Period Doubling Routes to Chaos in a Family of Unimodal Maps

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Definition 1. *Given topological spaces X and Y and a map $f : X \to Y$, f is a* local homeomorphism *at $x \in X$ if f is continuous at x and f^{-1} is continuous at $f(x)$ (in particular, f^{-1} exists in a neighborhood of $f(x)$).*

Lemma 1. *Given a continuous map $f : \mathbb{R} \to \mathbb{R}$ with a periodic point $x_0 = f^2(x_0)$, such that f is a local homeomorphism at x_0, then x_0 is an attracting (repelling) fixed point of $f^2 \iff \{x_0, f(x_0)\}$ is an attracting (repelling) period 2 orbit of f*

Proof. $\Leftarrow$ is obvious. To prove $\Rightarrow$, first consider the case where f^2 is attracting. Let U be an interval neighborhood of x_0 on which

$$\forall(x \in U) \lim_{n\to\infty} f^{2n}(x) = x_0.$$

Set $V = f(U)$. Since f is homeomorphic at x_0, we may choose U such that V is an open set. Then $\forall(y \in V)\ \exists(x \in U)\ f(x) = y$. So

$$\forall(y \in V) \lim_{n\to\infty} f^{2n}(y) = \lim_{n\to\infty} f^{2n+1}(x) = f\big(\lim_{n\to\infty} f^{2n}(x)\big) = f(x_0)$$

by continuity of f. It follows that $\{x_0, f(x_0)\}$ is a stable orbit of f.
We now consider the repelling case. Let U be an interval neighborhood of x_0 on which f^2 is repelling. Set $V = f(U) \cap f^{-1}(U)$. Again, we may choose U such that V is an open set. Thus for any $y \in V$ there exists $x \in U$ with $f(x) \in V$ and $f^2(x) \in U$. It is easy to show that f^2 is order preserving on U, meaning

$$y_1 < y_2 < x_0 < y_3 < y_4 \Rightarrow f(y_1) < f(y_2) < f(x_0) < f(y_3) < f(y_4)$$

It follows immediately that if f^2 is repelling on U, it is also repelling on V, and thus $\{x, f(x)\}$ is a repelling periodic orbit.

□

We state the following well-known results without proof. Given a sufficiently smooth function f such that x_0 is a fixed point and the following criteria hold:

$$f'(x_0) = 1,\ f^{(2)}(x_0) = f^{(3)}(x_0) = \ldots = f^{(k)}(x_0) = 0 \text{ and } f^{(k+1)}(x_0) \neq 0$$

we have the following:

1. When $k+1$ is even,there exists a neighborhood (a, b) of x_0 such that:
   (a) If $f^{(k+1)}(x_0) > 0$ then x_0 is attracting in $(a, x_0]$ and repelling in (x_0, b)
   (b) If $f^{(k+1)}(x_0) < 0$ then x_0 is repelling in (a, x_0) and attracting in $[x_0, b)$
2. When $k+1$ is odd, there exists a neighborhood U of x_0 such that:
   (a) If $f^{(k+1)}(x_0) > 0$ then x_0 is a repelling fixed point
   (b) If $f^{(k+1)}(x_0) < 0$ then x_0 is an attracting fixed point

We now consider the more difficult case of $f(x_0) = x_0,\ f'(x_0) = -1$. Thus, by Taylor's Theorem,

$$f(x_0 + x) = x_0 - x + ax^{k+1} + r(x)$$

where $a \neq 0$ and $r(x)$ is $o(|x^{k+1}|)$ as $x \to 0$.
Then

$$\begin{aligned} f^2(x_0 + x) &= x_0 - (-x + ax^{k+1} + r(x)) + a(-x + ax^{k+1} + r(x))^{k+1} + r \circ f(x) \\ &= x_0 + x - ax^{k+1} + a(-x)^{k+1} + q(x) \end{aligned}$$

for some error term q.

