Bounding the derived length of Lie Algebras of a special kind

By: Dustin Ragan and Geoff Tims

Under the supervision of Dr. Thomas Keller

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1 Introduction to Lie Algebras

Definition 1.1 *Let F be a field. Then a Lie algebra is an F-vector space V endowed with a bilinear operation $[\cdot,\cdot]$, called a Lie bracket, such that*

$$\begin{aligned} [u,u] &= 0 && \forall u \in V \\ [[u,v],w]+[[v,w],u]+[[w,u],v] &= 0 && \forall u,v,w \in V \end{aligned}$$

For convenience, we will write $[[u,v],w] = [u,v,w]$. Also note that the first property is equivalent to $[u,v] = -[v,u]$, so it is called *anti-symmetry*. As proof,

$$\begin{aligned} 0 &= [u+v,u+v] \\ &= [u+v,u]+[u+v,v] \\ &= [u,u]+[v,u]+[u,v]+[v,v] \\ &= [v,u]+[u,v] \end{aligned}$$

In linear algebra we learned that it is sufficient to know the action of a linear transformation on the basis of the space in order to know the action on the entire space. The same principle applies to Lie algebras. We need only know how each pair of basis vectors acts under the Lie bracket to know how any two arbitrary vectors act. For example, consider $u = u_1e_1+u_2e_2+\cdots+u_ne_n$ and $v = v_1e_1+v_2e_2+\cdots+v_ne_n$ located in some n-dimensional Lie algebra with basis $\{e_1,e_2,\cdots,e_n\}$ (and the u_i's and v_i's are coefficients in F). Then

$$\begin{aligned} [u,v] &= [u_1e_1+u_2e_2+\cdots+u_ne_n, v_1e_1+v_2e_2+\cdots v_ne_n] \\ &= \sum_{i=1}^{n} u_i[e_i, v_1e_1+v_2e_2+\cdots v_ne_n] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_iv_j[e_i,e_j] \end{aligned}$$

We define the *structural constants* to be the coefficients of the Lie bracket of two basis vectors; that is, they are the coefficients of

$$[e_i,e_j] = C^1_{i,j}e_1 + C^2_{i,j}e_2 + C^3_{i,j}e_3 + \cdots + C^n_{i,j}e_n$$

It is possible to write the two axioms in terms of the structural constants.

$$C^k_{i,i} = 0$$

$$\sum_{m=1}^{n} \left(C^m_{i,j} C^\ell_{m,k} + C^m_{j,k} C^\ell_{m,i} + C^m_{k,i} C^\ell_{m,j} \right) = 0$$

The first is obvious, but the second requires some justification:

$$\begin{aligned}
[e_i, e_j, e_k] &= [\sum_{m=1}^{n} C^m_{i,j} e_m, e_k] \\
&= \sum_{m=1}^{n} C^m_{i,j} [e_m, e_k] \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} C^m_{i,j} C^\ell_{m,k} e_\ell
\end{aligned}$$

Since the $e_\ell, \ell = 1, 2, \ldots n$ are linearly independent, the result follows immediately. Just as we can look at the Lie bracket on a finer, more detailed level with the structural constants, we may also look on it as an operation on sets. If $U, V \leq L$ for some Lie algebra L, then we define a Lie product of the two sets as

Notice that $[U, V] = [V, U]$. More interestingly, $[U, U]$ is not necessarily 0. This is called the *derived set of* U and is written U'. As in calculus, we extend this to further derivations, so that U'' makes sense as well. For higher order derivations, we use the notation $U^{(3)}, U^{(4)}, \ldots$ etc.

We call a subalgebra U of a Lie algebra L *normal* if $[L, U] \subseteq U$. Normal subalgebras act like ideals, in that they are closed under an operation with respect to the entire set.

2 Lie Algebras Satisfying a Special Hypothesis

We now restrict ourselves to a special class of Lie algebras, defined by the following condition (and henceforth referred to as condition (*)):

(*): Let L be a Lie algebra on $\mathbb{Z}_p$, such that $L = \langle e_1, e_2, \ldots, e_n \rangle, n \leq p$. Define

$L_i = \langle e_i, e_{i+1}, \ldots, e_n \rangle$ (so that $L = L_1$). Then we say that L satisfies condition

* if

$$[L, L_i] = [e_1, L_i] = L_{i+1}, \qquad i \geq 2$$

where $[e_1, L_i] = \{[e_1, v] | v \in L_i\}$.

