Asymptotic Sign-Solvability and Multiple Objective Linear Programming

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Abstract

A sign-solvable system is a linear system of equations such that the sign pattern of the solution can be determined by only knowing the signs of the coefficients of the data. Using optimization techniques, we show that the set of possible sign patterns of a solution vector of a time-dependent linear system stabilizes, under certain mild assumptions. Under these same assumptions, we use this result to show that the set optimal partitions of a multiple objective asymptotic linear program stabilizes.

Keywords: Sign-Solvable, linear programming, multiple objective programming

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1 Introduction

The main goal throughout our research was to extend a previous result in [5] showing that the optimal partition of an linear program stabilizes. Our extension is to the case of multiple objectives. We first approached this problem by trying to extend the technique used in the proof of the previous result. When this method proved unsuccessful, we looked for another way to approach the problem and found that sign-solvable systems provided the insight needed to prove the result that we wanted. We worked mainly with linear programming throughout our project.

Linear programming is an important tool used to solve optimization problems, with many real world applications in economics and management. A linear program (LP) consists of a linear objective funtion to be maximized or minimized with respect to a set of linear constraints. We work with asymptotic linear programs dependent on time. The asymptotic linear program is

$$LP(t): \qquad \min\{c^T(t)x : A(t)x = b(t),\ x \geq 0\},$$

where $A(t) : \mathbb{R} \to \mathbb{R}^{m\times n}$, $b(t) : \mathbb{R} \to \mathbb{R}^m$, and $c(t) : \mathbb{R} \to \mathbb{R}^n$. Additionally, every LP has an associated dual,

$$LD(t): \qquad \max\{b^T(t)y : A^T(t)y + s = c(t),\ s \geq 0\}.$$

A vector x satisfying the constraints of the primal linear program $LP(t)$ is called *feasible*. The *feasible region* for $LP(t)$ is $\mathcal{P}(t) = \{x : A(t)x = b(t),\ x \geq 0\}$, and $\mathcal{P}^0(t) = \{x \in \mathcal{P}(t) : x > 0\}$ is the strict interior. $\mathcal{P}^*(t)$ denotes the set of optimal solutions to the primal linear program. Similarly, $\mathcal{D}(t)$ denotes the dual feasible region, $\mathcal{D}^0(t) = \{(y, s) \in \mathcal{D}(t) : s > 0\}$ is the strict interior of the dual, and $\mathcal{D}^*(t)$ is the optimal dual set. Due to the Strong Duality Theorem of linear programming, we know that the feasible elements x and y are optimal if, and only if, $c^T(t)x = b^T(t)y$. Thus, the following three equations are necessary and sufficient conditions for optimality:

$$\begin{aligned} A(t)x &= b(t),\ x \geq 0, && (1)\\ A^T(t)y + s &= c(t),\ s \geq 0,\ \text{and} && (2)\\ x^T s &= 0. && (3) \end{aligned}$$

An early result by Goldman and Tucker [4] guarantees that every solvable linear program has a *strictly complementary* solution, meaning that $(x^*)^T s^* = 0$ and $x^* + s^* > 0$. This leads us to the idea of the *optimal partition*, which identifies the variables that are allowed to be positive at optimality and those that must be zero. The unique optimal partition is denoted $(B|N)$, and is defined by the strictly complementary solution x^* as follows,

$$B(t) = \{i : x_i^*(t) > 0\},\ \text{and}$$

$$N(t) = \{1, 2, 3, \ldots, n\} \backslash B(t).$$

In other words, $B(t)$ is the set of primal variables that can be positive at optimality and $N(t)$ is the set of primal variables that are zero in every optimal solution. Hasfura-Buenaga, Holder and Stuart [5] proved under mild assumptions about $A(t), b(t),$ and $c(t)$ that the optimal partition *stabilizes* after some time T. This means that there exists a time T, such that for all $t \geq T$, $(B(t)|N(t)) = (B(T)|N(T))$. We call $(B(T)|N(T))$ the *asymptotic optimal partition.* In this paper, we extend these results to the optimization of linear programs with multiple objective functions. The *multiple objective linear programs* (MOLPs) in this paper are of the form

