Sufficient Conditions to Guarantee a Globally Attractive 2-Cycle of the Non-autonomous Quadratic Family

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Abstract: While much of the behavior of the generic quadratic family of dynamical systems is understood, extending it to the realm of non-automous dynamical systems left the long-range behavior of the system yet to be determined. In this paper, we produce an abstract lemma that will provide sufficient conditions to guarantee a globally asymptotic fixed point. With this lemma in hand, we are able to disprove a previously made conjecture [3] by finding a bound which will guarantee a globally attractive 2-cycle under the infinite composition of two quadratic functions f and g.

I. Introduction

For our purposes, a dynamical system is the iterative process of composing a continuous function $f : I \to I$, where I is an interval. The composition of two functions is denoted by $f \circ g(x) = f(g(x))$, and thus, the n-fold composition of f with itself is the function $f^n(x) = f \circ \ldots \circ f(x)$.

In the area of discrete dynamical systems, one application of f corresponds to the passage of one unit of time, t. For any such time-dependent process, the goal of dynamical systems is the following: Given an initial state $\bar{x}$, we want to understand the eventual behavior of this iteration in the distant future ($t \to \infty$) and determine what, if any, asymptotic properties it has. Differential equations with time as the independent variable can be thought of as dynamical systems with continuous, rather than discrete, time.

Outside the purely mathematical realm, dynamical systems occur in fields such as physics, economics, and biology. Biologists, for example, construct mathematical models to monitor changes over time in populations, taking into account several factors, which include predators, climate, food, overcrowding, birth rate, and death rate. Biologists hope to determine whether a population tends toward zero over time, causing species extinction; whether the population grows exponentially large, leading to overcrowding; whether the population fluctuates periodically, such as with the seasons; or whether it fluctuates in a random, even chaotic, manner.

The simplest population model, both in discrete and continuous time, assumes that the growth rate of a population is directly proportional to the present population. Regardless of whether time is discrete or continuous, this assumption implies that the population grows, or decays, exponentially leading to unmonitored growth or extinction. Solving the differential equation $\frac{dP}{dt} = kP$ yields the solution $P(t) = P_0 e^{kt}$, where k is the proportionality constant and $P_0 = P(0)$ is the initial population. A positive k leads to unlimited population expansion since $P(t) \to \infty$ as $t \to \infty$, while a negative k leads to population depletion since $P(t) \to 0$ as $t \to \infty$.

Analagous to the above continuous model is the discrete difference equation $P_{n+1} = kP_n$ in which the next generation's population is directly proportional to the present generation's population. In this case, k is again the proportionality constant and P_n represents the population after n generations ($n \in \mathbf{N}$). This leads to the system

$$\begin{aligned}
&P_1 = kP_0 \\
&P_2 = kP_1 = k^2P_0 \\
&P_3 = kP_2 = k^3P_0 \\
&\quad\vdots \\
&P_n = kP_{n-1} = k^nP_0
\end{aligned}$$

where $P_n \to \infty$ if $k > 1$, and $P_n \to 0$ if $0 < k < 1$.

Rewriting this system of equations in terms of the function $f(x) = kx$, where $x = P_0$, yields

$$\begin{aligned}
&P_1 = f(x) = kx \\
&P_2 = f(f(x)) = k^2x \\
&P_3 = f(f(f(x))) = k^3x \\
&\quad\vdots \\
&P_n = f^n(x) = k^nx
\end{aligned}$$

allowing the eventual behavior of the population to be viewed in terms of a dynamical system.

By taking into account a limiting factor, L, to more accurately represent reality, the population model becomes $\frac{dP}{dt} = kP(L - P)$. For $k > 0$, the population either remains constant if $P = L$ ($\frac{dP}{dt} = 0$), decreases if $P > L$ ($\frac{dP}{dt} < 0$), or increases if $P < L$ ($\frac{dP}{dt} > 0$). While this equation can be explicitly solved for, the solution to the corresponding difference equation yields a most complex dynamical system that is still not entirely understood today.

To simplify the model further, the population is viewed in terms of a percentage, P_n, with the limiting factor $L = 1$: $P_{n+1} = kP_n(1 - P_n)$. As before, the equation is rewritten as a function, $f(x) = kx(1 - x)$, where $x = P_0$, producing the system of equations

$$\begin{aligned}
&P_1 = f(x) \\
&P_2 = f(f(x)) \\
&P_3 = f(f(f(x))) \\
&\quad\vdots \\
&P_n = f^n(x).
\end{aligned}$$

Now the population's behavior can be monitored by studying the logistics of this quadratic function. Many of the features found in dynamical systems are illustrated by the dynamics of this logistic family causing it to be a central focus of modern mathematical research.

For $1 < \mu < 3$, each element of the quadratic family $F_\mu(x) = \mu x(1-x)$ has two fixed points: a repelling fixed point at 0 and an attracting fixed point at $p_\mu = \frac{\mu-1}{\mu}$. As $n \to \infty$ for $F_\mu^n(x)$, points that lie outside the unit interval $[0,1]$ approach $-\infty$ while p_μ is globally attracting for all points within the open interval.

Values of $\mu > 3$ lead to very rich and increasingly complex dynamical behavior. In particular, there is no globally attracting fixed point. However, this will not be discussed here as the focus is strictly $1 < \mu < 3$. For a more detailed discussion, refer to [1].

