Asymptotic Multiple Objective Linear Programming

Lawrence Cayton* Jesse Holzer†

August 2, 2002

Abstract

In this paper we extend a theorem on stability of the optimal partition of a dynamic linear program to multiple objective linear programs. To do this we briefly review the area of linear programming and the concept of the optimal partition. We discuss optimality in a multiple objective linear program in the sense of pareto optimality and generalize the optimal partition to this sense of optimality. Furthermore we derive conditions on the parameters of a dynamic multiple objective linear program under which the optimal partition stabilizes and apply this result to an economic model where the multiple objective structure is readily apparent. Finally we discuss a different approach that yields further interesting insights into the asymptotic structure of multiple objective linear programs.

Keywords: Multiple objective linear program, asymptotic programming, computational economics

*Visiting from Washington University, St. Louis; email: lac3@cec.wustl.edu

†Visiting from Carleton College; email: holzerj@carleton.edu

1 An Introduction to Linear Programming

Linear programming is one of the most prominent areas in optimization because of its applicability to numerous problems in economics, management, and the physical sciences. A linear program (LP) in standard form is formulated as a linear objective function that is to be minimized subject to a set of linear constraints,

$$\text{LP:} \quad \min_x \{c^T x : Ax = b, x \geq 0\}, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Any vector x that satisfies the above constraints is called *feasible*, and the feasible region is denoted by $\mathcal{P}$. The optimal set is denoted by $\mathcal{P}^*$, and x^* denotes a vector in $\mathcal{P}^*$.

Each LP has an associated dual,

$$\text{LD:} \quad \max_y \{b^T y : A^T y + s = c, s \geq 0\}, \tag{2}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$. The dual feasible region is denoted by $\mathcal{D}$, the optimal dual set by $\mathcal{D}^*$, and (y^*, s^*) denotes an element of $\mathcal{D}^*$. The Strong Duality Theorem of Linear Programming states that if x and y are feasible, x and y are optimal if, and only if, $c^T x = b^T y$. This means that the following three equations are necessary and sufficient for optimality —i.e. $\mathcal{P}^* \times \mathcal{D}^*$ is the collection of solutions to

$$\begin{aligned} Ax &= b, \quad x \geq 0, && (3) \\ A^T y + s &= c, \quad s \geq 0, \text{ and,} && (4) \\ c^T x - b^T y &= s^T x = 0. && (5) \end{aligned}$$

From (5) we see that $x_i > 0$ implies that $s_i = 0$ and that $s_i > 0$ implies that $x_i = 0$. Note that if, for some i, $x_i^* > 0$, then $s_i^* = 0$ for all s^*. So, we actually have a slightly stronger condition; namely, if $x_i^* > 0$ for **some** optimal solution, $s_i^* = 0$ in **every** optimal solution. Similarly, if $s_i^* > 0$ for some optimal solution, $x_i^* = 0$ in every optimal solution. These relationships define the unique *optimal partition* of a linear program, denoted $(B|N)$, as follows,

$$\begin{aligned} B &= \{i : x_i^* > 0, \text{ some } x^* \in \mathcal{P}^*\}, \text{ and} \\ N &= \{i : s_i^* > 0, \text{ some } s^* \in \mathcal{D}^*\} \\ &= \{1, \ldots, n\} \setminus B. \end{aligned}$$

For a matrix A, we define A_B to be the matrix composed only of the columns A_{*i} where $i \in B$. Alternatively, A_B is the sub-matrix of A with the columns listed in N removed. Similarly, for a vector x, x_B is the vector consisting only of components x_i where $i \in B$. As an example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}, \text{ and } B = \{1, 3\}, \text{ then } A_B = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}.$$

If the optimal partition is known, the necessary and sufficient conditions for optimality may be rewritten as a linear system. For (3), we have that $Ax = A_B x_B + A_N x_N = A_B x_B$, so we rewrite these constraints as

$$A_B x_B = b, \; x_B \geq 0. \tag{6}$$

The dual constraints in (4) separate into $A_B^T y + s_B = A_B^T y = c_B$ and $A_N^T y + s_N = c_N$. So we have that

$$\begin{aligned} A_B^T y &= c_B, \; s_B = 0, \text{ and} \qquad (7)\\ A_N^T y + s_N &= c_N, \; s_N \geq 0. \qquad (8) \end{aligned}$$

Note that we get (5) for free, since $x^T s = x_B^T s_B + x_N^T s_N = x_B^T(0) + 0^T s_N = 0$, so the transformed system is entirely linear, whereas the previous system contained the bilinear equation $s^T x = 0$.