There are a few properties which follow immediately from this definition, which we list here:

Lemma 2.1 *Let L satisfy (*). Then $[L_i, L_j, L_k] \subseteq [L_j, L_k, L_i] + [L_k, L_i, L_j]$.*

Proof:

$$\begin{aligned}
[L_i, L_j, L_k] &= \langle [u, v, w] | u \in L_i, v \in L_j, w \in L_k \rangle \\
&= \langle -[v, w, u] - [w, u, v] | u \in L_i, v \in L_j, w \in L_k \rangle \\
&\subseteq \langle [v, w, u] | u \in L_i, v \in L_j, w \in L_k \rangle + \langle [w, u, v] | u \in L_i, v \in L_j, w \in L_k \rangle \\
&= [L_j, L_k, L_i] + [L_k, L_i, L_j]
\end{aligned}$$

□

Lemma 2.2 *Let L satisfy (*). Then L_i is normal in L.*

Proof:

This is clear because

$$[L, L_i] = L_{i+1} \subseteq L_i$$

Lemma 2.3 *Let L satisfy (*). Then $[L_i, L_j] \subseteq L_{i+j}$.*

Proof:

Let $u \in L_i, v \in L_j$. By the normality of L_i and L_j, $[e_1, u] \in L_i$ and $[e_1, v] \in L_j$. Then,

$$\begin{aligned}
[e_1, [u, v]] &= -[u, v, e_1] \\
&= -[v, e_1, u] - [e_1, u, v] \\
&= [e_1, v, u] - [e_1, u, v]
\end{aligned}$$

Thus $[e_1, [u, v]] \in [L_i, L_j]$. Since Lie multiplication by e_1 projects a vector across exactly one dimension, we must have a consecutive spanning set of vectors.

We proceed by induction on i. By definition, the assertion holds for $i = 1$. Now let $i \geq 1$ and suppose that $[L_i, L_j] \subseteq L_{i+j}, \forall j$. Then, by Lemma 2.1

$$\begin{aligned}
[L_{i+1}, L_j] &= [L_1, L_i, L_j] \\
&\subseteq [L_i, L_j, L_1] + [L_j, L_1, L_i] \\
&= [L_i, L_j, L_1] + [L_{j+1}, L_i] \\
&\subseteq L_{i+j+1}
\end{aligned}$$

as desired.

Lemma 2.4 *Let L satisfy (*). Then $[L_i, L_j] = L_k$, for some integer k.*

Proof:

Let $u \in L_i, v \in L_j$. By the normalcy of L_i and L_j, $[e_1, u] \in L_i$ and $[e_1, v] \in L_j$. Then,

$$\begin{aligned}[e_1, [u, v]] &= -[u, v, e_1] \\ &= -[v, e_1, u] - [e_1, u, v] \\ &= [e_1, v, u] - [e_1, u, v]\end{aligned}$$

Thus $[e_1, [u, v]] \in [L_i, L_j]$. Since Lie multiplication by e_1 projects a vector across exactly one dimension, we must have a consecutive spanning set of vectors, which generates some L_k.

Lemma 2.5 *Let L satisfy (*). Then $L_i' = [L_i, L_{i+1}]$.*

Proof:

Clearly $[L_i, L_i] \supseteq [L_i, L_{i+1}]$. Now let $[u, v] \in [L_i, L_i]$. Then

$$\begin{aligned}[u, v] &= [u_i e_i + u_{i+1} e_{i+1} + \cdots + u_n e_n, v_i e_i + v_{i+1} e_{i+1} + \cdots + v_n e_n] \\ &= [u_i e_i + u_{i+1} e_{i+1} + \cdots + u_n e_n, v_i e_i] \\ &\quad + [u_i e_i + u_{i+1} e_{i+1} + \cdots + u_n e_n, v_{i+1} e_{i+1} + \cdots + v_n e_n] \\ &= [u_i e_i, v_i e_i] + [u_{i+1} e_{i+1} + \cdots + u_n e_n, v_i e_i] \\ &\quad + [u_i e_i + u_{i+1} e_{i+1} + \cdots + u_n e_n, v_{i+1} e_{i+1} + \cdots + v_n e_n] \\ &= [-v_i e_i, u_{i+1} e_{i+1} + \cdots + u_n e_n] + [u_i e_i + u_{i+1} e_{i+1} + \cdots + u_n e_n, v_{i+1} e_{i+1} + \cdots + v_n e_n]\end{aligned}$$