Once the dynamics of a single μ value are understood, it is desirable to vary the value of μ with time. For example, varying the values of μ cyclically emulates the periodic nature of many environmental factors. To illustrate this, from [3], we have the non-autonomous dynamical system

$$f_n(x) = \mu_n f_{n-1}(x)\big(1 - f_{n-1}(x)\big) \tag{1}$$

given $x \in (0,1)$, $\mu_{n+p} = \mu_n$ for all $n \in \mathbf{N}$, and $f_{n+p} = f_n$ with $f_0(x)$ defined as x. Conjecture 2.6 claims equation 1 has a globally asymptotically stable p-cycle if and only if $1 < \min\ \mu_i \leq \max\ \mu_i \leq 3$. Our following research disproves this statement by considering the case where $p = 2$.

In our argument, we will consider the case where there are two functions with different μ values, A and B, that are periodic in a 2-cycle. We define them as $F(x) = g(f(x))$ and $G(x) = f(g(x))$ where $f(x) = Ax(1-x)$ and $g(x) = Bx(1-x)$ with $1 < A, B < 4$. Also, let $h(x) = x$.

II. Theorems and Definitions

It is necessary to begin by stating a number of theorems and definitions which will be used throughout our argument.

A. Dynamical Systems:

Definition 1. *The point x is a fixed point for f if $f(x) = x$.*

Definition 2. *The orbit of x is the set of points* $\{x,\ f(x),\ f^2(x),\ \ldots,\ f^n(x),\ \ldots\}$.

Definition 3. *The point x is a periodic point of period n if $f^n(x) = x$.*

Definition 4. *A point x is eventually periodic of period n if there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geq m$. That is, $f^i(x)$ is periodic for $i \geq m$.*

Definition 5. *A two-cycle of the two-periodic non-autonomous dynamical system* $\{f(x) = Ax(1-x), g(x) = Bx(1-x)\}$ *is a pair* $\{x_0, f(x_0)\}$ *such that* $g(f(x_0)) = x_0$ *is globally attracting if* $x \in (0,1) \Rightarrow$ $\lim_{n\to\infty} F^n(x) = \lim_{n\to\infty} (g \circ f)^n(x) = x_0$.

Theorem 1. Sarkovskii's Theorem: *Suppose* $f : \mathbf{R} \to \mathbf{R}$ *is continuous. Suppose* f *has a periodic point of prime period* k*. If* $k \rhd l$ *in the ordering*

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots$$
$$\rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd 2^3\cdot 7 \rhd \cdots\cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

then f *also has a periodic point of period* l.

B. Numerical Analysis:

Theorem 2. Archimedes' Principle: *Let* ϵ *and* M *be any two positive real numbers. There exists a* k *in* $\mathbf{N}$ *such that* $\mathrm{M} < k\epsilon$.

Definition 6. *Let* S *be a nonempty set of real numbers that is bounded above. A* supremum *or* least upper bound *of* S*, denoted* sup S*, is a real number* μ *such that*

(i) $x \leq \mu$*, for all* x *in* S.

(ii) If M *is an upper bound for* S*, then* $\mu \leq \mathrm{M}$.

Definition 7. *Let* S *be a nonempty set of real numbers that is bounded below. An* infimum *or* greatest lower bound *of* S*, denoted* inf S*, is a real number* ν *such that*

(i) $\nu \leq x$*, for all* x *in* S.

(ii) If m *is any lower bound for* S*, then* $\nu \geq \mathrm{m}$.

Theorem 3. The Completeness Axiom for $\mathbf{R}$: *If* S *is a nonempty set of real numbers that is bounded above, then sup* S *exists in* $\mathbf{R}$. (*Similarly for inf* S *if* S *is bounded below.*)

Theorem 4. *Let* S *be a nonempty set of real numbers and let* μ *and* ν *be real numbers.*

(i) Suppose that S *is bounded above. Then* $\mu = sup$ S *if and only if* μ *is an upper bound for* S *and, for every* $\epsilon > 0$*, there exists an* x *in* S *such that* $\mu - \epsilon < x \leq \mu$.

(ii) Suppose that S *is bounded below. Then* $\nu = inf$ S *if and only if* ν *is a lower bound for* S *and, for every* $\epsilon > 0$*, there exists an* x *in* S *such that* $\nu \leq x < \nu + \epsilon$.

Theorem 5. Intermediate Value Theorem:

(i) Suppose $f : [a, b] \to \mathbf{R}$ *is continuous. Suppose that* $f(a) = u$ *and* $f(b) = v$*. Then for any* z *between* u *and* v*, there exists* c, $a \leq c \leq b$, *such that* $f(c) = z$.

(ii) If f *is a continuous, real-valued function on* $[a, b]$ *and if* $f(a) \cdot f(b) < 0$, *then* f *must have value 0 at some point in* (a, b).

Theorem 6. The Mean Value Theorem: *Suppose that* f *is continuous on* $[a, b]$ *and differentiable on* (a, b)*. There exists a point* c *in* (a, b) *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Theorem 7. Racehorse Theorem: *Suppose that* f *and* g *are differentiable on* $[a, \infty)$, *that* $f(a) = g(a)$, *and that* $f'(x) < g'(x)$ *for all* x *in* (a, ∞)*. Then* $f(x) < g(x)$ *for all* x *in* (a, ∞)*. (Similarly for intervals of the form* $[a, b]$*.)*

Theorem 8. Chain Rule: *If* f *and* g *are differentiable functions, then* $(f \circ g)'(x) = f'(g(x))g'(x)$*. In particular, if* $h(x) = f^n(x)$*, then* $h'(x) = f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdot \ldots \cdot f'(x)$.

Theorem 9. L'Hôpital's Rule: *Suppose that* f *and* g *are each continuous on* $[a, b]$ *and differentiable on* (a, b)*. Suppose that* c *is a point of* $[a, b]$ *such that* $f(c) = g(c) = 0$*. Suppose also that, on some deleted neighborhood* $N'(c)$, g' *does not vanish. If* $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ *exists, then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *also exists and equals* L.