Lemma 2. *In the above computation, the error term $q(x)$ is $o(|x^{k+1}|)$ as $x \to 0$.*

Proof. This is a matter of checking that all terms in the expansion of f^2 included in q are $o(|x^{k+1}|)$ as $x \to 0$. Specifically, it suffices to show that
1) $\forall (n \geq 0)\ x^n \cdot r(x)$ is $o(|x^{k+1}|)$ as $x \to 0$
2) $\forall (n > 0)\ (r(x))^n$ is $o(|x^{k+1}|)$ as $x \to 0$
3) $r \circ f(x)$ is $o(|x^{k+1}|)$ as $x \to 0$

Cases 1 and 2 are straightforward. For case 3, we know that for any $\epsilon > 0$ there is a neighborhood U of x_0 on which $|f(x)| \leq (1+\epsilon)|x|$. Defining $y = f(x)$, it is clear that $|y| < |x|$ and so $y \to 0$ as $x \to 0$. Moreover, by the Inverse Function Theorem, we may choose U such that f is invertible on U. So

$$\lim_{x \to 0} \frac{|r \circ f(x)|}{|x^{k+1}|} = \lim_{y \to 0} \frac{|r(y)|}{|(f^{-1}(y))^{k+1}|} \leq \lim_{y \to 0} \frac{|r(y)|}{|y^{k+1}|} = 0$$

and we have proven case 3. □

We can now say something about the local behavior of $g = f^2$. If $k+1$ is odd, then we have

$$g(x) = x - 2ax^{k+1} + q(x)$$

and we know that g is asymptotically stable (unstable) if $a > 0$ ($a < 0$). By Lemma 1, this implies that f has a corresponding stable (unstable) period-2 orbit.
In the case where $k+1$ is even,

$$g(x) = x + q(x)$$

and the situation is undetermined without further knowledge of the behavior of the error term r of f. For example, if we take $r(x) = bx^{k+2}$ with $b \neq 0$, then we have (notice that $k+2$ is odd)

$$g(x) = x - 2bx^{k+2} - a^2x^{2k+1} + q(x)$$

where q is $o(|x^{k+2}|)$ as $x \to 0$. In the case $k = 1$, we have

$$g(x) = x + \frac{1}{3}(S(f))(x) + q(x)$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\frac{(f'')^2}{(f')^2}$$

is called the *Schwartzian derivative* of f. As we will see, the Schwarzian derivative is a useful tool in the analysis of periodic behavior.

Now consider a continuous map f from a compact interval $I \subset \mathbb{R}$ to itself. Suppose f has finitely many periodic orbits, and the period of these orbits is no larger than $K = 2^N$ for some N. Then by Sarkovsky's Theorem, f has periodic orbits of all periods 2^i for $i \in \{1, 2, .., N\}$, and no other periods. Define $F = f^K$. Then F has finitely many fixed points, and no periodic orbits of period greater than 1.

Definition 2. *f is said to be* turbulent *if there exist compact subintervals J, K of I with at most one point in common such that*

$$J \cup K \subset f(J) \cap f(K)$$

Definition 3. *For a point $x \in I$, the orbit of x under f is* alternating *if for all k even and j odd, $f^k(x) < f^j(x)$ or if the same holds for all k odd and j even. If the orbit of x is alternating, we make the following definitions:*

$$U = \{f^n(x) : f^{n+1}(x) > f^n(x)\}$$
$$D = \{f^n(x) : f^{n+1}(x) < f^n(x)\}$$

Clearly, U contains exactly the odd or exactly the even iterates of the orbit, and D contains the complement.

The following two results are proven in [1].

Lemma 3. *If f is turbulent, then f has periodic points of all periods.*

Thus F is not turbulent, nor are any of its higher iterates.

Theorem 1. *If f^2 is not turbulent and for some $n > 1$, $f^n(x) \leq x < f(x)$ or $f(x) < x \leq f^n(x)$, then the orbit of x is alternating.*

We also have the following.

Lemma 4. *Consider a continuous map $g : I \to I$ with no periodic orbits except for fixed points, and any $x \in I$. If there is no n such that*

$$g^n(x) \leq x < g(x) \text{ or } g(x) < x \leq g^n(x)$$

then the orbit of x under g converges in I.