$$MOLP(t): \qquad \min\{C(t)x : A(t)x = b(t), x \geq 0\},$$

where $C(t) : \mathbb{R} \to \mathbb{R}^{q \times n}$, meaning that there are q objective functions to minimize. Since the multiple objective functions do not lie in a completely ordered set, a solution to the MOLP is defined as an efficient point or *pareto optimum.* A feasible solution x is pareto optimal if there does not exist a $y \in \mathcal{P}(t)$ such that $C(t)y \leq C(t)x$, with strict inequality holding for at least one component. The set of all pareto optimal solutions is called the *efficient frontier* and is denoted by $\mathcal{E}$. From Matthias Ehrgott's *Multicriteria optimization* [3], we know that $x \in \mathcal{E}$ if, and only if, there exists a strictly positive *weighting vector* w such that x minimizes $\{wC^T(t)x : x \in \mathcal{P}\}$. This linear program is denoted by

$$LP(w,t): \qquad \min\{wC^T(t)x : x \in \mathcal{P}(t)\}.$$

For $MOLP(t)$, we need slightly different conditions to ensure an optimal solution. The following equations are an extension of (1)-(3) and are the necessary and sufficient conditions for optimality in the multiple objective case:

$$\begin{aligned} A(t)x &= b(t),\ x \geq 0, && (4)\\ A^T(t)y + s &= C^T(t)w,\ s \geq 0,\ w > 0, \text{ and} && (5)\\ x^T s &= 0. && (6) \end{aligned}$$

For $MOLP(t)$, the optimal partition is also slightly different than that for a single objective linear program. $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t))$ is defined by

$$\overset{\text{molp}}{N}(t) = \{i : x_i(t) = 0,\ \forall\, x \in \mathcal{E}\} \text{ and}$$

$$\overset{\text{molp}}{B}(t) = \{1, 2, \ldots, n\} \backslash \overset{\text{molp}}{N}(t).$$

This means that $\overset{\text{molp}}{N}(t)$ is the collection of indices such that the corresponding variables are zero over the entire efficient frontier. Note that since each $i \in \{1, \ldots, n\}$

is an element of either $B(t)$ or $N(t)$, there are 2^n possible two-set partitions. We let $\{(B^1|N^1), (B^2|N^2), \ldots, (B^{2^n}|N^{2^n})\}$ be the collection of all such partitions. Let $\mathcal{L}(t) = \{i : B^i \text{ is a sub-partition of } \overset{\text{molp}}{B(t)}\}$. Note that $\overset{\text{molp}}{B(t)} = \bigcup_{i \in \mathcal{L}(t)} B^i(t)$. Thus our goal of showing that $(\overset{\text{molp}}{B(t)}|\overset{\text{molp}}{N(t)})$ stabilizes after some time T is equivalent to showing that $\mathcal{L}(t)$ stabilizes.

This paper is organized as follows. In section 2, we introduce sign solvable systems and show that after some time T, the sign patterns of solutions to a time dependent linear system become fixed. In section 3, we apply this result to multiple objective asymptotic linear programs and show that there exists a time T, such that for all $t \geq T$, $(\overset{\text{molp}}{B(t)}|\overset{\text{molp}}{N(t)}) = (\overset{\text{molp}}{B(T)}|\overset{\text{molp}}{N(T)})$. The last section consists of continuity results for $(x(t), y(t), s(t), w(t))$.