III. Research Results

Proving sufficent bounds to guarantee the globally asymptotic 2-cycle will require an abstract tool – one which will guide the search for locating such bounds. The following lemma does just that, while also applying itself to a whole class of functions.

Lemma 1. *For a function* F *with the with the following properties:*

(i) $x < x_0 \Rightarrow F(x) > x$

(ii) $x > x_0 \Rightarrow F(x) < x$

(iii) $|F'(x_0)| < 1$

(iv) $F(0) = 0$,

the fixed point x_0 *is globally attracting if and only if* F *has no other periodic orbits.*

Proof. $\Rightarrow$ (By Contradiction)

Let F have a globally attracting fixed point $x_0 \in (0, 1)$:

$\forall x \in (0, 1), \lim_{n\to\infty} F^{(n)}(x) = x_0$.

Assume F has a periodic point $y \in (0,1)$, $y \neq x_0$, of order m.
Since $y \in (0,1)$, $\lim_{n\to\infty} F^{(n)}(y) = x_0$.

Fix $0 < \varepsilon < |x_0 - y|$.
By the definition of the limit of a sequence, $\exists k_o \in \mathbf{N}$ such that $k \geq k_0$, $|x_0 - F^{(k)}(y)| < \varepsilon$.

By Archimedes' Principle, $\exists p \in \mathbf{N}$ such that $k_0 < m \cdot p$.
By the definition of a periodic point, $F^{(mp)}(y) = y$,
yet $|x_0 - F^{(mp)}(y)| = |x_0 - y| > \varepsilon$
$\Rightarrow\Leftarrow \lim_{n\to\infty} F^{(n)}(y) = x_0$.

$\therefore F$ has no periodic points.

$\Leftarrow$ (By Contrapositive)
Assume there is a point $\bar{x}$ whose orbit $\{\bar{x}, x_1 = F(\bar{x}), x_2 = F(x_1), \ldots\}$ does not converge to x_0.

Case 1: The orbit of $\bar{x}$ is eventually strictly to the left of x_0.

$\exists n \in \mathbf{N}$ such that for $k \geq n$, $F^k(\bar{x}) < x_0$.
By *(i)*, $F^n(\bar{x}) < F^{n+1}(\bar{x}) < F^{n+2}(\bar{x}) < \ldots < x_0$.
The sequence above is bounded and monotone increasing,
so $\left(\lim_{n\to\infty} F^n(\bar{x}) = \bar{\bar{x}}\right)$.
By continuity,

$$F(\bar{\bar{x}}) = F\left(\lim_{n\to\infty} F^n(\bar{x})\right) = \lim_{n\to\infty} F^{n+1}(\bar{x}) = \bar{\bar{x}}$$

Thus, $\bar{\bar{x}}$ is a new fixed point.

Case 2: The orbit of $\bar{x}$ is eventually strictly to the right of x_0.

$\exists n \in \mathbf{N}$ such that for $k \geq n$, $F^k(\bar{x}) > x_0$.
By *(ii)*, $F^n(\bar{x}) > F^{n+1}(\bar{x}) > F^{n+2}(\bar{x}) > \ldots > x_0$.
The above sequence is bounded and monotone decreasing,
so $\left(\lim_{n\to\infty} F^n(\bar{x}) = \bar{\bar{x}}\right)$.
By continuity,

$$F(\bar{\bar{x}}) = F\left(\lim_{n\to\infty} F^n(\bar{x})\right) = \lim_{n\to\infty} F^{n+1}(\bar{x}) = \bar{\bar{x}}$$

Thus, $\bar{\bar{x}}$ is a new fixed point.

Case 3: The orbit is infinitely often to the left and right of x_0.

Let $x' = sup\{F^n(\bar{x}) : F^n(\bar{x}) < x_0\}$,
and $x'' = inf\{F^n(\bar{x}) : F^n(\bar{x}) > x_0\}$.
Claim: $F(x') \geq x''$.

Assume $F(x') \in (x', x_0)$.

F is continuous, so choose $\varepsilon > 0$ small enough such that for $x \in N(x', \varepsilon), F(x) \in N(F(x'), \delta)$ where $0 < \delta < min\{|F(x') - x'|, |F(x') - x_0|\}$.

By Theorem 4, $\exists i \in \mathbf{N}$ such that $F^i(\bar{x}) \in N(x', \varepsilon)$.

Hence $F^{i+1}(\bar{x}) \in N(x_0, \delta) \subseteq (x', x_0)$

$\Rightarrow\Leftarrow x' = sup\{F^n(\bar{x}) : F^n(\bar{x}) < x_0\}$.

$\therefore F(x') \notin (x', x_0)$

Assume $F(x') = x_0$.

By *(iii)*, x_0 is a locally attracting fixed point, so $\exists \gamma > 0$ such that for $x \in N(x_0, \gamma)$, $\lim_{n\to\infty} F^n(x) = x_0$.

F is continuous so choose $\varepsilon > 0$ small enough such that for $x \in N(x', \varepsilon)$, $F(x) \in N(x_0, \delta)$ for $0 < \delta < \gamma$.

By Theorem 4, $\exists F^i(\bar{x}) \in N(x', \varepsilon)$. Hence,

$F^{i+1}(\bar{x}) \in N(x_0, \delta) \subseteq N(x_0, \gamma)$ and $\lim_{n\to\infty} F^n(\bar{x}) = x_0$.

$\Rightarrow\Leftarrow$ definition of $\bar{x}$.

$\therefore F(x') \neq x_0$

Assume $F(x') \in (x_0, x'')$.