Proof. For brevity, define $x_m = g^m(x)$. If $\{x_m\}$ is eventually monotonic, then it converges. Otherwise there is a least 'turning index' k_1 such that for $m < k_1$, $x_m < x_{m+1}$ and $x_{k_1+1} < x_{k_1}$ or such that the same holds with the inequalities reversed (depending on whether the trajectory of x is initially increasing or decreasing). Similarly, for $\{x_m\}$, $m > k_1$ there must be another least turning index k_2, and so on, giving an infinite sequence $\{x_{k_n}\}$, $n > 0$, of the "turning points" of the orbit of x. By our hypothesis, we must have

$$x_{k_2} < x_{k_4} < \ldots < x_{k_{2i}} < \ldots < x_{k_{2i+1}} < \ldots < x_{k_3} < x_{k_1}$$

or the same arrangement with the inequalities reversed. We will assume the above ordering without loss of generality. Let $y = \lim_{i\to\infty} x_{k_{2i}}$ and $z = \lim_{i\to\infty} x_{k_{2i+1}}$. If $y \neq z$ then we must have $\{y,z\}$ a periodic orbit, by the following. First of all, there can be no $x_n \in (y, z)$ since otherwise for all $i \geq 0$,

$$x_{k_{2i}} < x_n < x_{n+1} \text{ or } x_{n+1} < x_n < x_{k_{2i}+1}$$

and for $k_{2i} > n+1$, our hypothesis is violated. If $f(y) < z$, then for suffiicently large i, $x_{k_{2i}+1} < z$, contradicting what we've just shown. If $f(y) > z$, let U and V be disjoint interval neighborhoods of z and $f(y)$, respectively. Choose $x_{k_{2i}+1} \in U$. Then there exists an $x_{k_{2j}+1}$ with $j > i$ such that $x_{k_{2j}+2} \in V$. But then

$$x_{k_{2i}+2} < x_{k_{2i}+1} < x_{k_{2j}+2}$$

contradicting our hypothesis. So we must have $f(y) = z$ and by the same argument $f(z) = y$. But of course this contradicts our assumption that F has no periodic orbits besides fixed points, and so we must have $x_n \to y = z$. □

Lemma 5. *Given an alternating orbit of f with associated sets U and D, there is a fixed point y "between" U and D, meaning*

$$\forall(x \in U)\ \forall(z \in D)\ x < y < z$$

Proof. Let $a = sup(U)$ and $b = inf(D)$. Then by continuity of f, $f(a) \geq b$ and $f(b) \leq a$. So $[a, b] \subset f([a, b])$, and the result follows from the intermediate value theorem. □

We can now prove

Theorem 2. *Given a continuous map $f : I \to I$ with only finitely many periodic points, all orbits of f converge to periodic orbits.*

Proof. It is sufficient to prove that trajectories of F converge to fixed points. The idea is to show that if this is not the case, then F must have infinitely many fixed points, contradicting our hypotheses. Consider any $x_1 \in I$. If the orbit of x_1 converges, we are done. Otherwise, by Lemma 4, there is some x' in the orbit of x_1 satisfying the hypotheses of Theorem 1, and we may partition the orbit of x' into sets U_1 and D_1. Using Lemma 5, let y_1 be a fixed point between U_1 and D_1. Choose some $x_2 \in U_1$ and set $F_1 = F^2$. Repeat this procedure on x_2 with F_1 to subdivide U_1 (with the possible exception of finitely many points) into U_2 and D_2, find a fixed point y_2 between these subsets, and so on, with $F_n = F_{n-1}^2 = F^{2^n}$. Since none of the F_n have perioic orbits, the sets U_n and D_n have an infinite number of points, and this process will only terminate if $x_n \in U_{n-1}$ has an oribt under F_n which converges to a fixed point. But this means $f^{k2^n}(x)$ converges as $k \to \infty$, so the orbit of x converges to a periodic orbit. Moreover, the y_n are distinct, by the following.

Given y_i, y_j, with $i < j$, any point $d_j \in D_j$ satisfies $y_j < d_j < y_i$. So the algorithm must eventually terminate, or we would have an infinite set $\{y_i\}$, $i = 1, 2, \ldots$ of periodic points of F. $\square$

This gives us information about how individual orbits behave, but does not tell us whether nearby points converge to the *same* periodic orbit. If we could show that a point x is eventually mapped into a stable neighborhood (or half-neighborhood) U of a periodic orbit $\mathcal{O}$, then we would know that all points in some neighborhood of x converge to the same orbit, as the open set

$$\bigcup_{n \geq 0} f^{-n}(U)$$

would converge to $\mathcal{O}$. So to complete our characterization of the orbits of a map f with the above hypotheses, we must also examine the local behavior at the fixed points. To be precise, we state the following.