Throughout this paper, we are concerned with *asymptotic linear programs*. Asymptotic linear programming is the study of the long-term behavior of linear programs whose objective function and constraints vary with time. This type of analysis parallels the long-term solution analysis of differential equations, but instead of population dynamics, asymptotic linear programs are typically of interest because of their economic interpretations [5, 6, 7].

$$\begin{aligned} LP(t): &\qquad \min_x \{c^T(t)x : A(t)x = b(t), x \geq 0\}, \text{ and}\\ LD(t): &\qquad \max_y \{b^T(t)y : A(t)y + s = c(t), s \geq 0\}, \end{aligned}$$

where $A(t) : \mathbb{R} \rightarrow \mathbb{R}^{m\times n}, b(t) : \mathbb{R} \rightarrow \mathbb{R}^m,$ and $c(t) : \mathbb{R} \rightarrow \mathbb{R}^n$.

In asymptotic linear programs, the primal and dual optimal sets are functions of time. As a result, the optimal partition also varies with time. However, in [4] it is shown that under mild restrictions the optimal parition *stabilizes* —i.e., there exists a time T, such that for all $t \geq T$, $(B(t)|N(t)) = (B(T)|N(T))$. $(B(T)|N(T))$ is defined as the *asymptotic optimal partition*. The primary goal of this paper is to extend the notion of an asymptotic optimal partition to the case when there are multiple objective functions.

A *multiple objective linear program* (MOLP) is a linear program where there are q objective functions to minimize. This paper deals with asymptotic MOLPs of the form

$$\text{MOLP(t):} \qquad \min_x \{C(t)x : A(t)x = b(t), x \geq 0\},$$

where $C(t) : \mathbb{R} \rightarrow \mathbb{R}^{q\times n}$

For $q \geq 2$, $\mathbb{R}^q$ does not have a complete ordering, so "minimization" is not uniquely defined. In this paper, we are concerned with *pareto* optimality. A feasible solution x is pareto optimal if there does not exist a $y \in \mathcal{P}$ such that $Cy \leq Cx$, with strict inequality holding for at least one component. The *efficient frontier*, denoted by $\mathcal{E}$, is the set of all pareto optimal solutions. As shown in [2], $x \in \mathcal{E}$ if, and only if, there exists a strictly positive *weight* w such that x minimizes $\{w^T C(t)x : x \in \mathcal{P}\}$. We denote this as

$$LP(w,t): \quad \min_x \{w^T C(t)x : x \in \mathcal{P}(t)\}.$$

For multiple objective linear programs the definition of the optimal partition is slightly different from the single objective case [5]. For a MOLP, $(\overset{\text{molp}}{B} \mid \overset{\text{molp}}{N})$ is defined by

$$\begin{aligned} \overset{\text{molp}}{N} &= \{i : x_i = 0 \text{ for all } x \in \mathcal{E}\} \text{ and} \\ \overset{\text{molp}}{B} &= \{1,2,\ldots,n\} \setminus \overset{\text{molp}}{N}. \end{aligned}$$

In other words, $\overset{\text{molp}}{N}$ corresponds to the indices of components of x that are zero over the entire efficient frontier.

This paper is organized as follows. In section 2, we show that the multiple objective optimal partition stabilizes. In section 3, we present some economic implications of the stabilization result. The last section contains some properties of the optimal partition and it's relation to the choice of w.

2 The Main Result

In this section we show that the multiple objective optimal partition remains constant after some time T. First, we let $\{(B^1|N^1), (B^2|N^2), \ldots, (B^{2^n}|N^{2^n})\}$ be the collection of all possible two-set partitions of $\{1,\ldots,n\}$. Recall that for a MOLP, x is pareto optimal at time t if, and only if, there exists a w such that x minimizes $LP(w,t)$. This linear program, $LP(w,t)$, has an optimal partition $(B^i(t)|N^i(t))$, for some $i = 1,2,\ldots,2^n$, and thus $B^i(t)$ is a *sub-partition* of $\overset{\text{molp}}{B}(t)$. Let $\mathcal{L}(t) = \{i : B^i \text{ is a sub-partition of } \overset{\text{molp}}{B}(t)\}$. Note that $\overset{\text{molp}}{B}(t) = \bigcup_{i \in \mathcal{L}(t)} B^i(t)$, so we can restate the goal of this section as showing that $\mathcal{L}(t)$ stabilizes. To do this, we show that any sub-partition of the multiple objective optimal partition remains a sub-partition after a certain time.

