B}(\mathbb{Z}_n)$ is also. □

Lemma 7. *Let G be a finite abelian group and S a nonempty subset of G. Let $\alpha = g_1^{x_1} \dots g_t^{x_t}$ be an irreducible in $\mathcal{B}(G,S)$ of length $D_S(G)$, where the g_i are all distinct. Suppose $g_1^{-1}, \dots, g_t^{-1} \in S$ and $x_1 = |g_1| - 1$. Then $\mathcal{R}(\mathcal{B}(G,S)) = \mathbb{Q} \cap [1, \frac{D_S(G)}{2}]$ so $\mathcal{B}(G,S)$ is fully elastic.*

Proof. First note that if $D_S(G) = |g_1|$ and $g_1^{|g_1|-1}g_2$ is an irreducible of length $D_S(G)$ then we must have $g_2 = (g_1^{|g_1|-1})^{-1} = g_1$ so g_1 and g_2 are not distinct. Thus, the hypotheses imply that that $D_S(G) > |g_1|$.

Let $\bar{\alpha} = (g_1^{-1})^{x_1} \dots (g_t^{-1})^{x_t}$, $\beta = g_1^{|g_1|}$, and $\bar{\beta} = (g_1^{-1})^{|g_1|}$. Then $\alpha, \bar{\alpha}, \beta$, and $\bar{\beta}$ are irreducible in $\mathcal{B}(G,S)$. Let u, v be nonnegative integers not both zero and consider the block

$$B = \alpha^u \bar{\alpha}^u \beta^v \bar{\beta}^v.$$

We claim the given factorization of B has minimal length. Suppose F is a factorization of B of length at most $2u + 2v$. For all $\gamma \in \mathcal{B}(G,S)$, let $v_{\hat{g}_1}(\gamma)$

denote the total number of the elements g_1 and g_1^{-1} in γ. Note that if γ is irreducible then $v_{\hat{g_1}}(\gamma) \le |g_1|$. Let δ be the number of irreducible factors γ in F with $v_{\hat{g_1}}(\gamma) < |g_1|$, and let σ be the number of such factors with $v_{\hat{g_1}}(\gamma) = |g_1|$. Then

$$\delta(|g_1|-1) + \sigma|g_1| \ge v_{\hat{g_1}}(B) = 2u(|g_1|-1) + 2v|g_1|.$$

Since the length of the factorization F is $\delta + \sigma \le 2u + 2v$, it follows that $\sigma \ge 2v$. Now note that all factors counted by σ have length $|g_1|$, and those counted by δ have length at most $D_S(G)$. Thus,

$$\delta D_S(G) + \sigma|g_1| \ge |B| = 2uD_S(G) + 2v|g_1|.$$

Since $\delta + \sigma \le 2u + 2v$, it follows that $\delta(D_S(G) - |g_1|) \ge 2u(D_S(G) - |g_1|)$, whence $\delta \ge 2u$. Hence, $\delta + \sigma \ge 2u + 2v$, and the given factorization of B is minimal as claimed (i.e. $l(B) = 2u + 2v$).

The factorization $B = (g_1 g_1^{-1})^{ux_1 + v|g_1|}(g_2 g_2^{-1})^{ux_2} \dots (g_t g_t^{-1})^{ux_t}$ has maximal length since all factors have minimal length 2. Thus, $L(B) = u(x_1 + \dots + x_t) + v|g_1| = uD_S(G) + v|g_1|$.