F is continuous, thus choose $\varepsilon > 0$ small enough such that for $x \in N(x', \varepsilon)$, $F(x) \in N(x_0, \delta)$ where $0 < \delta < min\{|F(x') - x_0|, |F(x') - x''|\}$.

By Theorem 4, $\exists i\ F^i(\bar{x}) \in N(x', \varepsilon)$.

Hence, $F^{i+1}(\bar{x}) \in N(x_0, \delta) \subseteq (x_0, x'')$

$\Rightarrow\Leftarrow x'' = inf\{F^n(\bar{x}) : F^n(\bar{x}) > \bar{x}\}$.

$\therefore F(x') \notin (x_0, x'')$.

$\therefore F(x') \geq x''$.

By Theorem 4, $\forall\ \varepsilon > 0$, $\exists\ F^i(\bar{x}) \in N(x', \varepsilon)$. Thus, by the continuity of F^k, $k \in \mathbf{N}$, choose $\varepsilon > 0$ small enough such that for $F^j(\bar{x}) \in N(x', \varepsilon)$,

$F^{j+1}(\bar{x}) \in N\big(F(x'), \delta_1\big)$ and

$F^2(x') \in N(F^{i+1}\big(\bar{x}), \delta_2\big)$

$F^3(x') \in N(F^{i+2}\big(\bar{x}), \delta_3\big)$

$\vdots$

$F^n(x') \in N(F^{i+n-1}\big(\bar{x}), \delta_n\big)$

where $0 < \delta_n < \bar{x} - x'$ and $F^{i+n-1}(\bar{x}) \in (0, \bar{x})$ which is guaranteed to exist by our case assumption.

Then $\exists\ n \in \mathbf{N}$, $n > 1$, such that

$F^n(x') \in N\big(F^{i+n-1}(\bar{x}), \delta_n\big) \subseteq (0, \bar{x})$.

Claim: $F^n(x') \leq x'$.

Assume $F^n(x) \in (x', x_0)$. F is continuous, so F^k is continuous for $k \in \{1, 2, \ldots, n\}$. Hence, choose $\varepsilon > 0$ small enough such that for $x \in N(x', \varepsilon)$, the following occurs:

$F(x) \in N(F(x'), \delta_1)$
$F^2(x) \in N(F^2(x'), \delta_2)$
$\vdots$
$F^n(x) \in N(F^n(x'), \delta_n)$
with $\delta_n < min\{|F^n(x') - x'|, |F^n(x') - x_0|\}$.

By Theorem 4, $\exists i \in \mathbf{N}$ such that $F^i(\bar{x}) \in N(x', \varepsilon)$.
Hence, $F^{n+i}(\bar{x}) \in N(F^n(x'), \delta_n) \subseteq (x', x_0)$
$\Rightarrow\Leftarrow x' = sup\{F^n(\bar{x}) : F^n(\bar{x}) < x_0\}$.
$\therefore F^n(x') \notin (x', x_0)$.

$\therefore F^n(x') \leq x'$.

Claim: $\exists \tilde{x} > 0$ such that $\tilde{x} < F^n(\tilde{x}) < x'$.

By *(iv)*, $F(0) = 0$, thus $F^n(0) = 0$. F is continuous at 0, so F^k is continuous at 0 for $k \in \{1, 2, \ldots, n\}$. There exists an open interval $P_1 = (0, x^{(1)})$ such that for all $x \in P_1$, $F(x) > x$. Define

$P_2 = F^{-1}(P_1) \cap P_1 = (0, x^{(2)})$
$P_3 = F^{-1}(P_2) \cap P_2 = (0, x^{(3)})$
$\vdots$
$P_n = F^{-1}(P_{n-1}) \cap P_{n-1} = (0, x^{(n)})$.

The inverse image of each open set is open since F is continuous, and the intersection of 2 open sets is open, thus each P_i, $i \in \{1, 2, \ldots, n\}$, is a nonempty open set. For any $x \in P_n = (0, x^{(n)})$,

$x < F(x) < F^2(x) < \ldots < F^n(x)$.

By continuity, choose an $\tilde{x} \in P_n$ small enough such that $F^n(\tilde{x}) < x'$.

$\therefore \exists \tilde{x} > 0$ such that $\tilde{x} < F^n(\tilde{x}) < x'$.

Note that on the interval $[\tilde{x}, x']$, $F^n(\tilde{x}) > \tilde{x}$ and $F^n(x') \leq x'$.

Claim: F has a periodic point of period $n > 1$ in $[\tilde{x}, x'] \subseteq (0, x_0)$.

Assume $F^n(x') = x'$.
Then x' is a fixed point of F^n.
$\therefore x'$ is a periodic point of F of period $n > 1$ in $[\tilde{x}, x'] \subseteq (0, x_0)$.

Assume $F^n(x') < x'$.
Define $I(x) = F^n(x) - x$. Since $F^n(\tilde{x}) > \tilde{x}$ and

$F^n(x') < x'$, $I(\tilde{x}) = F^n(\tilde{x}) - \tilde{x} > 0$ and
$I(x') = F^n(x') - x' < 0$. Hence, by the Intermediate Value Theorem, $\exists x^* \in (\tilde{x}, x')$ such that
$I(x^*) = F^n(x^*) - x^* = 0$.
$\therefore F^n(x^*) = x^*$ and x^* is a fixed point of F^n in $(\tilde{x}, x')$.
$\therefore x^*$ is a periodic point of F of period $n > 1$ in $[\tilde{x}, x'] \subseteq (0, x_0)$.
$\therefore F$ has a periodic point of period $n > 1$ in $[\tilde{x}, x'] \subseteq (0, x_0)$.
$\therefore F$ has another periodic orbit.
$\therefore$ If F has no other periodic orbits, then the fixed point x_0 is globally attracting. □

With the above lemma, it is now evident what needs to be true about F and G with various values of $1 < A, B < 4$. First off, it is necessary to prove the existence of such fixed points for F and G.