Since we are dealing with sub-partitions, we can make use of the necessary and sufficient conditions for a linear program. This follows since Goldman and Tucker [3] showed that every linear program has a strictly complementary solution, meaning that $x_B^* > 0$ and $s_N^* > 0$. The consequence of this result is that $(B^i|N^i)$ is the optimal partition for $LP(w,t)$ if, and only if, the following system is consistent,

$$\begin{aligned} A_{B^i}(t)x_{B^i} &= b(t),\ x_{B^i} > 0 && (9) \\ A_{N^i}^T(t)y + s_{N^i} &= C_{N^i}^T(t)w,\ s_{N^i} > 0, \text{ and} && (10) \\ A_{B^i}^T(t)y &= C_{B^i}^T(t)w && (11) \end{aligned}$$

where $w > 0$. We rewrite these conditions in matrix form to simplify the notation. For $B^i(t)$, let

$$H_i(t) = \begin{bmatrix} A_{B^i}(t) & 0 & 0 & 0 \\ 0 & A_{B^i}^T & -C_{B^i}^T(t) & 0 \\ 0 & A_{N^i}^T & -C_{N^i}^T(t) & I_{|N^i|} \end{bmatrix}, h(t) = \begin{pmatrix} b(t) \\ 0 \\ 0 \end{pmatrix}, \text{ and } v = \begin{pmatrix} x_{B^i} \\ y \\ w \\ s_{N^i} \end{pmatrix}.$$

Then (9), (10), and (11) may be written as $H_i(t)v = h(t)$, where v must be *sufficiently positive*, meaning that the x_B, w, and s_N components of v must be strictly positive.

Lemma 2.1. *B^i is a sub-partition of $\overset{molp}{B(t)}$ if, and only if, $H_i(t)v = h(t)$ has a sufficiently positive solution.*

Proof. If $H_i(t)v = h(t)$ has a sufficiently positive solution, then (9), (10), and (11) are all satisfied. Since these three equations provide sufficient conditions for optimality, $(B^i(t)|N^i(t))$ must be the optimal partition for $LP(w,t)$, which means that $B^i(t)$ is a sub-partition of $\overset{\text{molp}}{B(t)}$.

Now assume $B^i(t)$ is a sub-partition of $\overset{\text{molp}}{B(t)}$. Then, there exists a w such that $(B^i(t)|N^i(t))$ is the optimal partition for $LP(w,t)$. Thus, equations (9), (10), and (11) must hold, but these equations are the same as $H_i(t)v = h(t)$, with v sufficiently positive. $\square$

In order to get the result, we need to ensure that the ranks of $H_i(t)$ and $h(t)$ stabilize at some point. The following assumption provides this property as Lemma 2.2 will establish.

Assumption 1. *There exists a time T such that for all $t \geq T$ the determinants of all square sub-matrices of*

$$\begin{bmatrix} A(t) & 0 & b(t) & 0 \\ 0 & A^T(t) & 0 & C(t) \end{bmatrix} \tag{12}$$

have no zeros or become constant. Additionally, we require that all submatrices of (12) are continuous after T.

Note that a large class of functions satisfy Assumption 1. In particular, rational and exponential functions are in this class. As a result of this assumption, we get the following lemma.

Lemma 2.2 (Hasfura-Buenaga, Holder, and Stuart [4]). *Given Assumption 1, for all $t > T$, rank$(M(t))$ = rank$(M(T))$, where $M(t)$ is any submatrix of (12). In particular, the ranks of $H_i(t)$ and $[H_i(t)|h(t)]$ stabilize.*

In addition to the continuity of submatrices of (12), we need the continuity of the Moore-Penrose pseudo-inverses of the sub-matrices. The following lemma shows that Assumption 1 provides this property.

Lemma 2.3 (Campbell and Meyer [1]). *Let M(t) be a matrix function. Then,* $M^+(t)$*, the Moore-Penrose pseudo-inverse of M(t), is continuous at* t_0 *if, and only if,* $M(t)$ *is continuous at* t_0 *and* $\text{rank}(M(t_0)) = \text{rank}(M(t))$ *for t sufficiently close to* t_0.