We have written $[u, v]$ as the sum of two elements of $[L_i, L_{i+1}]$. Thus $[u, v]$ *in* $[L_i, L_{i+1}]$, and so $L_i' \subseteq [L_i, L_{i+1}]$, and the proof is complete.

3 The Standard Example

For Lie Algebras over the field $F = Z_p$, there is an example known as the standard example which was discovered by B.A. Panferov in 1980.

Definition 3.1 *Let $L =< e_1, e_2, \ldots, e_p >$ and define $[e_i, e_j]$ as $(i-j)e_{i+j}$ if $i+j/leqp$ and 0 if $i+j$. This example is called the standard example.*

The standard example is of interest because it satisfies (*) and gives an upper bound on a function which will be described in the next section.

4 Statement of Problem

Definition 4.1 *Suppose L is a Lie Algebra with n generators satisfying (*). We define a new function n(L)=$\{L'_1$,L'_2, $\ldots$,$L'_n\}$. That is, n(L) is the number of different derived Lie subalgebras of L.*

Trivially, if L'_i=0 for all i, there is one derived Lie Algebra for L. Also trivially, since there are at most n different L'_is, there can be at most n different derived Lie Algebras of L. Hence

$1 \leq n(L) \leq n$

Problem: Among all those L satisfying (*) where dl(L)=k, what is the smallest possible value n(L) that occurs? That is, we are interested in the values of the following function $f(k)$.

The function $f(k)$ can easily be bounded below and above. If L satisfies (*) and has a derived length of k, then there must be at least k different derived Lie Algebras. Further, since f(k) is just the minimum of all values of n(L) for a given derived length, then the standard example with n=2^k-1 gives an upper bound on f(k) of 2^{k-1}.

$k \leq f(k) \leq 2^{k-1}$

Conjecture: It is conjectured that the function f(k) more closely fits the upper bound 2^{k-1}. That is, $k \leq A\log(f(k)) + B$ for some A,B $\in \mathbf{R}$.

However, before attempting to prove a general conjecture, f(k) should be evaluated for small k. The bounds given on f(k) give f(1) and f(2) because the left and right hand sides are equal.
f(1)=1
f(2)=2

For f(3), it is not so easy. We have $3 \leq f(3) \leq 4$. This leaves two possibilities. To increase the lower bound, one must prove that it is not possible for any L satisfying (*) with dl(L)=3 to have n(L)=3. Instead, to decrease the upper bound, one must find an example of L satisfying (*) such that dl(L)=3 and n(L)=3. Since f(k) is the minimum of all possible values of n(L), this would show that $3 \leq f(3) \leq 3$.

It turns out that it is very easy to find an example(see the first example of section 7) of a Lie Algebra satisfying (*) with dl(L)=3 and n(L)=3. Thus
f(3)=3

Next comes f(4) where we know $4 \leq f(4) \leq 8$ from our bound on any f(k). Again there are two ways to improve these bounds. The first is to prove that it is not possible to have a Lie Algebra satisfying (*) where dl(L)=4 and n(L)=4. The second possibility is to find an example where dl(L)=4 and $n(L) \leq 7$.

5 Solving This Problem

At this point, for the ease of notation, we introduce a new operator on the elements of the Lie algebra. We define the operator d of a vector in the algebra as:

$$d(v) = [e_1, v]$$

We consider a negative power of d to be 0.

The linearity of d follows from the linearity of the Lie bracket. This definition greatly simplifies the notation of many of our results.

A second simplification we can make is to choose a particular basis that is convenient for our use. It is often particularly convenient for us to choose a basis such that $[e_1, e_i] = e_{i+1}$, and fortunately for us this is always possible. However, for a particular Lie algebra this may not be the easiest representation; but in the abstract we can represent ignore those difficulties and subsume the coefficients into a single symbol.