2 Asymptotic Sign-Solvability

For a linear system $Ax = b$, it is often important to make conclusions based on strictly qualitative information. For a matrix A and vector b, there are times when we do not know the exact value of each entry but do know what sign each entry has. For example, suppose we know that $\hat{A}$ and $\hat{b}$ take the form,

$$\hat{A} = \begin{bmatrix} - & + & 0 & + \\ - & - & + & 0 \end{bmatrix} \text{ and } \hat{b} = \begin{bmatrix} + \\ - \end{bmatrix}.$$

The question now becomes, is there a vector $\hat{x}$ that solves $\hat{A}x = \hat{b}$, and if so, what properties does it have? We need some notation to better analyze these systems. We define the *sign* of a real number a as

$$\text{sign}(a) = \begin{cases} +1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

The *sign pattern* of a matrix A is a $(0, 1, -1)$-matrix obtained by replacing each entry a_{ij} with $\text{sign}(a_{ij})$. We denote the sign pattern of a matrix A with $\sigma(A)$. In addition, we define the following time dependent set,

$$\Sigma(t) = \{\sigma(x) : A(t)x = b(t)\}. \tag{7}$$

The set $\Sigma(t)$ contains all of the possible sign patterns that solve $A(t)x = b(t)$. In this section, we show that $\Sigma(t)$ stabilizes under some mild assumptions, which is to say that $\Sigma(t)$ becomes constant after some time T. To analyze asymptotic sign-solvability, we first examine asymptotic linear programs in order to use some results proved in [5]. We make assumptions on the behavior of $A(t)$ and $b(t)$ in order to make conclusions about the behavior of the solutions to $A(t)x = b(t)$. We make these assumptions to ensure that $A(t)$ and $b(t)$ each preserve their respective ranks, meaning the ranks of each matrix become constant after some time T_0. By establishing constant ranks of $A(t)$ and $b(t)$, we can determine whether or not the system $A(t)x = b(t)$ is consistent for all $t \geq T_0$. Once the stability and consistency of $A(t)$ and $b(t)$ are established, we proceed to show that $\Sigma(t)$ stabilizes. The following assumption is key to establishing the stability of $A(t)$ and $b(t)$.

Assumption 1 *There exists a time T_0 such that for all $t \geq T_0$, the determinants of all square sub-matrices of $[A(t) \,|\, b(t)]$ are either constant or have no roots. We also assume that for all $t \geq T_0$, the functions $A(t)$ and $b(t)$ are continuous.*

This assumption is the basis for two important lemmas proved in [5]. These results are optimization results that we present in order to understand the behavior of sign pattern solutions to $A(t)x = b(t)$. The first result provides a foundation for studying asymptotic linear systems in general. The purpose of the following lemma is to establish whether the system $A(t)x = b(t)$ is consistent.

Lemma 1 (Hasfura-Buenaga, Holder, Stuart [5]) *Let $M(t)$ be a matrix function whose component functions have the property that there exists a time T, such that for all $t \geq T$, the determinants of all square sub-matrices are either constant or have no roots. Then,* $\text{rank}(M(t))$ *stabilizes.*

Lemma 1 implies that the ranks of $A(t)$ and $[A(t)|b(t)]$ stabilize. We know that the system $Ax = b$ is consistent if and only if $\text{rank}(A) = \text{rank}([A|b])$. Since the ranks stabilize, we know that if $\text{rank}(A(t)) = \text{rank}([A(t)|b(t)])$ then the system is consistent for all $t \geq T$, and if those ranks are not equal then the system will be inconsistent for all $t \geq T$.