Coupled with Assumption 1, this lemma implies that the Moore-Penrose psuedo-inverse of H_i is continuous after some time T. We now proceed to show that the multiple objective optimal partition stabilizes. The proof is split between Lemma 2.4, where we show that $\mathfrak{L}(t)$ becomes constant over local open intervals, and Theorem 2.5, which demonstrates the asymptotic stability of $\mathfrak{L}(t)$.

Lemma 2.4. *Let* t_0 *be large enough to satisfy Assumption 1. Then, the multiple objective optimal partition is constant over some neighborhood of* t_0*. Equivalently,* $\mathfrak{L}(t_0) = \mathfrak{L}(t)$ *for all t in a sufficiently small neighborhood of* t_0.

Proof. At any time t, $H_i(t)v = h(t)$ has solution if, and only if,

$$\text{rank}([H_i(t)|h(t)]) = \text{rank}(H_i(t)).$$

But, Lemma 2.2 gives us that both $H_i(t)$ and $[H_i(t)|h(t)]$ have stable ranks as long as $t > T$. So, if $H_i(t_0)v = h(t_0)$ has a solution for some $t_0 > T$, we know it that remains consistent for all $t > T$.

Let $i \in \mathfrak{L}(t_0)$, so that $(B^i(t_0)|N^i(t_0))$ is a sub-partition of $(\overset{\text{molp}}{B}(t_0)|\overset{\text{molp}}{N}(t_0))$. Then we have that

$$H_i(t_0)v(t_0) = h(t_0),$$

where $v(t_0)$ is sufficiently positive. Then $v(t_0) = H_i^+(t_0)h(t_0) + q_0$, for some $q_0 \in \text{Null}(H_i(t_0))$. Similarly, at time t, we have that $v(t) = H_i^+(t)h(t) + q$, where $q \in \text{Null}(H_i)$. Because any q in the nullspace of H_i satisfies the previous equation, we take $q = (I - H_i^+(t)H_i(t))q_0$. So we have that

$$v(t) = H_i^+(t)h(t) + (I - H_i^+(t)H_i(t))q_0.$$

Lemma 2.3 guarantees that $H_i^+(t)$ is continuous, and Assumption 1 guarantees the continuity of $H_i(t)$ and $h(t)$. Thus, as $t \to t_0$,

$$\begin{aligned} v(t) &= H_i^+(t)h(t) + (I - H_i^+(t)H_i(t))q_0 \\ &\to H_i^+(t_0)h(t_0) + q_0 = v(t_0), \end{aligned}$$

which is sufficiently positive. So, for t in some open nieghborhood $\mathcal{N}_{\epsilon^i}(t_0)$, $v(t)$ is a sufficiently positive solution to $H_i(t)v = h(t)$.

Let $\epsilon = \min_{i \in \mathfrak{L}(t_0)} \{\epsilon^i\}$. Then we have that for all $t \in \mathcal{N}_\epsilon$, $\mathfrak{L}(t) = \mathfrak{L}(t_0)$. Thus, for $t \in \mathcal{N}_\epsilon$,

$$(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t)) = (\overset{\text{molp}}{B}(t_0)|\overset{\text{molp}}{N}(t_0))$$

. □

Fortunately, the local stability of the multiple objective optimal partition implies the asymptotic stability. The following theorem proves this result.

Theorem 2.5. *Under Assumption 1, there exists a T such that for all $t > T$, $(\overset{molp}{B}(t)|\overset{molp}{N}(t)) = (\overset{molp}{B}(T)|\overset{molp}{N}(T))$.*

Proof. Lemma 2.4 guarantees the stability of $(\overset{\text{molp}}{B}(t_0)|\overset{\text{molp}}{N}(t_0))$ for all $t \in \mathcal{N}_\epsilon(t_0)$.

Now, we define another neighborhood, $\mathcal{N}^2$, which contains all points to the right of T where the partition remains the same:

$$\mathcal{N}^2 = \{T + \hat{\delta} : (\overset{\text{molp}}{B}(t+\delta)|\overset{\text{molp}}{N}(t+\delta)) = (\overset{\text{molp}}{B}(T)|\overset{\text{molp}}{N}(T)), \delta \in [0, \hat{\delta}]\}.$$