Lemma 5.1 *Let L be a Lie algebra satisfying $*$. Then it has some basis $\{e_1, e_2, \ldots, e_n\}$ such that*

$$[e_1, e_i] = e_{i+1}, \qquad 2 \leq i \leq n$$

Proof:

Let $\langle d_1, d_2, \ldots, d_n \rangle$ be a basis of L. Take $e_1 = d_1$ and $e_2 = d_2$. Then we may define

$$e_i = [e_1, e_{i-1}]$$

which satisfies our conclusion by construction.

Also, because of the Jacobi identity, there is a great deal of inter-relations between the Lie products of various basis vectors. Because of this, we were motivated to find a convenient basis for the Lie bracket itself, and to find a particular choice of brackets that suffice to define others. To that end,

Lemma 5.2 *Let L be a Lie algebra satisfying condition $*$. Then the $[e_i, e_{i+1}]$ bracket terms fully define the Lie algebra, for a specified dimension. Also,*

$$[e_i, e_{i+j+1}] = d\,[e_i, e_{i+j}] - [e_{i+1}, e_{i+j}]$$

and

$$[e_i, e_{i+j+1}] = \sum_{k=0}^{n-i} (-1)^k \binom{j-k}{k} d^{j-2k}\,[e_{i+k}, e_{i+k+1}]$$

Proof:

The equation follows immediately from the Jacobi identity $[e_1, e_i, e_j]+[e_i, e_j, e_1]+[e_j, e_1, e_i] = 0$, and along with the anticommutivity serves to define every $[e_i, e_j]$. For the second, we proceed by induction. The equation is clearly satisfied if $j = 0$. Now, assume the formula holds for some j. Then,

$$
\begin{aligned}
[e_i, e_{i+j+2}] &= d[e_i, e_{i+j+1}] - [e_{i+1}, e_{i+j+1}] \\
&= d\left(\sum_{k=0}^{n-i} d\,(-1)^k \binom{j-k}{k} d^{j-2k}\left[e_{i+k}, e_{i+k+1}\right]\right) \\
&\qquad - \sum_{k=0}^{n-i-1} (-1)^k \binom{j-1-k}{k} d^{j-2k-1}\left[e_{i+1+k}, e_{i+k+2}\right] \\
&= \sum_{k=0}^{n-i} (-1)^k \binom{j-k}{k} d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right] \\
&\qquad - \sum_{k=1}^{n-i} (-1)^{k-1} \binom{j-k}{k-1} d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right] \\
&= d^{j+1}[e_i, e_{i+1}] \\
&\qquad + \sum_{k=1}^{n-i} (-1)^k \binom{j-k}{k} d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right] \\
&\qquad + \sum_{k=1}^{n-i} (-1)^k \binom{j-k}{k-1} d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right] \\
&= d^{j+1}[e_i, e_{i+1}] \\
&\qquad \sum_{k=1}^{n-i} (-1)^k \left(\binom{j-k}{k} + \binom{j-k}{k-1}\right) d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right] \\
&= \sum_{k=0}^{n-i} (-1)^k \binom{j+1-k}{k} d^{j-2k+1}\left[e_{i+k}, e_{i+k+1}\right]
\end{aligned}
$$

Another technique which will be useful is a means of deriving a simpler Lie algebra from a more complex one. There are two methods which we employ. First, we may remove the final generator from a Lie algebra. Second, we may remove the second generator. The Jacobi identity and anti-commutativity are clearly still satisfied, and we have closure because no Lie produce can ever produce a vector containing e_2. Since they are both just restrictions of a function, the Lie bracket must retain all of its properties.

The next theorem is entirely non-obvious. It required looking at many examples and seeing how and where they failed or succeeded in order to see. However, the result is quite useful.

Lemma 5.3 *Let L satisfy (*). If $L_2' = L_3' = L_m$, and if L has dimension less than or equal to $2m-5$, then $L_4' \geq L_{2m-5}$.*

Proof:

Suppose that there exists some L such that $L'_2 = L'_3 = L_m$, L has dimension less than or equal to $2m-5$, and $L'_4 < L_{2m-5}$. Since there are only $2m-5$ dimensions, $L'_4 = 0$.