The next lemma presents a tool to analyze the partition of solution variables x, which is described by the following notation. For this notation, assume that $\sigma = \sigma(x)$ for some time t. Let

$$\begin{aligned} \Lambda_+(\sigma) &= \{i : \sigma_i = +1\} \\ \Lambda_-(\sigma) &= \{i : \sigma_i = -1\} \\ \Lambda_0(\sigma) &= \{i : \sigma_i = 0\}. \end{aligned}$$

A sign pattern $\tilde{\sigma}$ is a solution if there exists a vector x with $\sigma(x) = \tilde{\sigma}$ such that x solves $Ax = b$. With the above notation we turn the problem of finding sign pattern solutions into a linear feasibility problem. A feasibility problem for our purposes is a linear program that minimizes the zero function. This is called a feasibility problem because with the objective function fixed at zero, any feasible solution is also an optimal solution. Consider the following program:

$$\min \{0^T x : A(t)x = b(t),\ x \geq 0\}. \tag{8}$$

A potential problem for an asymptotic sign-solvable system is that solutions with both negative and positive components are allowed, whereas a linear program requires solutions to be non-negative. Without loss in generality, it is possible to take all components in $\Lambda_-(\sigma)$ and reverse the sign in the corresponding column of A. Thus, $B(t)$ contains all indices of components.

Despite the fact that the objective function is fixed in (8), we still use optimization techniques to solve the program. It is important to remember that for any linear program, there exists a strictly complementary solution. A strictly complementary solution induces a *maximal* optimal partition, which means that $B(t)$ contains *all* indices whose corresponding variables are not required to be zero. The reader is directed to Roos, Terlaky, and Vial [7] as well as Wright [8] for more information on strictly complementary solutions and maximal partitions. With the knowledge of maximal partitions we use the following result.

Lemma 2 (Hasfura-Buenaga, Holder, Stuart [5]) *Assume that $(A(t), b(t))$ satisfy Assumption 1. Then, there exists a time T, such that for all $t \geq T$, the optimal partition of (8) stabilizes, $(B(t)|N(t)) = (B(T)|N(T))$.*

Lemma 2 asserts that the optimal partition of the feasibility problem stabilizes. This result is important in our proof that $\Sigma(t)$ stabilizes. We use the stability of the optimal partition to show the stability of the sign patterns of an asymptotic linear system. The stability of each sign pattern solution then implies the overall stability of $\Sigma(t)$. We now have the necessary tools to prove our result.

Theorem 1 *Assume that $(A(t), b(t))$ satisfy Assumption 1. Then there exists a time T, such that for all $t \geq T$, $\Sigma(t) = \Sigma(T)$.*

Proof: Let $t_0 \geq T_0$ be large enough to satisfy Assumption 1. From Lemma 1 we know that for all $t \geq t_0$, $\text{rank}([A(t)|b(t)]) = \text{rank}([A(t_0)|b(t_0)])$. Also note that $A(t_0)x = b(t_0)$ has a solution if and only if $\text{rank}(A(t_0)) = \text{rank}([A(t_0)|b(t_0)])$. If $A(t_0)x = b(t_0)$ has no solution, then $\Sigma(t_0) = \emptyset$. Since the ranks have stabilized, the system $A(t)x = b(t)$ is inconsistent for all $t \geq t_0$, which implies that $\Sigma(t) = \emptyset$ for all $t \geq t_0$. Thus, $\Sigma(t)$ stabilizes.

Now assume that the system does have a solution. Let $\tilde{\sigma} \in \Sigma(t_0)$. We proceed to show that $\tilde{\sigma} \in \Sigma(t)$ for all $t \geq t_0$. Consider the following program:

$$\min \{0^T x : A(t_0)x = b(t_0),\ \sigma(x) = \sigma\}. \tag{9}$$

This is not a linear program because $\sigma(x)$ is not a linear function. However, this program is represented by a linear program such that every optimal solution to the nonlinear program is also optimal for the linear program. We have that (9) can be represented by:

$$\min \{0^T x : A(t_0)x = b(t_0),\ x_{\Lambda_+(\tilde{\sigma})} \geq 0,\ x_{\Lambda_-(\tilde{\sigma})} \leq 0,\ x_{\Lambda_0(\tilde{\sigma})} = 0\}. \tag{10}$$