Definition 4. 1) *A fixed point x is* stable *if there exists a neighborhood U of x such that for all $y \in U$, $f(y) \in U$ and $f^k(y) \to x$ as $k \to \infty$.*
2) *x is* left (right) stable *if there exists a left (right) neighborhood $U = (a, x)$ ($U = (x, a)$) of x such that for all $y \in U$, $f(y) \in U$ and $f^k(y) \to x$ as $k \to \infty$.*
3) *x is* repelling *if there exists a neighborhood U of x such that for all $y \in U$, if $f(y) \in U$ then $|f(y) - x| > |y - x|$.*
4) *x is* left (right) repelling *if there exists a left (right) neighborhood $U = (a, x)$ ($U = (x, a)$) of x such that for all $y \in U$, if $f(y) \in U$ then $|f(y) - x| > |y - x|$.*
5) *x is* semi-stable *if x is left stable and right repelling or left repelling and right stable.*
6) *A periodic orbit $\mathcal{O}$ with period n is stable, repelling, or semi stable if $x \in \mathcal{O}$ is stable, repelling, or semi-stable under f^n.*

Remarks:
1) Not all local behavior is characterized by one of the above definitions. For example, consider $f(x) = x$ or $f(x) = |x| sin(x^{-1})$.
2) The definition of repelling makes use of the metric properties of $\mathcal{R}$.
3) In item (6), $x \in \mathcal{O}$ is stable, repelling, or semi-stable under f^p whenever n divides p.

We make the further assumption that f has a unique turning point.

Definition 5. *A map $f : I \to I$ is* unimodal *if there exists a unique $c \in I$ such that $f(x)$ is strictly monotonically increasing for $x < c$ and strictly monotonically decreasing for $x > c$.*

Lemma 6. *For a unimodal map $f : I \to I$ with only finitely many periodic points, and a periodic point x of f, exactly one of the following holds for the orbit $\mathcal{O}$ of x:*
1) $\mathcal{O}$ is stable
2) $\mathcal{O}$ is repelling
3) $\mathcal{O}$ is semi-stable

Proof. We will again work with $F = f^K$, where K is the maximal periodicity of orbits of f. Since F has finitely many fixed points, we may consider a neighborhood (a, b) of x in which F has no fixed points besides x. Thus $F > id$ or $F < id$ on (a,x) and on (x,b). Furthermore, it is clear from the unimodality of f that the set $C = \{c\} \cup f^{-1}(c) \cup \ldots \cup f^{-K}(c)$ is finite, so either $x \in C$, in which case x is a turning point of F, or F is strictly monotonically increasing or decreasing in a neighborhood of x. In either case, we may choose (a, b) such that $F > x$ or $F < x$ on (a,x) and on (x,b). Then in (a, b) we may list the various possibilities as to the position of F.

1. On (a, x):
   (a) $F > x$
   (b) $F < x$ and $F > id$
   (c) $F < id$
2. On (x, b)
   (a) $F < x$
   (b) $F > x$ and $F < id$
   (c) $F > id$

One can easily verify the following:
(1a,2b) or (1b,2b) or (1b,2a) $\Rightarrow$ Case 1
(1a,2c) or (1c,2c) or (1c,2a) $\Rightarrow$ Case 2
(1b,2c) or (1c,2b) $\Rightarrow$ Case 3

For the alternating orbits (1a,2a), we consider F^2, which satisfies all of the same conditions in as F in a suitably restricted neighborhood (a', b') of

x. Moreover, $F^2 > x$ on (a', x) and $F^2 < x$ on (x, b'), so either (1b,2b) or (1c,2c) for F^2, so F^2 is stable or repelling at x, and combined with the proof of Lemma 1 we know that the same holds for F. But then the orbits of F in V are converging or repelling alternating orbits, so Case 3 holds. By Remark 3 above, the result holds for any periodic point with period dividing K. And by Sarkovsky's Theorem, these are all the periodic orbits of f. □

Lemma 7. *Unimodal maps defined on compact intervals with finitely many periodic points must have at least one periodic orbit which is stable or semi-stable.*

Proof. This follows immediately from Theorem 2 and Lemma 6. □

We now turn to a specific one parameter family of unimodal maps, the so-called Ricker maps

$$R_p(x) = xe^{(p-x)}$$

For $p \leq 1$, the dynamics are trival: 0 is a globally attracting fixed point. For $p > 1$, we will restrict our attention to the invariant interval $I = [0, R_p(1)]$. Notice that $x_c = 1$ is the unique critical point of R_p for any $p \in \mathbb{R}^+$. Also, a direct calculation will show that $S(R_p) < 0$ on I. The starting point for our discussion will be Singer's theorem, as formulated in [3].