We observe that m must be at least 7, since $[L_3, L_4] \subseteq L_7$.

Then, since L only has $2m-5$, $L'_4 = 0$.

First, consider $[e_2, e_3, e_4]$. Clearly $[e_2, e_3] \in L_m \subseteq L_5$, so $[e_2, e_3, e_4] \in [L_5, L_4] = 0$, and so $[e_2, e_3, e_4] = 0$.

Second, consider

$$
\begin{aligned}
[e_3, e_4, e_2] &= [\sum_{k=m}^{n} C_{3,4}^{m} e_m, e_2] \\
&= -\sum_{k=m}^{n} C_{3,4}^{m} [e_2, e_m] \\
&= -\sum_{k=m}^{n} C_{3,4}^{m} \left(\sum_{j=0}^{n-2} \binom{k-3-2j}{j} d^{k-3-2j} [e_{2+j}, e_{3+j}] \right) \\
&= -C_{3,4}^{m} (d^{m-3}[e_2, e_3] + (m-4) d^{m-5} [e_3, e_4] \\
&= -(m-4) \left(C_{3,4}^{m} \right)^2 e_{2m-5}
\end{aligned}
$$

The last equality follows from the fact that since $[e_2, e_3] \in L_m, d^{m-3}[e_2, e_3] \in L_{2m-3} = 0$. A similar argument leaves only one term left of $(m-4)d^{m-5}[e_3, e_4]$.

Thirdly, consider

$$
\begin{aligned}
[e_4, e_2, e_3] &= -[e_3, [e_2, e_4]] \\
&= -[e_3, d[e_2, e_3]] \\
&= -\sum_{k=m}^{n} [e_3, e_{k+1}] \\
&= -\sum_{k=m}^{n} \left(\sum_{j=0}^{n-3} (-1)^j \binom{m-4-j}{j} d^{m-4} [e_3, e_4] \right) \\
&= 0
\end{aligned}
$$

The last equality holds from an argument like the above. By the Jacobi identity,

$$\begin{aligned}0 &= [e_2, e_3, e_4] + [e_3, e_4, e_2] + [e_4, e_2, e_3] \\ &= -(m-4)\left(C^m_{3,4}\right)^2 e_{2m-5}\end{aligned}$$

Since $m \geq 7, (m-4) \neq 0$, and this implies that $C^m_{3,4} = 0$.

Now,

$$\begin{aligned}L_m &= L'_3 \\ &= \langle [e_i, e_j] | i, j \geq 3 \rangle \\ &= \langle [e_3, e_j] | j \geq 4 \rangle \\ &= \langle [e_3, e_4], [e_3, e_5], [e_3, e_6], \ldots, [e_3, e_{2m-5}] \rangle \\ &= \left\langle [e_3, e_4], d[e_3, e_4], d^2[e_3, e_4], \ldots, d^{m-5}[e_3, e_4] \right\rangle\end{aligned}$$

This forces $C^m_{3,4} \neq 0$, a contradiction.

Theorem 5.4 *Let L satisfy $(*)$. If*

$$L'_{2+i} = L'_{3+1} = L_m$$

then

$$L'_{4+i} \supseteq L_{2m-5+i}$$

Proof:

If L has dimension less than or equal to $2m-5+i$, then the proof is trivial.

Otherwise, suppose that there existed L such that $L'_{2+i} = L'_{3+1} = L_m$ and $L'_{4+i} \not\supseteq L_{2m-5+i}$. Then let

$$M = \langle e_1, e_{2+i}, e_{3+i}, \ldots, e_{2m-5+i} \rangle$$

Then M is a Lie algebra satisfying the hypothesis of the previous lemma, but violating the conclusion. Thus M, and so L, do not exist.

This result allows us to actually generally increase the lower bound on our function, $f(k)$.