Now, let $\hat{t} = \inf\{t > T : (\overset{\text{molp}}{B}(T)|\overset{\text{molp}}{N}(T)) \neq (\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t))\}$. Suppose towards a contradiction that $\hat{t} < \infty$. Then, from Lemma 2.4, we have that there exists an open neighborhood, $\mathcal{N}^3$ about $\hat{t}$ such that for $t \in \mathcal{N}^3$, $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t)) = (\overset{\text{molp}}{B}(\hat{t})|\overset{\text{molp}}{N}(\hat{t}))$. But, $\mathcal{N}^2 \cap \mathcal{N}^3 \neq \emptyset$, so for any $t \in \mathcal{N}^2 \cap \mathcal{N}^3$, we obtain the contradiction that

$$(\overset{\text{molp}}{B}(T)|\overset{\text{molp}}{N}(T)) = (\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t)) = (\overset{\text{molp}}{B}(\hat{t})|\overset{\text{molp}}{N}(\hat{t})).$$

Hence, the multiple objective optimal partition for all $t > T$ is $(\overset{\text{molp}}{B}(T)|\overset{\text{molp}}{N}(T))$. □

We have shown that the multiple objective optimal partition stabilizes given Assumption 1. The next section discusses some economic applications of this result.

3 Economic Interpretations

In this section we demonstrate the usefulness of Theorem 2.5 by showing how it illuminates an otherwise puzzling economic model. Suppose that at any given time, the economy can transform certain amounts of n input commodities and some amount of labor into different amounts of the same n commodities as outputs. Suppose that this production takes place by means of m processes, each of which yields constant returns to scale. Suppose also that the rate of profit through production is bounded. Given this information we want to determine the price of each commodity on the market, the wage of labor, and the rate at which to use each process.

In elementary economic models, the prevailing rate of interest on loaned money is derived from the particular climate of risk in the economy, and in turn, this interest rate induces a maximum rate of profit. Intuitively speaking, if some industry yields a higher rate of profit than the interest rate, then new firms will enter that industry, which increases the supply, lowers the market price, and finally lowers the rate of profit. Thus we are justified in assuming that there is some maximum rate of profit that any choice of prices, wages, and use of capital may yield. In the language of economics, the maximum rate of profit is exogenous to the model.

We assume that the technology available to the economy allows it *constant returns to scale*, meaning that changing the inputs to a process by some factor changes the outputs by the same factor. This hypothesis is the subject of constant debate in economics, but it is not unreasonable. After all, if you have one factory that produces some amount, should you not be able to produce twice that amount if you buy another identical factory?

We assume that each commodity is produced by at least one process. Equivalently, labor can be considered the only nonproducible commodity. We assume also that no process operates free of labor. We also assume that for every commodity there is a process that can produce only that commodity and that every process that produces more than one commodity does not yield any savings of commodity inputs over the most efficient processes that produce those commodities individually. This assumption is a weakening of another standard economic assumption, that of *single production*, which requires that no process produce more than one commodity. Although we do not go as far as to allow joint production to make more efficient use of commodity inputs, we do allow it do make better use of labor. We will call this the asuumption of *joint production without commodity savings*.

Given a choice of prices, a wage, and process utilization, each process incurs a *cost* equal to the sum of the quantity of each input commodity times its price plus the quantity of labor employed times its wage. If we increase the utilization of that process, we increase its cost. In fact, the cost increases linearly since prices and wages do not vary with process utilization. Thus the *marginal cost*, the derivative with respect to process utilization of the cost of any particular process is constant. Similarly, given a choice of prices and a wage, for each process we can calculate the *marginal revenue* and *marginal profit* associated with running that process, where *revenue* is the price of the output times the quantity produced, and the *profit* is the difference between revenue and cost. The maximum rate of profit that we assume induces a maximum profit and thus a maximum marginal profit. If the marginal profit is strictly less than the maximum marginal profit, then we say that that process incurs *extra costs*. If the opposite strict inequality is true, then that process yields *extra profits*. Consider what happens if a process yields extra profits. By the assumption of constant returns to scale, the revenue from that process may exceed the sum of its profit and cost by any amount if the process is run at sufficiently high intensity. That is, any level of wealth can be achieved by the economy. In order to avoid this absurd outcome, we assume that no process yields extra profits.

So far we have not mentioned any specific units. In particular, without a monetary unit, any choice of prices can only indicate relative prices, the price of each commodity in terms of some amount of the others. It is convenient to specify a standard of value, called a *numeraire* in the language of economics. Specifically, at any time, let the numeraire consist of a particular bundle of commodities, and let us suppose that the prices are to be scaled in such a way that the total value of the numeraire is one monetary unit. To make this definition more concrete, note that the total value of the numeraire is the sum of the values of all the commodities in numeraire and that the value of a quantity of any one commodity is the product of its price and that quantity.