Now let $\frac{p}{q} \in \mathbb{Q} \cap [\frac{|g_1|}{2}, \frac{D_S(G)}{2}]$, and take $u = 2p - |g_1|q$ and $v = D_S(G)q - 2p$. Note that the restrictions on $\frac{p}{q}$ ensure that u, v are nonnegative and not both zero. With this choice of u and v, we have

$$\begin{aligned}\rho(B) = \frac{uD_S(G) + v|g_1|}{2u + 2v} &= \frac{(2p - |g_1|q)D_S(G) + (D_S(G)q - 2p)|g_1|}{2(2p - |g_1|q) + 2(D_S(G)q - 2p)} \\ &= \frac{2(D_S(G) - |g_1|)p}{2(D_S(G) - |g_1|)q} = \frac{p}{q}.\end{aligned}$$

Thus, $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathbb{Q} \cap [\frac{|g_1|}{2}, \frac{D_S(G)}{2}]$.

Now let $n = |g_1|$. Then $\langle g_1 \rangle \cong \mathbb{Z}_n$ is a subgroup of G. Using Lemmas 5 and 6, $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathbb{Q} \cap [1, \frac{n}{2}] = \mathbb{Q} \cap [1, \frac{|g_1|}{2}]$, and so $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathbb{Q} \cap [1, \frac{D_S(G)}{2}]$. By [3, Proposition 3], $\rho(\mathcal{B}(G,S)) = \frac{D_S(G)}{2}$ so $\mathcal{R}(\mathcal{B}(G,S)) = \mathbb{Q} \cap [1, \rho(\mathcal{B}(G,S))]$ and $\mathcal{B}(G,S)$ is fully elastic. □

Theorem 8. *Let G be a finite abelian group. Then*

$$\mathcal{R}(\mathcal{B}(G)) \supseteq \mathbb{Q} \cap [1, \frac{M(G)}{2}].$$

Proof. Let $G = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$, where $1 < n_1 | \dots | n_k$. Let

$$g_1 = (1, 0, \dots, 0), \dots, g_k = (0, 0, \dots, 1)$$

be the standard basis elements of G, and let $g_* = (1, 1, \dots, 1)$. Let $S = \{g_1, \dots, g_k, g_*, g_1^{-1}, \dots, g_k^{-1}, g_*^{-1}\}$. We next show $D_S(G) = M(G)$.

First note that if G is a finite cyclic group then $D_S(G) = D(G) = M(G)$, so suppose G has rank $k \ge 2$. Since the block $g_1^{n_1 - 1} \dots g_k^{n_k - 1} g_*$ has length $M(G)$, we know $D_S(G) \ge M(G)$. We show $D_S(G) \le M(G)$. Suppose B is an irreducible of length $D_S(G)$. Then B cannot contain both g and g^{-1} for any $g \in S$. Without loss of generality, we can assume B contains g_* but not

g_*^{-1}. So suppose $B = \hat{g_1}^{x_1} \dots \hat{g_k}^{x_k} g_*^{x_*}$, where $\hat{g_i} \in \{g_i, g_i^{-1}\}$ for all $1 \le i \le k$, and $x_1, \dots, x_k, x_* \in \mathbb{N}_0$.

Now let j be the smallest index such that $n_j = n_k$. If $\hat{g_i} = g_i$ for any $i \ge j$ then we can assume by some isomorphism that $\hat{g_k} = g_k$. It follows that $x_k = n_k - x_*$, and so

$$\begin{aligned} |B| &= x_1 + \dots + x_k + x_* = x_1 + \dots + x_{k-1} + (n_k - x_*) + x_* \\ &\le (n_1 - 1) + \dots + (n_k - 1) + 1 = M(G). \end{aligned}$$

Now assume $\hat{g_i} = g_i^{-1}$ for all $i \ge j$. Then $x_i = x_*$ for all $i \ge j$. If $j = 1$ then we must have $x_1 = \dots = x_k = x_* = 1$ since B is irreducible, so $|B| = k + 1 \le M(G)$. So suppose $j > 1$.