Theorem 5.5 *Let L by a Lie algebra satisfying $(*)$ such that $dl(L) = 4$. Then $n(L) \neq 4$.*

Proof: Suppose that there exists a Lie algebra L satisfying $(*)$ such that $dl(L) = 4$ and $n(L) = 4$. Then we may reduce it (by repeatedly removing e_2) to M such that

$$
\begin{aligned}
M_1' &= M_3 \\
M_2' &= M_3' = M_m \\
M_4' &= M_5' = \cdots = M_m' = M_n \\
M_{m+1}' &= 0
\end{aligned}
$$

where n is the dimension of M. Clearly $n \geq 2m+1$, but by the above theorem $n \leq 2m-5$. This is a contradiction.

Theorem 5.6 *Let L by a Lie algebra satisfying $(*)$ such that $dl(L) = k$. Then $n(L) \geq \frac{3}{2}k - 1$.*

Proof:

Let L by a Lie algebra satisfying $(*)$ such that $dl(L) = k$. Then

$$
\begin{aligned}
L_1' &= L_{m_1} \\
L_{m_1}' &= L_{m_2} \\
L_{m_2}' &= L_{m_3} \\
L_{m_3}' &= L_{m_4} \\
&\vdots \\
L_{m_k}' &= 0
\end{aligned}
$$

Consider L_{m_i-1}. Suppose that $L_{m_i-1} = L_{m_i}, i \neq k$. Then, by Theorem , there must exist some j such that $L_{m_i+1}' = L_j \neq L_{m_{i+1}}$. Thus for at most half of the L_{m_i}'s can we have $L_{m_i-1}' = L_{m_i}'$, excluding the final one. Thus $n(L) \geq \frac{3}{2}k - 1$.

6 Future of the Problem

6.1 Raising the Lower Bound

Given the above theorem, the simple part of raising the lower bound has been accomplished. But to show that $f(4) \neq 5$ requires a more difficult sort of proof. There are two possibly ways in which a Lie algebra of derived length 4 and $n(L) = 5$ could exist:

$$\begin{aligned} L'_1 &= L_3 \\ L'_2 &= L'_3 = L_{m_1} \\ L'_4 &= \cdots = L'_j = L_{m_2} \\ L'_{j+1} &= \cdots = L'_{m_2} = L_n \end{aligned}$$

or

$$\begin{aligned} L'_1 &= L_3 \\ L'_2 &= L_j \\ L'_3 &= L_m \\ L'_4 &= \cdots = L'_m = L_n \end{aligned}$$

Our experimentations have so for led to neither a successful example or insight as to why one cannot exist, so clearly more work is required.

6.2 Lowering the Upper Bound

Note: In this section, at times it will be said that a Lie Algebra is not consistent. These are, in reality, not even Lie Algebras if they are not consistent, but here an inconsistent Lie Algebra will be a tried but failed example.

Lowering the upper bound for our problem in general means finding examples of Lie Algebras satisfying our Hypothesis such that $f(k) < 2_{k-1}$. Currently we are hoping to find an example where dl(L)=4 and $n(L) < 8$. One problem in doing this is that there are a great deal of calculations to be done in checking the consistency of Lie Algebras as n, the number of generators, grows large. Even at n=10, the task can be time consuming by hand, but to get a derived length of 4, we must have at least 15 generators and possibly more.

So far, no example of a Lie Algebra satisfying (*) with $dl(L) \geq 4$ is known other than the standard example. Although examples of dl(L)=3 are very easily found, examples with dl(L)=4 are extremely hard to find. The reason for this is that the higher the number of generators grows, the more Jacobis must then be satisfied for the Lie Algebra to be consistent. Still, if an example does exist, it can be found even if it does take a long time. On the other hand, if an example does not exist, it must be proven that there are no examples other than the standard example.

To help combat this, we have a computer program written in Mathematica. The program takes as input n, $[e_1,e_i]$ for all i, and $[e_j,e_{j+1}]$ for all j. The program then calculates all other Brackets from these original ones and uses those to calculate all Jacobis. If all of the Jacobis are 0, the program calculates the derived length and n(L) over the integers and gives them as output. Otherwise, if not all Jacobis are 0, the program informs the user that the Lie Algebra is not consistent over the integers. Further, it does several calculations to check if the given Lie Algebra is instead consistent for some p, prime. If so, like the case of consistency over the integers, the program calculates the derived length and n(L) of the new Lie Algebra over $\mathbb{Z}_p$.