In a model such as this one, a choice of prices, a wage, and capital use that achieves exactly the maximum rate of profit is called an equilibrium point of the economy. The goal of economic equilibrium models is to show that there is such a stable point, that in the economy, prices, wages and industrial production are not random but are subject to governing forces, guided, in Adam Smith's phrase, by an invisible hand [9]. In our model the particular equilibrium concept we use is that of the *long-period solution.* This means that for each time, we require positive wages, nonnegative prices, and nonnegative utilization of the processes such that the output of each commodity is positive, no process yields extra profits, and any process that incurs extra costs is not used.

Kurz and Salvadori [8] show that, for each value of time, this model, under the assumption of no joint production, has a long period solution if, and only if, a certain dual pair of linear programs is consistent. Hasfura-Buenaga, Holder, and Stuart [4] extend this result to the model under joint production without commodity savings. We keep the assumption of joint production without commodity savings but examine the more realistic situation where labor is allowed to be heterogeneous—i.e. where there are q types, or sources, of labor, and we are to find a positive wage for each type.

We show that for each value of time and for each choice of wages, the exisitence of a long period solution is once again equivalent to the consistency of a dual pair of linear programs. In particular, the model does not determine unique prices, wages, and process intensities but rather shows the existence of prices and process intensities for all positive wages. It should be noted that even the case of a single homogenous labor source does not yield a unique equilibrium. But with many types of labor, we find that the relative wages of the various types are completely undetermined, and this is exactly the problem of multiple objective mathematical programming. So we see that introducing multiple labor sources makes the model difficult to interepret economically. However, Theorem 2.5 shows that after some time, the collection of processes that can be run in a long period solution for some choice of wages stabilizes.

The assumption of constant returns to scale means that for every time t, every process i, and every commodity j, there is a real number $a^i_j(t)$ equal to the number of units of commodity j required to run process i at unit intensity. Similarly let $b^i_j(t)$ be the number of units of commodity j yielded by a unit of process i, and $l^i_k(t)$ be the number of units of labor type k required to use process i at unit intensity. With multiple labor sources, the assumption that no process operates free of labor is taken to mean that no process can be run without some labor of some type. Then for each process i there exists some labor type k such that $l^i_k(t)$ is positive. For each time t, let $r(t)$ be the rate of maximum profit associated with the economy at time t and $d(t)$ be a vector of commodities representing the numeraire chosen for time t. For every time t, we construct the matrices $A(t)$, $B(t)$, and $L(t)$ with entries $a^i_j(t)$, $b^i_j(t)$, $l^i_k(t)$ respectively, the rows being indexed by i and the columns by j or k. In general we represent a choice of prices, wages, and utilization of processes by vectors p,

w, x, respectively.

Suppose we choose particular values for prices, wages, and utilization of processes. Then for any process i, $x_i(Bp)_i$ is equal to the revenue from the sale of the commodity produced by that process, so the marginal revenue of process i is $(B(t)p)_i$. Likewise $x_i(A(t)p + wL(t))_i$ is the cost of the commodities and labor required by process i, so the marginal cost of process i is $(A(t)p+wL(t))_i$. Lastly $x_i(r(t)A(t)p)_i$ is the profit yielded by process i, so the marginal profit of process i is $(r(t)A(t)p)_i$. Then since no process yields extra profits we have that $(B(t)p)_i \leq ((1 + r(t))A(t)p + wL(t))_i$. Thus

$$[B(t) - (1 + r(t))A(t)]p \leq wL(t).$$

Furthermore, if x, p and w are long period solutions at time t, then since no process with extra costs is used, the actual profit must equal the maximum possible profit, so

$$x^T[B(t) - (1 + r(t))A(t)]p = x^T L(t)w.$$

Since each commodity is to be produced,

$$x^T B(t) > 0.$$

Also, in order that prices and wages reflect the appropriate numeraire, we require that

$$d(t)^T p = 1.$$

Thus the economic model is to find x, p and w for each $t \geq 0$ such that

$$\left.\begin{array}{rcl} [B(t) - (1 + r(t))A(t)]p & \leq & L(t)w \\ x^T[B(t) - (1 + r(t))A(t)]p & = & x^T L(t)w \\ x^T B(t) & > & 0 \\ d(t)^T p & = & 1 \\ x, p & \geq & 0 \\ w & > & 0. \end{array}\right\} \quad (13)$$