Since $x_k = x_*$,

$$|B| = x_1 + \dots + x_{k-1} + 2x_* \le (n_1 - 1) + \dots + (n_{k-1} - 1) + 2x_*.$$

Suppose $|B| > M(G)$. It follows from the last equation that $x_* > \frac{n_k}{2} \ge n_{j-1}$. By the minimality of B, we must have $B = (g_j^{-1})^{n_{j-1}} \dots (g_k^{-1})^{n_{j-1}} g_*^{n_{j-1}}$ since this is a zero-subsequence of B. Thus $|B| = (k - j + 2)n_{j-1}$, and hence

$$\begin{aligned} M(G) &= (n_1 - 1) + \dots + (n_{j-1} - 1) + (k - j + 1)(n_k - 1) + 1 \\ &\ge n_{j-1} + (k - j + 1)(2n_{j-1} - 1) \\ &= (k - j + 2)n_{j-1} + (k - j + 1)(n_{j-1} - 1) \ge (k - j + 2)n_{j-1} = |B|. \end{aligned}$$

This is a contradiction, so in all cases $D_S(G) = |B| \le M(G)$.

Now let $\alpha = g_1^{n_1-1} \dots g_k^{n_k-1} g_*$. Then α is an irreducible in $\mathcal{B}(G, S)$ of length $D_S(G)$. By Lemma 7, $\mathcal{R}(\mathcal{B}(G)) \supseteq \mathcal{R}(\mathcal{B}(G, S)) = \mathbb{Q} \cap [1, \frac{D_S(G)}{2}] = \mathbb{Q} \cap [1, \frac{M(G)}{2}]$. □

Theorem 9. *Let G be a finite abelian group. If*

(i) *$D(G) = M(G)$, or*
(ii) *there exists a maximal length irreducible in $\mathcal{B}(G)$ which contains $g^{|g|-1}$ for some $g \in G$,*

then $\mathcal{B}(G)$ is fully elastic.

Proof. For (i), if $D(G) = M(G)$ then it follows immediately from Theorem 8 that $\mathcal{B}(G)$ is fully elastic.

For (ii), suppose $G \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$, where $1 < n_1 | \dots | n_k$. Suppose $\alpha \in \mathcal{A}(\mathcal{B}(G))$ has length $D(G)$ and contains $g^{|g|-1}$. If α contains $g^{|g|}$ then $D(G) = |g| \le n_k$, and so $M(G) = 1 + \sum_{i=1}^{k} (n_i - 1) \le D(G) \le n_k$. Thus, $\sum_{i=1}^{k-1} (n_i - 1) \le 0$ so G is cyclic, and hence fully elastic by Lemma 6.

Now suppose α contains $g^{|g|-1}$ and no higher power of g. Write $\alpha = g^{|g|-1} g_2^{x_2} \dots g_t^{x_t}$. Letting $S = G$ and applying Lemma 7, we have $\mathcal{R}(\mathcal{B}(G)) = \mathbb{Q} \cap [1, \frac{D_G(G)}{2}] = \mathbb{Q} \cap [1, \frac{D(G)}{2}]$ so $\mathcal{B}(G)$ is fully elastic. □

3.2. **Infinite abelian groups.** We now turn our attention to block monoids $\mathcal{B}(G)$ where G is an infinite (but finitely generated) abelian group. In particular, we show all such block monoids are fully elastic.

Lemma 10. *$\mathcal{B}(\mathbb{Z})$ is fully elastic.*

Proof. Let $x = \frac{p}{q} \in [1, \infty)$, where $p, q \in \mathbb{N}$. It suffices to show $\rho(B) = x$ for some $B \in \mathcal{B}(\mathbb{Z})$. First, if $x = 1$ then $\rho(\alpha) = 1 = x$ for any irreducible $\alpha \in \mathcal{A}(\mathcal{B}(\mathbb{Z}))$. Now suppose $x > 1$. Let $m, s, t \in \mathbb{N}$ such that $t \geq s$ and $m > 1$, and consider the block