Lemma 2. *F has a fixed point x_0 in the open interval* $(0, 1)$.

Proof. Let h(x)=x.
Both $F(0) = 0 = h(0)$.
Furthermore,
$F'(0) = g'(f(0)) \cdot f'(0)$ by the chain rule
$= g'(0) \cdot f'(0)$
$= A \cdot B > 1 = h'(0)$ for A and B as specified.
So by the Racehorse Theorem, there exists a neighborhood about 0 such that $(*)$ for $x \in (0, \varepsilon)$, $F(x) > x$ for some $0 < \varepsilon < 1$.
Let $c \in (0, \varepsilon)$.
Define $I(x) = F(x) - x$.
$I(c) > 0$ by $(*)$
$I(1) = F(1) - 1 = -1 < 0$
Thus by the Intermediate Value Theorem, there exists a point $m \in (c, 1) \subset (0, 1)$ such that $I(m) = F(m) - m = 0$, and hence $F(m) = m$.
$\therefore F$ has a fixed point $x_0 \in (0, 1)$. □

Lemma 3. *G has a fixed point in the open interval* $(0, 1)$.

Proof. By the same argument as Lemma 2, G has a fixed point in the interval $(0, 1)$. □

Knowing such fixed points exist, it is important to understand how these points are related under the alternate composition of f with g. Although the below lemma is written for the alternate composition of only two functions, it can be extended to the periodic composition of n functions of the type described in Lemma 1 with the same concepts and techniques.

Lemma 4. *If x_0 is the only fixed point of F, then $f(x_0)$ is the only fixed point of G.*

Proof. Let x_0 be the only fixed point of F.

Claim: $f(x_0)$ is a fixed point of G.

Consider $G(f(x_0))$:

$$\begin{aligned} G(f(x_0)) &= f(g(f(x_0))) \\ &= f(F(x_0)) \\ &= f(x_0) \end{aligned}$$

since x_0 is a fixed point of F.

Thus, $f(x_0)$ is a fixed point of G.

Claim: $f(x_0)$ is the only fixed point of G.

Let $\bar{x}$ be any fixed point of G.

Consider $F(g(\bar{x}))$:

$$\begin{aligned} F(g(\bar{x})) &= g(f(g(\bar{x}))) \\ &= g(G(\bar{x})) \\ &= g(\bar{x}) \end{aligned}$$

since $\bar{x}$ is a fixed point of G.

Thus, $g(\bar{x})$ is a fixed point of F and $x_0 = g(\bar{x})$.

Consequently,

$$\begin{aligned} f(x_0) &= f(g(\bar{x})) \\ &= G(\bar{x}) \\ &= \bar{x} \end{aligned}$$

since $\bar{x}$ is a fixed point of G.

This makes $f(x_0)$ the only fixed point of G. □

It is interesting to note that the maximum value of F is independent of our value of A; it depends solely on the chosen value of B. This will come into play in the various calculations to come.

Lemma 5. *F has a maximum at $x = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}$ for $A > 2$ with $F(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}) = \frac{B}{4}$.*

Proof. F will have a local maximum where $F'(x) = 0$.

$$F(x) = ABx(1-x)\big(1 - Ax(1-x)\big)$$

$= ABx\big(1-(A+1)x+2Ax^2-Ax^3\big)$
$= ABx\cdot p(x)$ from Claim 1.
Thus, by the Chain Rule,
$F'(x) = AB\big(p(x)+xp'(x)\big)$
$= AB\Big(1-(A+1)x+2Ax^2-Ax^3+x\big(-(A+1)+4Ax-3Ax^2\big)\Big)$
$= AB\big(1-2(A+1)x+6Ax^2-4Ax^3\big)$
Hence, we are interested in those x values for which
$1-2(A+1)x+6Ax^2-4Ax^3=0$
$(x-\frac{1}{2})(-4Ax^2+4Ax-2)=0$
$(x-\frac{1}{2})(-4A)(x^2-x+\frac{1}{2A})=0.$
Using the quadratic equation yields the roots $x=\frac{1}{2}$ or
$x=\frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{2}{A}}$ for $A>2$.
Evaluating F at these values of x gives
$F(\frac{1}{2}) = AB(\frac{1}{2})(1-\frac{1}{2})(1-A(\frac{1}{2})(1-\frac{1}{2}))$
$= \frac{AB}{4}(\frac{4-A}{4})$
$= \frac{AB(4-A)}{16}$
$F(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}})$
$= AB\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big(1-\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)\cdot$
$\Big(1-A\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big(1-\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)\Big)$
$= AB\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\cdot$
$\Big(1-A\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)$
$= AB\big(\frac{1}{2A}\big)\big(1-A\frac{1}{2A}\big)$
$= \frac{B}{4}$
$F(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}})$
$= AB\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big(1-\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)\cdot$
$\Big(1-A\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big(1-\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)\Big)$
$= AB\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\cdot$
$\Big(1-A\big(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\big(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}\big)\Big)$
$= AB\big(\frac{1}{2A}\big)\big(1-A\frac{1}{2A}\big)$
$= \frac{B}{4}$
Thus F has a local maximum at $x=\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2}{A}}$. Since $F(0)=$

$F(1) = 0 < \frac{B}{4}$, $\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{B}{4}\right)$ is also an absolute maximum of F on $x \in (0, 1)$. □

Below is a rather lengthy lemma that will establish our sufficiency conditions. The concept will work as follows: After choosing a value of $A > 3$, a bound $\mathbf{P}_B$ will be found such that for values of B falling between 1 and $\mathbf{P}_B$ will guarantee one requirement established in Lemma 1 as well as conditions necessary to ensure the other properties. This bound is determined by finding values of B for which each of the conditions is met separately and by then putting together a piecewise function that will guarantee the properties will happen simulataneously.