For each t and positive w, (13) is equivalent to that described by Hasfura-Buenaga, Holder, and Stuart [4]. They prove that for each t and positive w the model is equivalent to the primal/dual pair of linear programs

$$\min\{w^T L(t)^T x : x^T[B(t) - (1 + r(t))A(t)] \geq d(t)^T, x \geq 0\}, \quad (14)$$

$$\max\{d(t)^T y : [B(t) - (1 + r(t))A(t)]y \leq L(t)w, y \geq 0\}. \quad (15)$$

If x^* and y^* are optimal for (14) and (15), then $x = x^*$ and $p = (1/d^T y^*)y^*$ are long-period solutions to the model.[1] Thus for each w we have essentially a separate model with a separate solution set. This multiplicity of solutions,

[1] This characterization suggests a particularly satisfying interpretation of the model. If we view d as a vector of final demands for the commodities, then (14) seeks to minimize the total labor cost while satisfying the demand for each good.

combined with the fact that w is not determined even up to a scalar multiple, makes the model economically unsatisfactory.

However, we can analyze this model by noting that its equivalence to the primal/dual pair of linear programs (14) and (15) allows us to apply Theorem 2.5. Carefully making Assumption 1, we find that the collection of processes that can be run in a long-period solution for some choice of wages stabilizes. Specifically, the following theorem is immediate.

Theorem 3.1. *Under Assumption 1, where matrix 12 has the form*

$$\begin{bmatrix} B^T(t)-(1+r(t))A^T(t) & -I & 0 & -d(t) & 0 \\ 0 & 0 & B(t)-(1+r(t))A(t) & 0 & -L^T(t) \\ 0 & 0 & -I & 0 & 0 \end{bmatrix},$$

after some time the collection of processes that can be run in a long period solution for some choice of wages stabilizes.

To evaluate this assumption economically requires further research.

We have shown that this economic model is difficult to interepret, because of the nature of multiple objective mathematical programs. Asymptotic analysis, however, gives some insights that would otherwise be missed.

4 Convexity Results

In this section, we consider a fixed set of weights that yield a particular single objective optimal partition. We are concerned with the stabilization time of the optimal partition as a function of the individual weights. Originally, we attempted to prove Theorem 2.5 by showing that for each set of weights, the time required for the single objective optimal paritition to stabilize was bounded. Though we were unable to prove Theorem 2.5 using this technique, we were able to show that the sets of weights are convex, and that the stabilization time is quasi-convex function of the weights.

Recall that a vector x is pareto optimal at t if there exists a positive weight w such that x minimizes $LP(w,t)$. We denote the set of all positive weights by W. For $w \in W$, let $(B_w(t)|N_w(t))$ be the optimal parition of $LP(w,t)$. Then, we define

$$T(w) = \inf\{T : \forall t > T,\ (B_w(t)|N_w(t)) = (B_w(T)|N_w(T))\}.$$

This function maps a particular weight w to the time when the optimal paritition for $LP(w,t)$ stabilizes. In [5], it is shown that for any fixed w, the optimal partition of $LP(w,t)$ stabilizes, so $T(w)$ is guaranteed to exist for all w.

For every two-set partition $(B^i|N^i)$ of the set $\{1,...,n\}$, let

$$V^i = \{w \in W : (B^i|N^i) \text{ is the asymptotic optimal partition of } LP(w,t)\}.$$

These sets partition W into regions that yield the same asymptotic optimal partition. With these definitions in place, we proceed to show the convexity of each V^i and the quasi-convexity of $T(w)$.

Theorem 4.1. *Let $(B^i|N^i)$ be a two-set partition of $\{1,\ldots,n\}$. Then V^i is convex and T is a quasi-convex function on V^i.*

Proof. Let $\{w^1, w^2, \cdots, w^p\} \subseteq V^i$. Let $t \geq \max\{T(w^j) : j = 1,\ldots,p\}$. Then, for $j = 1,\ldots,p$, there exist x, y^j, and s^j such that:

$$\begin{aligned} A_{B^i}(t)x_{B^i} &= b(t),\ x_{B^i} > 0 && (16)\\ A^T_{N^i}(t)y^j + s^j_{N^i} &= C^T_{N^i}(t)w^j,\ s^j_{N^i} > 0, \text{ and} && (17)\\ A^T_{B^i}(t)y^j &= C^T_{B^i}(t)w^j. && (18) \end{aligned}$$