$$B = (1^m \cdot (-m))^s ((-1)^m \cdot m)^t.$$

The only possible irreducible divisors of B are $(1^m \cdot (-m))$, $((-1)^m \cdot m)$, $(1 \cdot (-1))$, and $(m \cdot (-m))$, which have lengths $m+1$ and 2. Since the given factorization of B contains only maximal length irreducible factors, it has minimal length so $l(B) = s + t$. Also,

$$B = (1 \cdot (-1))^{ms} ((-m) \cdot m)^s ((-1)^m m)^{t-s}.$$

Since this factorization of B contains the greatest possible number of irreducible factors of length 2, it has maximal length and $L(B) = ms + t$. Hence,

$$\rho(B) = \frac{ms + t}{s + t}.$$

Take $s = 1, t = 2q - 1 \geq s$, and $m = 2p - 2q + 1 > 1$. Then

$$\rho(B) = \frac{(2p - 2q + 1) + (2q - 1)}{1 + (2q - 1)} = \frac{2p}{2q} = x,$$

which completes the proof. □

Corollary 11. *If G is a finitely generated infinite abelian group then $\mathcal{B}(G)$ is fully elastic.*

Proof. Since G is finitely generated, we can write $G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$ for some $r, n_1, \ldots, n_k \in \mathbb{N}$. Since G is infinite, we know $r > 0$ so G contains a subgroup isomorphic to $\mathbb{Z}$. It follows from Lemmas 5 and 10 that $\mathcal{R}(\mathcal{B}(G)) \supseteq \mathcal{R}(\mathcal{B}(\mathbb{Z})) = \mathbb{Q} \cap [1, \infty)$, so $\mathcal{B}(G)$ is fully elastic. □

3.3. **Asymptotic elasticity in $\mathcal{B}(G)$.** In this section we first present several results that relate asymptotic elasticity in $\mathcal{B}(G)$ to the cross number $\mathbf{k}$, and then use results of the preceding sections to characterize the set of asymptotic elasticities of $\mathcal{B}(G)$.

Theorem 12. *Let G be an abelian group. Let $x \in \mathcal{B}(G)$ and $y \in \mathcal{A}(\mathcal{B}(G))$. Then*

(i) $\bar{\rho}(y) \geq \max\{\mathbf{k}(y), \frac{1}{\mathbf{k}(y)}\}$,
(ii) $\bar{\rho}(x) = 1$ *if and only if every irreducible divisor α of the collective powers of x has $\mathbf{k}(\alpha) = 1$.*

Proof. (i) Let $n \in \mathbb{N}$. Since y^n is a product of n irreducibles, we know $l(y^n) \leq n \leq L(y^n)$. Dividing through by n and taking limits yields $\bar{l}(y) \leq 1 \leq \bar{L}(y)$. From [6, Proposition 7], we also know $\bar{l}(y) \leq \mathbf{k}(y) \leq \bar{L}(y)$. Taking the appropriate quotients yields the desired result.

The proof of (ii) is adapted from [6, Lemma 3].

($\Rightarrow$) Suppose $\bar{\rho}(x) = 1$ so that $\bar{L}(x) = \bar{l}(x)$. By [6, Lemma 1(5)], every factorization of x^n into irreducibles has the same length, for all $n \in \mathbb{N}$. Let α be an irreducible divisor of x^t for some $t \in \mathbb{N}$. Then for every $s \in \mathbb{N}$, $\alpha^s | x^{ts}$, and since x^{ts} has unique irreducible factorization length, it follows that α^s does as well.