Lemma 6. *There exists $A' \in (3, 4)$ such that for $1 < B < \mathbf{P}_B(A)$ with*

$$\mathbf{P}_B(A) = \begin{cases} 2 & A \in (3, A') \\ \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2} & A \in (A', 4) \end{cases}$$

F has the following properties:

(i) $F(x)$ has exactly one fixed point $x_0 \in (0, 1)$

(ii) $|F'(x_0)| < 1$

(iii) F has a maximum value of $\frac{B}{4} < \frac{1}{2}$.

Proof. Claim 1: For $0 < B < \frac{9-2A+(6-2A)\sqrt{1-3/A}}{4-A}$ with $A \in (3, 4)$, $F(x) = ABx(1-x)\big(1 - Ax(1-x)\big)$ has only one fixed point in $(0, 1)$.

F has fixed points exactly where

$F(x) = x$

$ABx(1-x)\big(1 - Ax(1-x)\big) = x$

$ABx\big(1 - (A+1)x + 2Ax^2 - Ax^3\big) = x$ and thus

$1 - (A+1)x + 2Ax^2 - Ax^3 = \frac{1}{AB}$ since $A, B, x \neq 0$.

Define $p(x) = 1 - (A+1)x + 2Ax^2 - Ax^3$.

Facts about $p(x)$:

[1a] $p(0) = 1$

[1b] $p(1) = 1 - A - 1 + 2A - A = 0$

[1c] p has an inflection point at $x = \frac{2}{3}$:

$p'(x) = -(A+1) + 4Ax - 3Ax^2$

$p''(x) = 4A - 6Ax$

p has an inflection point where

$p''(x) = 0$

$4A - 6Ax = 0$

$4A = 6Ax$
$x = \frac{2}{3}$

[1d] p has a local maximum and minimum $\in (0,1)$ for $A > 3$:

$$p'(\tfrac{2}{3}) = -(A+1) + 4A(\tfrac{2}{3}) - 3A(\tfrac{2}{3})^2$$
$$= -A - 1 + (\tfrac{8}{3})A - (\tfrac{4}{3})A$$
$$= -1 + (\tfrac{1}{3})A$$

$p'(\frac{2}{3}) < 0$ for all $A < 3$
$p'(\frac{2}{3}) > 0$ for all $A > 3$

[1e] The local minimum and maximum occur at
$x = \frac{2-\sqrt{1-\frac{3}{A}}}{3}$ and $x = \frac{2+\sqrt{1-\frac{3}{A}}}{3}$ respectively:

Local minimums and maximums occur where
$p'(x) = -(A+1) + 4Ax - 3Ax^2 = 0.$
The quadratic formula yields

$$x = \frac{-4A \pm \sqrt{16A^2 + 4(-3A)(A+1)}}{-6A}$$
$$= \frac{2}{3} \pm \frac{\sqrt{16A^2 - 12A^2 - 12A}}{-6A}$$
$$= \frac{2}{3} \pm \frac{2\sqrt{A^2 - 3A}}{-6A}$$
$$= \frac{2 \pm \sqrt{1-\frac{3}{A}}}{3} \quad \text{for } A > 3.$$

Since, by [1d], p is increasing at $x = \frac{2}{3}$, there is a local minimum at $\frac{2-\sqrt{1-\frac{3}{A}}}{3}$ and a local maximum at $x = \frac{2+\sqrt{1-\frac{3}{A}}}{3}$.

[1f] The local maximum is $\left(\frac{2+\sqrt{1-\frac{3}{A}}}{3}, \frac{9-2A-6\sqrt{1-\frac{3}{A}}+2A\sqrt{1-\frac{3}{A}}}{27}\right)$:

$$p\left(\frac{2+\sqrt{1-\frac{3}{A}}}{3}\right)$$
$$= 1-(A+1)\left[\frac{2+\sqrt{1-\frac{3}{A}}}{3}\right]+2A\left[\frac{2+\sqrt{1-\frac{3}{A}}}{3}\right]^2-A\left[\frac{2+\sqrt{1-\frac{3}{A}}}{3}\right]^3$$
$$= 1 - \frac{2}{3}A - \frac{1}{3}A\sqrt{1-\frac{3}{A}} - \frac{2}{3} - \frac{1}{3}\sqrt{1-\frac{3}{A}} + \frac{8}{9}A +$$
$$\frac{8}{9}A\sqrt{1-\frac{3}{A}} + \frac{2}{9}A - \frac{2}{3} - A\left(\frac{4+4\sqrt{1-\frac{3}{A}}+(1-\frac{3}{A})}{9}\right)\left(\frac{2+\sqrt{1-\frac{3}{A}}}{3}\right)$$
$$= 1 - \frac{18}{27}A - \frac{9}{27}A\sqrt{1-\frac{3}{A}} - \frac{2}{3} - \frac{9}{27}\sqrt{1-\frac{3}{A}} + \frac{24}{27}A +$$
$$\frac{24}{27}A\sqrt{1-\frac{3}{A}} + \frac{6}{27}A - \frac{18}{27} - \frac{8}{27}A - \frac{12}{27}A\sqrt{1-\frac{3}{A}} -$$
$$\frac{6}{27}A + \frac{18}{27} - \frac{1}{27}A\sqrt{1-\frac{3}{A}} + \frac{3}{27}\sqrt{1-\frac{3}{A}}$$
$$= \frac{9-2A-6\sqrt{1-\frac{3}{A}}+2A\sqrt{1-\frac{3}{A}}}{27}.$$