Theorem 3. *(Singer) Consider a piecewise monotone C^3 map f from a closed interval I to itself with local extrema $c_0 < \ldots < c_l$ (where c_0 and c_l are the endpoints of I). Furthermore, assume $S(f) < 0$ on I. Then f has at most $l+1$ periodic orbits $\mathcal{O}$ which are stable or nearly stable in the sense that*

$$-1 \leq (f^p)'(x) \leq 1$$

where $x \in \mathcal{O}$ and p is the period of $\mathcal{O}$. Any such orbit can be obtained as the limit of the successive images $f^k(c_i)$ as $k \to \infty$, where $0 \leq i \leq l$.

Certainly the R_p satisfy these hypotheses. In this setting, $l = 2$. Since $c_0 = 0$ is an unstable fixed point under R_p for $p > 1$, and $c_2 = f(c_1)$, we conclude that the Rickers maps have at most one stable or nearly stable orbit. And in the case of differentiable maps, semi-stable implies nearly stable, so R_p has at most one stable or semi-stable orbit. Define

$$P = \{p : R_p \text{ has finitely many periodic orbits}\}$$

The following result is taken from [2].

Lemma 8. *Suppose f has finitely many critical points and S(f)¡0. Then f has only finitely many periodic points of period m for any $m \in \mathbb{N}$.*

It follows immediately that

$P = \{p :$ the maximum periodicity of any orbit of R_p is 2^i, for some $i \in \mathbb{N}\}$

By Lemma 7, for $p \in P$, R_p has at least one stable or semi-stable periodic orbit, so for such p, R_p has exactly one such orbit. We would like to know what the period of this stable orbit is for a given p, and how transitions to different stable periodicities occur. The idea is to follow a stable periodic point in (p, x) phase space until stablility is lost. This can occur in a variety of ways. The following result is essential in limiting the types of bifurcations which may occur ([1], [4]).

Theorem 4. *(Block and Hart) Let $f \in C^1[I, I]$ and $x_0 \in I$. If there exists a sequence $\{f_n\} \subset C^1[I, I]$ such that $|f_n - f|_{C^1} \to 0$ as $n \to \infty$ and if each f_n has a periodic point x_n of period k with $x_n \to x_0$ as $n \to \infty$, then x_0 is a periodic point of f with period k or $k/2$.*
Moreover, if x_0 has period $k/2$, then $(f^{k/2})'(x_0) = -1$.

Definition 6. *Given topological spaces X and Y, a function $f : X \to \mathcal{P}(Y)$ (where $\mathcal{P}(Y)$ is the power set of Y) is* limit point continuous *if, for each $x_0 \in X$, and any sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$,*

$$\bigcap_n \overline{\bigcup_{i>n} f(x_i)} = f(x_0).$$

We define the functions $\gamma_k : P \to \mathcal{P}(\mathbb{R}^+)$, $k \in \mathcal{N}$ by

$$\gamma_k(p) = \{x \in I : x \text{ is a stable or semi-stable fixed point of } R_p^{2^k}\}$$

which is equivalent to

$\gamma_k(p) = \cup\{\mathcal{O} : \mathcal{O}$ is a (semi-) stable periodic orbit of R_p with period dividing $2^k\}$.

¿From our earlier considerations, if such an $\mathcal{O}$ exists for a given k, then it is unique.