Now write $\alpha = g_1 \dots g_\omega$ where $g_1, \dots, g_\omega \in G$. Let $k = lcm\{|g_1|, \dots, |g_\omega|\}$ and for each $1 \leq i \leq \omega$ let $k_i = \frac{k}{|g_i|}$. Then

$$\alpha^k = (g_1^{|g_1|})^{k_1} \dots (g_\omega^{|g_\omega|})^{k_\omega}$$

and setting $\gamma_i = g_i^{|g_i|}$, we have

$$\alpha^k = \gamma_1^{k_1} \dots \gamma_\omega^{k_\omega},$$

where $\gamma_i \in \mathcal{A}(\mathcal{B}(G))$ and $\mathbf{k}(\gamma_i) = 1$ for each i. By the properties of cross numbers, $\mathbf{k}(\alpha^k) = k\mathbf{k}(\alpha) = k_1 + \ldots + k_\omega$. Since α^k has unique irreducible factorization length, it follows that $k = k_1 + \ldots + k_\omega$, whence $\mathbf{k}(\alpha) = 1$.

($\Leftarrow$) Suppose $x \in \mathcal{B}(G)$ and $x^n = \beta_1 \dots \beta_s = \gamma_1 \dots \gamma_t$, where each β_i and γ_j is in $\mathcal{A}(\mathcal{B}(G))$. By the properties of cross numbers,

$$\mathbf{k}(x^n) = n\mathbf{k}(x) = \mathbf{k}(\beta_1) + \ldots + \mathbf{k}(\beta_s) = \mathbf{k}(\gamma_1) + \ldots + \mathbf{k}(\gamma_t).$$

Since all irreducible divisors of x^n have cross number 1, we have $n = s = t$. Thus, $l(x^n) = L(x^n)$ for all $n \in \mathbb{N}$, from which $\bar{L}(x) = \bar{l}(x)$ and $\bar{\rho}(x) = 1$. □

Theorem 13. *Let G be a finite abelian group. Then*

$$\bar{\mathcal{R}}(\mathcal{B}(G)) \supseteq \mathbb{Q} \cap [1, \frac{M(G)}{2}].$$

If $M(G) = D(G)$, or if there exists a maximal length irreducible in $\mathcal{B}(G)$ which contains $g^{|g|-1}$ for some $g \in G$ then

$$\bar{\mathcal{R}}(\mathcal{B}(G)) = \mathbb{Q} \cap [1, \frac{D(G)}{2}].$$

Proof. Let $B = \alpha\bar{\alpha}\beta\bar{\beta}$ be as in the proof of Lemma 7, and let $C = (1^n)^u((n-1)^n)^v$ be the block considered in the proof of Theorem 6. Then for all $k \in \mathbb{N}$,

$$\rho(B^k) = \frac{kuD_S(G) + kv|g_1|}{2ku + 2kv} = \frac{uD_S(G) + v|g_1|}{2u + 2v} = \rho(B)$$

and

$$\rho(C^k) = \frac{(n-1)ku + kv}{ku + kv} = \frac{(n-1)u + v}{u + v} = \rho(C).$$

By Theorem 2, $\bar{\rho}(B) = \rho(B)$ and $\bar{\rho}(C) = \rho(C)$. By the proof of Theorem 8, for any $x \in \mathbb{Q} \cap [1, \frac{M(G)}{2}]$ we can choose u and v such that either $\bar{\rho}(B) =$

$\rho(B) = x$ or $\bar{\rho}(C) = \rho(C) = x$. This proves the first statement. In a similar way, the second statement follows easily from Theorem 9. □

Theorem 14. *If G is a finitely generated infinite abelian group then*

$$\bar{\mathcal{R}}(\mathcal{B}(G)) = \mathbb{Q} \cap [1, \infty).$$

Proof. By the proof of Corollary 11, it suffices to show $\bar{\mathcal{R}}(\mathcal{B}(\mathbb{Z})) = \mathbb{Q} \cap [1, \infty)$. Let $B = (1^m \cdot (-m))^s((-1)^m \cdot m)^t$ be as in the proof of Lemma 10. Then for all $n \in \mathbb{N}$,

$$\rho(B^n) = \frac{mns + nt}{ns + nt} = \frac{ms + t}{s + t} = \rho(B),$$

so by Lemma 2, $\bar{\rho}(B) = \rho(B)$. By Lemma 10 for any $x \in \mathbb{Q} \cap [1, \infty)$, we can choose B such that $\bar{\rho}(B) = \rho(B) = x$, and the result follows. □