We know that all linear programs have a strictly complementary optimal solution, which means that there exists a solution such that the inequalities become strict (— i.e. $x_{\Lambda_+(\tilde{\sigma})} > 0$ and $x_{\Lambda_-(\tilde{\sigma})} < 0$). These strict inequalities induce a optimal partition that is maximal, which implies that the sign pattern of the solution x at time t_0 is equal to the chosen sign pattern $\tilde{\sigma}$, namely $\sigma(x) = \tilde{\sigma}$. We know from Lemma 2 that for all $t \geq t_0$, the optimal partition stabilizes. This means that we can always find a solution x that maintains the current sign pattern, or equivalently that $\sigma(x) = \tilde{\sigma}$ for all $t \geq t_0$. This implies that the sign pattern $\tilde{\sigma}$ remains a solution to $A(t)x = b(t)$ for all $t \geq t_0$, and hence $\tilde{\sigma} \in \Sigma(t)$ for all $t \geq t_0$.

For each $\tilde{\sigma} \in \Sigma(t)$ we have shown that $\tilde{\sigma}$ remains in $\Sigma(t)$ for all $t \geq t_0$, which implies that for all $t \geq t_0$, $\Sigma(t_0) \subseteq \Sigma(t)$. If at some time $t_1 \geq t_0$ a new sign pattern $\hat{\sigma}$ becomes a solution to $A(t)x = b(t)$, then by the same reasoning as before, that sign pattern remains in $\Sigma(t)$ for all $t \geq t_1$. $\Sigma(t)$ is a finite set bounded by 3^n, which is the number of possible sign patterns. Thus, since $\Sigma(t)$ can never lose any of its components and is bounded above, there must exist a time $T \geq t_0$ such that for all $t \geq T$, $\Sigma(t) = \Sigma(T)$. ■

This result is a useful application of optimization to a linear algebra problem. Intuition leads one to suspect that this result is true, but it is a hard result to prove directly through linear algebra techniques. The use of an optimization result shows the power of optimization techniques. In the next section, we extend the main result in [5] to multiple objective asymptotic linear programming.

3 Multiple Objective Asymptotic Linear Programming

In this section we show that the optimal partition $(\overset{\text{molp}}{B(t)}|\overset{\text{molp}}{N(t)})$ stabilizes under some mild assumptions on $A(t), b(t),$ and $C(t)$. First, we introduce some notation, followed by an explanation of how we extend Theorem 1. For $j \in \{1, 2, \ldots, 2^n\}$, let

$$H_j(t) = \begin{bmatrix} A_{B^j}(t) & 0 & 0 & 0 \\ 0 & A_{B^j}^T(t) & 0 & -C_{B^j}^T(t) \\ 0 & A_{N^j}^T(t) & I & -C_{N^j}^T(t) \end{bmatrix},\ v = \begin{pmatrix} x_{B^j} \\ y \\ s_{N^j} \\ w \end{pmatrix},\ \text{ and } h(t) = \begin{pmatrix} b(t) \\ 0 \\ 0 \end{pmatrix}. \tag{11}$$

A solution v to $H_j(t)v = h(t)$ satisfies the necessary and sufficient optimality conditions (4)-(6) for the multiple objective asymptotic linear program $(MOLP(t))$ provided that v is sufficiently positive, written $v \gtrdot 0$. A sufficiently positive solution v is such that $x_{B^j} > 0, s_{N^j} > 0$, and $w > 0$. The reader should verify that (11) is equivalent to (4)-(6), knowing that it is necessary for $x_{N^j} = 0$ and $s_{B^j} = 0$ in any optimal solution. The optimal partition stabilizes because the sign patterns of v stabilize. For Theorem 1, we made an assumption about the behavior of $A(t)$ and $b(t)$, so for the linear system $H_j(t)v = h(t)$, we need to make a similar assumption.