Next define

$$S_k = \{p \in P : \gamma_k(p) \neq \emptyset\}$$
$$T_0 = S_0$$
$$\text{for } k > 0,\ T_k = S_k - S_{k-1}$$

Lemma 9. *For all $k \in \mathbb{N}$, γ_k is limit point continuous in S_k.*

Proof. Given a sequence $\{p_n\} \subset S_k$ with $p_n \to p \in P$, let $\mathcal{O}_n = \gamma_k(p_n)$ be the (semi-) stable orbit of R_{p_n}. That $\bigcap_n \overline{\bigcup_{i>n} \mathcal{O}_n} \neq \emptyset$ is immediate from the limit point compactness of I. Choose any $z \in \bigcap_n \overline{\bigcup_{i>n} \mathcal{O}_n}$. Then there is a subsequence $\{p_{n_j}\} \subset \{p_n\}$, $j \in \mathbb{N}$, and a sequence $\{z_j\}$ with $z_j \in \mathcal{O}_{n_j}$ such that $z_j \to z$ as $j \to \infty$. By C^1 continuity in p and x, we have

$$\begin{aligned} &R_p^{2^k}(z) = \lim_{j\to\infty} R_{p_j}^{2^k}(z_j) = \lim_{j\to\infty} z_j = z \\ \text{and} \quad &(R_p^{2^k})'(z) = \lim_{j\to\infty} (R_{p_j}^{2^k})'(z_j) \in [-1, 1] \end{aligned}$$

since for all j, $(R_{p_j}^{2^k})'(z_j) \in [-1, 1]$.
So $z \in \mathcal{O}_0 = \gamma_k(p)$, and we have $\bigcap_n \overline{\bigcup_{i>n} \mathcal{O}_n} \subset \mathcal{O}_0$. Reverse containment follows from uniqueness of $\mathcal{O}_0$.

□

In this proof, we saw that $p \in S_k$ did not need to be assumed. Hence we have

Lemma 10. *The S_k are closed (in P).*

Lemma 11. *For all $k \in \mathbb{N}$, $S_k \subset int(S_{k+1})$.*

Proof. This follows immediately from Theorem 4. □

Lemma 12. *For $p \in \partial(S_k)$, the stable orbit $\gamma_k(p)$ undergoes a right (left) period doubling bifurcation if T_{k+1} is a right (left) neighborhood of p. Both types may occur for a given p.*

Proof. We will take $x_{q,n}$ to be some element of $\gamma_k(q)$ with period 2^n (so $n \leq k$). By theorem 4, $(R_p^{2^k})'(x_{p,k}) = -1$. For $q \in T_{k+1}$, $x_{q,k}$ must be unstable, and by C^1 continuity, $(R_q^{2^k})'(x_{q,k}) < -1$, and so $(R_q^{2^{k+1}})'(x_{q,k}) > 1$. By the Implicit Function Theorem, the $x_{q,k}$ must vary continuously with q in T_{k+1}. It is an easy consequence of limit point continuity that the $x_{q,k+1}$ can be chosen such that they vary continuously with q in T_{k+1} as well. Choose a particular branch $x(q) \in \gamma_{k+1}(q)$. By uniqueness of (semi-) stable orbits, we must have $\lim_{q\to p} x(q) = z$ for some $z \in \gamma_k(p)$. Consequently,

$$\begin{aligned} &\lim_{q\to p} x(q) = z \\ &\lim_{q\to p} R_q^{2^k}(x(q)) = R_p^{2^k}(z) = z \\ &\text{and so for } 1 \leq r < 2^k, \\ &\lim_{q\to p} R_q^r(x(q)) = \lim_{q\to p} R_q^{r+2^k}(x(q)) = R_p^r(z) \end{aligned}$$

Since $(R_p^{2^k})'(x_{q,k}) < -1$ for $q \in T_{k+1}$, $R_q^r(x(q)) < x_{q,k} < R_q^{r+2^k}(x(q))$ for appropriately chosen r. This completes the proof.

□

We are almost finished, but we need to rule out the possibility that unrelated bifurcations do not result in unstable periodic orbits (the period of which might "reach ∞" before the period-doubling process we have demonstrated).

Lemma 13. *Given $p \in P$, the maximum periodicity of a periodic point of R_p is equal to the cardinality of $\gamma_k(p)$.*

Proof. The creation of an isolated period-n point x at some p requires $(R_p^n)'(x) = 1$ (we have already shown in Lemma 6 that periodic points in P are isolated). This requires $n = 2^i$ for some i and $x \in \gamma_i(p)$. So any unstable periodic orbit must (1) be present over all of P, or (2) emerge from one of the period doubling bifucations discussed above.
(1) There are no such orbits; for example, $R_{1/2}$ has one stable fixed point and no other periodic points.
(2) By Theorem 4, for a period doubling from k to $2k$, any such orbit must have periodicity equal to k or $2k$.

□

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