While Theorems 13 and 14 show that all possible asymptotic elasticities are attained in block monoids over many finite groups and all finitely generated infinite groups, the values attained by the functions $\bar{L}$ and $\bar{l}$ are generally much more restricted. We illustrate this in the following

Example. Let $\bar{L}(\mathcal{B}(G))$ and $\bar{l}(\mathcal{B}(G))$ denote the images of $\mathcal{B}(G)$ under the functions $\bar{L}$ and $\bar{l}$, respectively. Consider the block monoid $\mathcal{B}(\mathbb{Z}_3)$. By Theorem 13, $\bar{\mathcal{R}}(\mathcal{B}(\mathbb{Z}_3)) = \mathbb{Q} \cap [1, \frac{3}{2}]$. In contrast, we show

$$\bar{l}(\mathcal{B}(\mathbb{Z}_3)) = \{\frac{n}{3} \mid n \in \mathbb{N}, n > 1\},$$

and

$$\bar{L}(\mathcal{B}(\mathbb{Z}_3)) = \mathbb{N}.$$

Proof. Suppose $B = 1^x 2^y \in \mathcal{B}(\mathbb{Z}_3)$. Since B is a block, $x \equiv y \pmod 3$. By the division algorithm, let $x = 3s + r$ and $y = 3t + r$, where $0 \le r < 3$, and consider the block B^{3n}. The factorization $B^{3n} = (1^3)^{nx}(2^3)^{ny}$ has minimal length since all factors are of maximal length. Thus $l(B^{3n}) = n(x + y)$ and

$$\bar{l}(B) = \lim_{n\to\infty} \frac{l(B^{3n})}{3n} = \frac{x + y}{3} = s + t + \frac{2r}{3} \in \{\frac{n}{3} \mid n \in \mathbb{N}, n > 1\}.$$

Now let $n \in \mathbb{N}, n > 1$ and write $n = 3q + a$ for $0 \le a < 3$. If $a \in \{0, 2\}$ take $r = \frac{a}{2}$ and $s + t = q$. If $a = 1$ take $r = 2$ and $s + t = q - 1$. For these values of x and y, $\bar{l}(B) = \frac{n}{3}$, and thus $\bar{l}(\mathcal{B}(\mathbb{Z}_3)) = \{\frac{n}{3} \mid n \in \mathbb{N}, n > 1\}$.

Now assume $m = \min\{x, y\}$ and write $m = 3q + r$. Then the factorization $B^{3n} = (1 \cdot 2)^{3nm}(1^3)^{n(x-m)}(2^3)^{n(y-m)}$ has maximal length since it contains the greatest possible number of factors of length 2. Thus $L(B^{3n}) = nx + ny + nm$, and so $\bar{L}(B) = \frac{x+y+m}{3} = s + t + q + r$. It follows that $\bar{L}(\mathcal{B}(\mathbb{Z}_3)) = \mathbb{N}$. □

4. Full elasticity in numerical monoids

Given that several large classes of block monoids are fully elastic, it becomes relevant to ask if all monoids and integral domains are fully elastic as well. To provide an answer, we focus now on the context of numerical monoids. Let $a_1, \ldots, a_t$ be positive integers. We define the *numerical monoid generated by* $a_1, \ldots, a_t$ to be

$$S = \langle a_1, \ldots, a_t \rangle = \{x_1 a_1 + \ldots + x_t a_t \mid x_1, \ldots, x_t \in \mathbb{N}_0\},$$

which is a submonoid of $\mathbb{N}_0$. Note that $0 \in \mathbb{N}_0$ acts as the identity in S, and if $1 \in \{a_1, \ldots, a_t\}$ then $S = \mathbb{N}_0$. It's a fact that every numerical monoid S has a minimal set of generators, which are precisely the atoms of S. S is also clearly commutative, cancellative, and atomic. For more information on numerical monoids, see [7]. Our first result gives the elasticity of an arbitrary numerical monoid.