Assumption 2 *There exists a time T_1 such that for all $t \geq T_1$, the determinants of all square sub-matrices of*

$$\begin{bmatrix} A(t) & 0 & b(t) & 0 \\ 0 & A^T(t) & 0 & C^T(t) \end{bmatrix} \tag{12}$$

are either constant or have no roots. We also assume that for all $t \geq T_1$, the functions $A(t)$, $b(t)$, and $C(t)$ are continuous.

Assumption 2 combined with Lemma 1 shows that the ranks of $H_j(t)$ and $[H_j(t)|h(t)]$ stabilize. This fact allows us to use Theorem 1. Later, we show that the stability of the sign pattern solutions to $H_j(t)v = h(t)$ implies that the optimal partition stabilizes. Now we present an example that gives an idea of how a multiple objective asymptotic linear program behaves.

Example 1 *Consider the following program:*

$$\min \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -\left(\frac{(t-2)^2-t}{3t^2+4t}\right) x_1 - x_2 \end{pmatrix}$$

$$\text{subject to} \qquad \begin{aligned} x_1 + x_2 &\leq 3, \\ x_1 &\leq 2, \\ x_2 &\leq 2, \\ x_1, x_2 &\geq 0. \end{aligned}$$

The data of this program in standard form is

$$A(t) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},\ b(t) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix},\ \text{and } C(t) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -\frac{(t-2)^2-t}{3t^2+4t} & -1 & 0 & 0 & 0 \end{bmatrix}.$$

First note that the matrix functions $A(t)$, $b(t)$, and $C(t)$ satisfy Assumption 2, with stabilization occurring after $T_0 = 4$. Also note that in this example $n = 5$, as there are two main variables and three slack variables. This example can be visualized in *two dimensions and the geometry makes the following results more clear. For* $t \leq 4$ *the optimal partition is different than for* $t > 4$*. To see this, consider* $t = 4$*, at which point* $z_2 = -x_2$*, which means that the second objective does not contribute to the overall objective since* $-x_2$ *is a component of the first objective. So for this case, there is only one objective to minimize and it leads to the unique optimal solution* $x_1 = 0, x_2 = 2, x_3 = 1, x_4 = 2,$ *-and* $x_5 = 0$*. Thus, for* $t = 4$*, the optimal partition is* $(\overset{\text{molp}}{B}(4)|\overset{\text{molp}}{N}(4)) = (\{2,3,4\}|\{1,5\})$*. However, when* $t > 4$*, the second objective function is* $-\epsilon x_1 - x_2$*, where* $\epsilon \in (0,1)$*. In this case, the objectives lead to different parts of the feasible region. Thus, by pareto optimality, the optimal partition is* $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t)) = (\{1,2,3,4,5\}|\{\emptyset\})$*.*

This example shows how quickly the optimal partition can change when the ranks of the matrices change, as when $t = 4$. With an idea of what is necessary for the optimal partition to stabilize, we now formally present the following theorem.

Theorem 2 *Assume that* $(A(t), b(t), C(t))$ *satisfy Assumption 2. Then there exists a time* T*, such that for all* $t \geq T$*,* $(\overset{\text{molp}}{B}(t)|\overset{\text{molp}}{N}(t)) = (\overset{\text{molp}}{B}(T)|\overset{\text{molp}}{N}(T))$*. Equivalently,* $\mathcal{L}(t) = \mathcal{L}(T)$*.*

Proof: Let $t_0 \geq T_1$ be large enough to satisfy Assumption 2. We know from Theorem 1 that for each $j \in \{1, 2, \dots, 2^n\}$, $\Sigma_j(t) = \{\sigma(v) : H_j(t)v = h(t)\}$ is constant for all $t \geq \tilde{T}$, where $\tilde{T} \geq t_0$. Let

$$\overset{\text{suff}}{\Sigma}_j(t) = \{\sigma(v) : H_j(t)v = h(t),\ v >\!| \, 0\}.$$