This program is very useful because it can do the calculations many times faster than any human and, at the same time, eliminates possible human errors in calculation. The program, however, is not perfect and is still under development. Apparently, the program does not always(or ever?) check the Lie Algebra mod primes when general coefficients, C[i][j][k] for C_{ij}^k, are used. Hopefully the program will soon be enhanced to take care of this problem and allow a more thorough investigation of the Lie Algebras input.

The program is useful in checking consistency, but thus far it seems as if finding a Lie Algebra satisfying our hypothesis with derived length 4 is extremely rare. The program can not find an example of derived length 4. All it can do is check consistency of the input given by the user and then checking the derived length. In the future it is hoped that either the mathematical theory or the program will be improved so that examples are easily found. However, as far as improving the program to do this, almost nothing is known. So this program is a great help, but not unless the user knows the mathematical theory behind it and has a bit of luck.

7 Examples of Lie Algebras Satisfying the Hypothesis

7.1 Example 1

This Lie Algebra is interesting more so than any other example given. However, the extra interest comes from an almost trivial point. What I mean is this example is consistent mod two different primes, both 37 and 223, and this is very interesting. But, almost all of the structural constants are divisible by 37 so being divisible by 37 isn't of great interest. Still, it shows that different primes may lead to different values for n(L).

n=13

$[e_2,e_3]$=37e_7
$[e_3,e_4]$=-74e_9
$[e_4,e_5]$=-111e_{11}
$[e_5,e_6]$=-37e_{11}
$[e_6,e_7]$=e_{13}

Over the integers, this Lie Algebra is not consistent.
This Lie Algebra is consistent mod 37.
dl(L)=3
n(L)=3
This Lie Algebra is consistent mod 223
dl(L)=3
n(L)=6

7.2 Example 2

Example 1 and this example are of interest for the same reason. In Example 1, none of the structural constants are divisible by 223, yet the Lie Algebra is consistent mod 223 and not the integers. This example is not consistent over the integers, but is mod 103 and none of the structural constants are divisible by 103.

n=16

$[e_2,e_3]$=e_7

$[e_3,e_4]=2e_9$
$[e_4,e_5]=28e_{11}$
$[e_5,e_6]=107e_{13}$
$[e_6,e_7]=-7e_{15}$
$[e_7,e_8]=2e_{15}$

Over the integers, this Lie Algebra is not consistent.
This Lie Algebra is consistent mod 103.
dl(L)=3
n(L)=7

7.3 Example 3

Example 3 is interesting because it shows an advantage of our method of defining the Brackets. Our method, as described in the paper, takes as input $[e_1,e_i]$ for all i and $[e_j,e_{j+1}]$ for all j and then defines all other Brackets based on these. What this really does is guarantee that all Jacobis, which have one term e_1 as one of the terms, are 0. Not until n=9 does any other Jacobi enter into our system and thus any system we define will be consistent if n¡9.

n=8

$[e_2,e_3]=C_{23}^5e_5+C_{23}^6e_6+C_{23}^7e_7+C_{23}^8e_8$
$[e_3,e_4]=C_{34}^7e_7+C_{34}^8e_8$

This example is consistent over the integers.
dl(L)=3
n(L)=4

7.4 Example 4

Example 4 is very general in that most of the coefficients are just C_{ij}^k for some i, j, and k, but not a specific number. It also has 11 generators and is not trivially consistent as Example 3. Notice that instead of C_{56}^{11}, there is $6C_{45}^9$ as the coefficient of e_{11} for $[e_5,e_6]$. Only with $C_{56}^{11}=6C_{45}^9$ is this example consistent.

n=11

$[e_2,e_3]=C_{23}^8e_8+C_{23}^9e_9+C_{23}^{10}e_{10}+C_{23}^{11}e_{11}$

$[e_3,e_4]=C_{34}^{9}e_9+C_{34}^{10}e_{10}+C_{34}^{11}e_{11}$
$[e_4,e_5]=C_{45}^{9}e_9+C_{45}^{10}e_{10}+C_{45}^{11}e_{11}$
$[e_5,e_6]=6C_{45}^{9}e_{11}$

This example is consistent over the integers.
dl(L)=3
n(L)=5