Theorem 15. *Let $S = \langle a_1, \ldots, a_t \rangle$ be a numerical monoid, where $a_1, \ldots, a_t \in \mathbb{N}$ is a minimal set of generators for S. Then $\rho(S) = \frac{a_t}{a_1}$.*

Proof. We may assume $a_1 < \ldots < a_t$. Let $n \in S$ and suppose $n = x_1 a_1 + \ldots + x_t a_t$. Then

$$\frac{n}{a_t} = \frac{a_1}{a_t} x_1 + \ldots + \frac{a_t}{a_t} x_t \leq x_1 + \ldots + x_t \leq \frac{a_1}{a_1} x_1 + \ldots + \frac{a_t}{a_1} x_t = \frac{n}{a_1}.$$

Thus $L(n) \leq \frac{n}{a_1}$ and $l(n) \geq \frac{n}{a_t}$ for all $n \in S$, from which $\rho(S) \leq \frac{a_t}{a_1}$. Also, $\rho(S) \geq \rho(a_1 a_t) = \frac{a_t}{a_1}$, so we have equality. □

Note that if $S = \langle a \rangle$ is generated by a single element, then S is a half-factorial monoid so S is trivially fully elastic. In the results that follow, we will assume the minimal number of generators of S is $t \geq 2$.

Theorem 16. *Let $S = \langle a_1, \ldots, a_t \rangle$ be a numerical monoid, where $a_1, \ldots, a_t \in \mathbb{N}$ minimally generate S and $t \geq 2$. Then S is not fully elastic.*

Proof. Suppose without loss of generality that $1 < a_1 < \ldots < a_t$. Let $n \in S$ with maximal length factorization $n = x_1 a_1 + \ldots + x_t a_t$. If $x_i \geq a_{i-1}$ for any $i \in \{2, \ldots, t\}$ then

$$n = x_1 a_1 + \ldots + (x_{i-1} + a_i) a_{i-1} + (x_i - a_{i-1}) a_i + \ldots + x_t a_t$$

is a factorization with longer length. Thus $x_2 < a_1, \ldots, x_t < a_{t-1}$ and $x_2 a_2 + \ldots + x_t a_t < a_1 a_2 + \ldots + a_{t-1} a_t$. Let $s = a_1 a_2 + \ldots + a_{t-1} a_t$. Then

$$L(n) = x_1 + \ldots + x_t \geq x_1 = \frac{n - (x_2 a_2 + \ldots + x_t a_t)}{a_1} > \frac{n - s}{a_1}.$$

Now suppose $n = y_1 a_1 + \ldots + y_t a_t$ is a factorization of minimal length. Then by a parallel argument we have $y_1 < a_2, \ldots, y_{t-1} < a_t$, and so $y_1 a_1 + \ldots + y_{t-1} a_{t-1} < s$. Thus,

$$y_1 + \ldots + y_{t-1} \leq \frac{a_1}{a_1} y_1 + \ldots + \frac{a_{t-1}}{a_1} y_{t-1} = \frac{y_1 a_1 + \ldots + y_{t-1} a_{t-1}}{a_1} < \frac{s}{a_1}.$$

Also,

$$y_t = \frac{n - (y_1 a_1 + \ldots + y_{t-1} a_{t-1})}{a_t} \leq \frac{n}{a_t},$$

and combining this with the previous result we have

$$l(n) = y_1 + \ldots + y_t < \frac{s}{a_1} + \frac{n}{a_t} = \frac{na_1 + sa_t}{a_1 a_t}.$$