Thus, for $F(x)$ to have exactly one fixed point, by [6a],

$$\frac{1}{AB} > \frac{9-2A-6\sqrt{1-\frac{3}{A}}+2A\sqrt{1-\frac{3}{A}}}{27}$$

$$\frac{1}{B} > \frac{9A-2A^2-6A\sqrt{1-\frac{3}{A}}+2A^2\sqrt{1-\frac{3}{A}}}{27}$$

$$B < \frac{27}{9A-2A^2-6A\sqrt{1-\frac{3}{A}}+2A^2\sqrt{1-\frac{3}{A}}}$$

$$= \frac{27}{A(9-2A)-(6A-2A^2)\sqrt{1-\frac{3}{A}}}$$

$$= \frac{27}{A(9-2A)-(6A-2A^2)\sqrt{1-\frac{3}{A}}} \cdot \frac{A(9-2A)+(6A-2A^2)\sqrt{1-\frac{3}{A}}}{A(9-2A)+(6A-2A^2)\sqrt{1-\frac{3}{A}}}$$

$$= \frac{27A\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{A^2(9-2A)^2-(6A-2A^2)^2(1-\frac{3}{A})}$$

$$= \frac{27A\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{A^2(81-36A+4A^2)-A(36A-24A^2+4A^3)(1-\frac{3}{A})}$$

$$= \frac{27\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{81A-36A^2+4A^3-36A+24A^2+4A^3+108-72A+12A^2}$$

$$= \frac{27\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{-27A+108}$$

$$= \frac{9-2A+(6-2A)\sqrt{1-\frac{3}{A}}}{4-A}, \qquad A \neq 4$$

with $\lim\limits_{A\to 4^-} \frac{9-2A+(6-2A)\sqrt{1-\frac{3}{A}}}{4-A}$

$$= \lim_{A\to 4^-} \frac{-2+(6-2A)(\frac{1}{2})\left(\frac{1}{\sqrt{1-\frac{3}{A}}}\right)(\frac{3}{A^2})-2\sqrt{1-\frac{3}{A}}}{-1}$$

by L'Hôpital's Rule

$$= \lim_{A\to 4^-} 2 - \frac{3(6-2A)}{2A^2\sqrt{1-\frac{3}{A}}} + 2\sqrt{1-\frac{3}{A}}$$

$$= 2 - \frac{3\left(6-2(4)\right)}{2(4)^2\sqrt{1-\frac{3}{4}}} + 2\sqrt{1-\frac{3}{4}}$$

$$= 2 + \frac{3}{8} + 1$$

$$= \frac{27}{8}$$

$$= 3.375$$

$\therefore$ For $0 < B < \frac{9-2A+(6-2A)\sqrt{1-\frac{3}{A}}}{4-A}$ with $A \in (3,4)$,

$F(x) = ABx(1-x)\big(1-Ax(1-x)\big)$ has exactly one fixed point.

Claim 2: For $0 < B < \frac{8}{4A-A^2}$ with $A \in (0,4)$, F has a fixed point in $(0, \frac{1}{2})$.

Define $I(x) = F(x) - x$. By the proof for Lemma 2, there exists an $0 < \varepsilon < 1$ such that for $x \in (0,\varepsilon)$, $F(x) > x$. Let $c \in (0,\varepsilon)$. Thus, $I(c) > 0$.

For $0 < B < \frac{8}{4A-A^2}$,

$8 > B(4A - A^2)$ since $4A - A^2 > 0$ for all $A \in (0,4)$

$$0 > AB(4-A) - 8$$

$$= 4AB\big(1 - \tfrac{1}{4}A\big) - 8$$

$$= 16\Big[\tfrac{1}{4}AB\big(1-\tfrac{1}{4}A\big) - \tfrac{1}{2}\Big]$$

$$= 16\Big[AB(\tfrac{1}{2})(1-\tfrac{1}{2})\big(1-A(\tfrac{1}{2})(1-\tfrac{1}{2})\big) - \tfrac{1}{2}\Big]$$

$= 16\Big[F(\tfrac{1}{2}) - \tfrac{1}{2}\Big]$
$= 16I(\tfrac{1}{2})$

Thus, $I(\frac{1}{2}) < 0$.

By the Intermediate Value Theorem, $\exists\ m \in (0, \frac{1}{2})$ such that $I(m) = F(m) - m = 0$, and hence, $F(m) = m$.

$\therefore$ F has a fixed point in $(0, \frac{1}{2})$ for $0 < B < \frac{8}{4A - A^2}$ with $A \in (0, 4)$.

Claim 3: $F'(x) < 1$ when the fixed point $x_0 \in (0, \frac{1}{2})$.

From the proof from Claim 1,
$F(x) = ABx \cdot p(x)$ with $p(x) = 1 - (A+1)x + 2Ax^2 - Ax^3$.
So by the product rule, $F'(x) = AB\big(p(x) + xp'(x)\big)$.
For $x = x_0$,

$F(x_0) = ABx_0 p(x_0)$
$p(x_0) = \frac{F(x_0)}{ABx_0}$
$= \frac{1}{AB}$ since $F(x_0) = x_0$

Thus, by substitution,
$F'(x_0) = AB\big(\frac{1}{AB} + x_0 p'(x_0)\big)$
$= 1 + ABx_0 p'(x_0)$.