Let $w = \sum_j \alpha_j w^j$, where $\sum_j \alpha_j = 1$. Let $y = \sum_j \alpha_j y^j$ and $s = \sum_j \alpha_j s^j$. We now show that w, y, and s satisfy the above constraints, which implies that $(B^i|N^i)$ is the optimal partiton for $LP(w,t)$. Since (16) doesn't depend on w, it is satisfied for any w. For (17),

$$\begin{aligned} A^T_{N^i}(t)y + s_{N^i} &= \sum_j \alpha_j A^T_{N^i}(t)y^j + s^j_{N^i}\\ &= \sum_j \alpha_j C^T_{N^i}(t)w^j\\ &= C^T_{N^i}(t)w. \end{aligned}$$

Furthermore, $s_{N^i} = \sum_j \alpha_j s^j_{N^i} > 0$, so constraint (17) is satisfied.

For, (18), we have that

$$\begin{aligned} A^T_{B^i}(t)y &= \sum_j \alpha_j A^T_{B^i}(t)y^j\\ &= \sum_j \alpha_j C^T_{B^i}(t)w^j\\ &= C^T_{B^i}(t)w. \end{aligned}$$

Thus, constraint (18) is satisfied.

Since it satisfies the three necessary and sufficient conditions for optimality, $(B^i|N^i)$ is the optimal partition for $LP(w,t)$. Because w is an arbitrary convex combination of elements from V^i, we have shown that V^i is convex.

Additionally, the three sufficient and necessary conditions hold for any $t \geq \max\{T(w^j) : j = 1,\ldots,p\}$, so $T(w) = \max\{T(w^j) : j = 1,\ldots,p\}$, which implies that T is a quasi-convex function. □

The function T is quasi-convex on each set V^i, so if T is unbounded, this unboundedness can only occur at a boundary of one of the V^i sets. In the proof of Theorem 2.5, where we show that any subpartition of $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t))$ remains a sub-partition for all sufficiently large t, we chose a possibly different w for each t.

If we could establish conditions under which T is bounded, then we would have a slightly stronger result; namely, each sub-partition of $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t))$ remains a sub-partition for $t > T$ *and* the set, V^i, that yields a particular sub-partition would remain constant after for $t > T$. However, this method of proof would require stronger assumption than the one we give in Section 2, since there are dynamic multiple objective linear programs that satisfy the premises of Theorem 2.5 but for which T is not bounded.

An interesting avenue of further research would be to characterize $\dim(V^i)$ in terms of the parameters of the multiple objective linear program. Note that since every $w \in V^i$ is required to be strictly positive, V^i is not a vector space. However, each V^i is contained in an affine space. Thus, in particular, there is an affine space of least dimension containing V^i, so the dimension of V^i is well-defined.

5 Conclusion

We have shown that under relatively weak conditions, the multiple objective optimal partition stabilizes. We then demonstrated the usefulness of this result by using it to analyze an economic equilibrium model with multiple labor sources. Finally, we proved the convexity of the set of weights that yield the same optimal paritition

All research for this paper was conducted at Trinity University in the Research Experience for Undergraduates program. We wish to acknowledge the help of Allen Holder, who provided both broad inspiration and mathematical assistance.

References

[1] Scott Campbell and Carl Meyer Jr. *Generalized Inverses of Linear Transformations.* Pitman Publishers, Inc., Boston, 1979.

[2] Matthias Ehrgott. *Multicriteria optimization.* Springer-Verlag, Berlin, 2000.

[3] A. Goldman and A. Tucker. Theory of linear programming. In H. Kuhn and A. Tucker, editors, *Linear Inequalities and Related Systems*, volume 38, pages 53–97, Princeton, 1956. Princeton University Text.

[4] Julio-Roberto Hasfura-Buenaga, Allen Holder, and Jeffrey Stuart. The asymptotic optimal partition and extensions of the nonsubstitution theorem. Technical report, Trinity University, May 2002.

[5] Allen Holder. Partioning multiple objective solutions with applications in radiotherapy designs. Technical report, Trinity University, May 2001.

[6] Robert Jeroslow. Asymptotic linear programming. *Operations Research*, 1972.

[7] Robert Jeroslow. Linear programs dependent on a single parameter. *Discrete Mathematics*, 1973.

[8] Heinz D. Kurz and Neri Salvadori. *Theory of Production.* Cambridge University Press, New York, 1995.

[9] Adam Smith. *The Wealth of Nations.* Alfred A. Knopf, Inc., New York, 1991.