Each j corresponds to a partition $(B^j|N^j)$. We divide the set of partitions into two classes: those that contain a sufficiently positive solution and those that do not. Let

$$\Omega(t) = \{j : \overset{\text{suff}}{\Sigma}_j(t) \neq \emptyset\},$$

$$\Psi(t) = \{j : \overset{\text{suff}}{\Sigma}_j(t) = \emptyset\}.$$

From Theorem 1 we know that for each partition $(B^j|N^j)$, there exists a time $\tilde{T} \geq t_0$ such that for all $t \geq \tilde{T}$, $\Sigma_j(t) = \Sigma_j(\tilde{T})$. That implies that for all $t \geq \tilde{T}$, $\Omega(t) = \Omega(\tilde{T})$ and $\Psi(t) = \Psi(\tilde{T})$. The set $\Omega(t)$ corresponds to all the partitions that contain a sufficiently positive solution and thus, are in the efficient frontier. Therefore, $\Omega(t) = \mathcal{L}(t)$ and since $\Omega(t)$ remains constant for all $t \geq \tilde{T}$, so must $\mathcal{L}(t)$. ■

4 Continuity results

As stated in our introduction, we began our research with the goal of showing that the optimal partition $(\overset{\text{molp}}{B(t)}|\overset{\text{molp}}{N(t)})$ stabilizes after some time T. Although we ended up using sign-solvability to show this, we tried several approaches beforehand. One of these methods was to show that there exists a continuous function of the weights for sufficiently large t that keeps the same optimal partition. Working on the proof to this conjecture resulted in the proof of a stronger theorem that shows that $(x(t), y(t), s(t), w(t))$ is continuous after a certain time T. For our proof, we first need the following lemma.

Lemma 3 (Campbell and Meyer [1]) *A Moore-Penrose pseudo inverse $M^+(t)$ is continuous at t_0 if, and only if,* $\text{rank}(M(t)) = \text{rank}(M(t_0))$ *for t sufficiently close to t_0.*

Theorem 3 *For sufficiently large t, there exists continuous $(x(t), y(t), s(t), w(t))$, such that*

$$x(t) \in \text{argmin}\{w^T(t)C(t)x : A(t)x = b(t), x \geq 0\} \textit{ and}$$
$$y(t), s(t)) \in \text{argmax}\{b^T(t)y : A^T(t)y + s = C^T(t)w(t), s \geq 0\}.$$

Proof: Let $t_0 \geq T_1$ satisfy Assumption 2, and let $(B^j|N^j)$ be the optimal partition for $LP(\hat{w}_j, t)$, where $w(t_0) = \hat{w}$. Because we satisfy this assumption, there exists a solution to $H_j(t_0)v(t_0) = h(t_0)$, such that $v(t_0)$ is sufficiently positive, where

$$H_j(t) = \begin{bmatrix} A_{B^j} & 0 & 0 & 0 \\ 0 & A^T_{B^j} & 0 & -C^T_{B^j}(t) \\ 0 & A^T_{N^j} & I & -C^T_{N^j}(t) \end{bmatrix}, \; v(t_0) = \begin{pmatrix} x_{B^j}(t_0) \\ y(t_0) \\ s_{N^j}(t_0) \\ w(t_0) \end{pmatrix}, \; \text{ and } \; h(t) = \begin{pmatrix} b(t) \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $v(t_0) = H_j^+(t_0)h(t_0) + q_0$, where H^+ is the Moore-Penrose pseudo inverse of H and $q_0 \in \text{Null}(H_j(t_0))$. We define $v(t) = H_j^+(t)h(t) + (I - H_j^+(t)H_j(t))q_0$, where $(I - H_j^+(t)H_j(t))q_0$ is the projection of q_0 onto the null space of $H_j(t)$. Let $t_k \to t_0$. From our definition of $v(t)$, we have that $v(t_k) = H_j^+(t_k)h(t_k) + (I - H_j^+(t_k)H_j(t_k))q_0$.