Hence,

$$\rho(n) = \frac{L(n)}{l(n)} > \frac{na_t - sa_t}{na_1 + sa_t}.$$

Let N be an integer greater than $\frac{2sa_t}{a_t - a_1}$ and define $m = \frac{Na_t - sa_t}{Na_1 + sa_t}$. Note by the choice of N that $m > 1$. Now if $n > N$ then

$$\rho(n) > \frac{na_t - sa_t}{na_1 + sa_t} > \frac{Na_t - sa_t}{Na_1 + sa_t} = m$$

since $\{\frac{na_t - sa_t}{na_1 + sa_t}\}_{n \in \mathbb{N}}$ is an increasing sequence. Thus there are at most N elements of S which have elasticity m or less. Since there are infinitely many rationals in $[1, m]$, this implies S is not fully elastic. □

Theorem 17. *Let $a_1, \ldots, a_t \in \mathbb{N}$ be a minimal set of generators for the numerical monoid $S = \langle a_1, \ldots, a_t \rangle$, where $t \geq 2$. Then the only limit point of $\mathcal{R}(S)$ is $\frac{a_t}{a_1}$.*

Proof. Suppose $1 < a_1 < \ldots < a_t$. First, if $n = k(a_1 a_t) + a_1$ for $k \in \mathbb{N}_0$ then $\rho(n) = \frac{L(n)}{l(n)} = \frac{ka_t + 1}{ka_1 + 1}$. It follows that $\rho(n) < \frac{a_t}{a_1}$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \rho(n) = \frac{a_t}{a_1}$. Thus, $\frac{a_t}{a_1}$ is a limit point of the set $\mathcal{R}(S)$.

We now show $\frac{a_t}{a_1}$ is the only limit point of this set. Let $r \in [1, \frac{a_t}{a_1})$, let $s = a_1 a_2 + \ldots + a_{t-1} a_t$, and take N to be an integer greater than $\frac{(r+1)sa_t}{a_t - ra_1}$ (which is positive by the restrictions on r). By the proof of Theorem 16, for all $n > N$,

$$\rho(n) > \frac{Na_t - sa_t}{Na_1 + sa_t}.$$

The reader can verify that

$$N > \frac{(r+1)sa_t}{a_t - ra_1} \implies \frac{Na_t - sa_t}{Na_1 + sa_t} > r$$

so that $\rho(n) > r$. Thus there are at most N elements of S which have elasticity r or less. Since there are a finite number of elasticities less than r there can be no limit points less than r. Since this is true of any $r \in [1, \frac{a_t}{a_1})$, there are no limit points other than $\frac{a_t}{a_1}$. □

Theorem 18. *Let $a_1, \ldots, a_t \in \mathbb{N}$ be a minimal set of generators for the numerical monoid $S = \langle a_1, \ldots, a_t \rangle$, where $t \geq 2$. Then $\bar{\mathcal{R}}(S) = \{\frac{a_t}{a_1}\}$.*

Proof. Suppose $1 < a_1 < \ldots < a_t$ and let $n \in S$. Then $n^{a_1 a_t} = (a_1 n)a_t = (a_t n)a_1$ are the minimal and maximal length factorizations of $n^{a_1 a_t}$, respectively, so $\rho(n^{a_1 a_t}) = \frac{a_t n}{a_1 n} = \frac{a_t}{a_1}$. By Lemma 1,

$$\frac{a_t}{a_1} = \rho(n^{a_1 a_t}) \leq \bar{\rho}(n^{a_1 a_t}) = \bar{\rho}(n) \leq \bar{\rho}(S) = \rho(S) = \frac{a_t}{a_1}$$

so $\bar{\rho}(n) = \frac{a_t}{a_1}$ and the result follows. □

From the proof of Theorem 17, there are elasticities in $\mathcal{R}(S)$ less than $\frac{a_t}{a_1}$ so in particular the last theorem tells us that $\bar{\mathcal{R}}(S) \subsetneq \mathcal{R}(S)$.

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