First, we show that $v(t_k)$ satisfies $H_j(t_k)v(t_k) = h(t_k)$, which means that our partition $(B^j|N^j)$ remains optimal. We have

$$\begin{aligned} H_j(t_k)v(t_k) &= H_j(t_k)[H_j^+(t_k)h(t_k) + (I - H_j^+(t_k)H_j(t_k))q_0] \\ &= H_j(t_k)H_j^+(t_k)h(t_k) + H_j(t_k)q_0 - H_j(t_k)H_j^+(t_k)H_j(t_k)q_0 \\ &= h(t_k) - H_j(t_k)q_0 + H_j(t_k)q_0 \\ &= h(t_k). \end{aligned}$$

Second, we show that $v(t_k) \to v(t_0)$, proving that v is continuous. Note that $H^+(t)$ is continuous from Assumption 2 and Lemma 3. So, as $t_k \to t_0$, $v(t_k) = H_j^+(t_k)h(t_k) + (I - H_j^+(t_k)H_j(t_k))q_0 \to H_j^+(t_0)h(t_0) + q_0 = v(t_0)$. This is because $(I - H_j^+(t_k)H_j(t_k))q_0$ is the projection vector onto the null space of $H_j(t_k)$ and $q_0 \in \text{Null}(H_j(t_0))$. Hence, as $t_k \to t_0$, the projection vector actually becomes a vector in the null space of $H(t_0)$.

Notice that $v(t_k) \succneq 0$ for large k because $v(t_k) \to v(t_0)$ and $v(t_0) \succneq 0$. The proof follows because $x_{N^j}(t) = 0$ and $s_{B^j}(t) = 0$, both of which are continuous. Hence, there exists a continuous $(x(t), y(t), s(t), w(t))$. ∎

Let $w_j(t)$ be the function of weights for the weighted multiple objective linear program

$$LP(w,t): \qquad \min\{w_j(t)C^T(t)x : A(t)x = b(t), x \geq 0\}.$$

Since w is a sub-vector of v, we get the following corollary.

Corollary 1 *For t sufficiently large, $w_j(t)$ is continuous and the partition $(B^j|N^j)$ is optimal for $LP(w,t)$.*

5 Conclusion and Final Remarks

We have shown under some mild assumptions that the set of sign patterns for asymptotic linear systems stabilizes. We then used this result to show that the optimal partition of an asymptotic multiple objective linear program stabilizes. The latter result substantially improves and extends the result in [5], which applied only to single objective linear programs.

There is also an economic interpretation to our MOLP result. An economic result known as the Nonsubstitution Theorem says that there is always a collection of (not necessarily unique) processes in an economy that are optimal, independent of demand. Hasfura-Buenaga, Holder, and Stuart [5] demonstrated that the collection of optimal processes stabilizes in a dynamic version of the Nonsubstitution Theorem. In this economic model, it is assumed that the labor source is homogeneous, —i.e. every potential employee has equal skills and thus would earn an equal wage. Our multiple objective result, however, says that the collection of optimal processes stabilizes even if we allow multiple labor sources with varying wages, a much more realistic claim. This extension is more thoroughly explored in [2].

There are further questions to study in this area. We first define the following:

$$\mathcal{W}_j(t) = \{w : \text{the optimal partition for } \min\{w^T c(t)x : A(t)x = b(t), x \geq 0\} \text{ is } (B^j|N^j)\}.$$

• Does $\dim(\text{Null}(A_{B^j}(t))) + \dim(W_j(t))$ become constant? Graphical analysis with $w = [w_1, w_2]^T$ intuitively leads to this conclusion, and it may hold true for any $w = [w_1, \ldots, w_q]$.

• What class of functions satisfy Assumption 2? The assumption holds for rational functions, but does not for all differentiable functions (—i.e. $sin(t)$). There are no known necessary conditions for this assumption to hold, and it is thus somewhat ambiguous how strong this assumption is.

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