By [1d], $p'(x) < 0$ for all $x \in (0, \frac{2}{3})$. Hence, $p'(x_0) < 0$ and $ABx_0 p'(x_0) < 0$.

$\therefore F'(x_0) < 1$.

Claim 4: For $0 < B < \frac{3\sqrt{3 - \frac{6}{A}}}{(A-2)^2}$ with $A \in (3, 4)$ and the fixed point $x_0 \in (0, \frac{1}{2})$, $F'(x_0) > -1$.

By calculations from Lemma 5,
$F'(x) = AB\big(1 - 2(A+1)x + 6Ax^2 - 4Ax^3\big)$.
For $F'(x) > -1$,
$AB\big(1 - 2(A+1)x + 6Ax^2 - 4Ax^3\big) > -1$ and thus,
$1 - 2(A+1)x + 6Ax^2 - 4Ax^3 > \frac{-1}{AB}$ since $A, B \neq 0$.
Define $q(x) = 1 - 2(A+1)x + 6Ax^2 - 4Ax^3$.
Facts about $q(x)$:

[4a] $q(0) = 1$
[4b] $q(1) = 1 - 2A - 2 + 6A - 4A = -1$
[4c] q has an inflection point at $x = \frac{1}{2}$:

$q'(x) = -2(A+1) + 12Ax - 12Ax^2$
$q''(x) = 12A - 24Ax$
q has an inflection point where $q''(x) = 0$

$12A - 24Ax = 0$
$12A = 24Ax$
$x = \frac{1}{2}$

[4d] $q(\frac{1}{2}) = 1 - 2(A+1)(\frac{1}{2}) + 6A(\frac{1}{2})^2 - 4A(\frac{1}{2})^3 = 0$

[4e] q has a local minimum $\in (0, \frac{1}{2})$ for $A > 2$:

$$q'(0) = -2(A+1) < 0$$

$$\begin{aligned} q'(\tfrac{1}{2}) &= -2(A+1) + 12A(\tfrac{1}{2}) - 12A(\tfrac{1}{2})^2 \\ &= -2A - 2 + 6A - 3A \\ &= A - 2 \end{aligned}$$

$q'(\frac{1}{2}) < 0$ for all $A < 2$

$q'(\frac{1}{2}) > 0$ for all $A > 2$

Thus, by the Intermediate Value Theorem, $\exists\, c \in (0, \frac{1}{2})$ such that $q'(c) = 0$ when $A > 2$, and c is a local minimum.

[4f] The local minimum occurs at $x = \frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}$:

Local minimums occur where $q'(x) = 0$

$$-2(A+1) + 12Ax - 12Ax^2 = 0$$

$$-(A+1) + 6Ax - 6Ax^2 = 0$$

The quadratic formula yields

$$\begin{aligned} x &= \frac{-6A \pm \sqrt{36A^2 + 4(-6A)(A+1)}}{-12A} \\ &= \frac{1}{2} \pm \frac{\sqrt{36A^2 - 24A^2 - 24A}}{-12A} \\ &= \frac{1}{2} \pm \frac{\sqrt{12A^2 - 24A}}{-12A} \\ &= \frac{1}{2} \pm \frac{1}{6}\sqrt{3 - \frac{6}{A}} \qquad \text{for } A > 2. \end{aligned}$$

Since, by [4e], the local minimum is in $(0, \frac{1}{2})$, it must occur at $x = \frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}$.

[4g] The local minimum is $\left(\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}, \frac{1}{9}(2 - A)\sqrt{3 - \frac{6}{A}}\right)$:

$$q\left(\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}\right)$$

$$= 1 - 2(A+1)\left[\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}\right] + 6A\left[\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}\right]^2 - 4A\left[\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}\right]^3$$

$$= 1 + (-2A - 2)\left[\frac{1}{2} - \frac{1}{6}\sqrt{3 - \frac{6}{A}}\right] + 6A\left[\frac{1}{4} - \frac{1}{6}\sqrt{3 - \frac{6}{A}} + \frac{1}{36}(3 - \frac{6}{A})\right] - 4A\left[\frac{1}{8} - \frac{1}{8}\sqrt{3 - \frac{6}{A}} + \frac{1}{24}(3 - \frac{6}{A}) - \frac{1}{216}(3 - \frac{6}{A})\sqrt{3 - \frac{6}{A}}\right]$$

$$= 1 - A + \frac{1}{3}A\sqrt{3 - \frac{6}{A}} - 1 + \frac{1}{3}\sqrt{3 - \frac{6}{A}} + \frac{3}{2}A - A\sqrt{3 - \frac{6}{A}} + \frac{1}{6}A(3 - \frac{6}{A}) - \frac{1}{2}A + \frac{1}{2}A\sqrt{3 - \frac{6}{A}} - \frac{1}{6}A(3 - \frac{6}{A}) + \frac{1}{54}A(3 - \frac{6}{A})\sqrt{3 - \frac{6}{A}}$$

$= (\frac{-1}{9})A\sqrt{3-\frac{6}{A}} + \frac{2}{9}\sqrt{3-\frac{6}{A}}$
$= \frac{1}{9}(2-A)\sqrt{3-\frac{6}{A}}$

[4h] $\frac{1}{9}(2-A)\sqrt{3-\frac{6}{A}}$ is the absolute minimum of q on $(0,\frac{1}{2})$:

By [4e] and [4f] it is conclusive that q is decreasing on $\left(0, \frac{1}{2}-\frac{1}{6}\sqrt{3-\frac{6}{A}}\right)$ and increasing on $\left(\frac{1}{2}-\frac{1}{6}\sqrt{3-\frac{6}{A}}, \frac{1}{2}+\frac{1}{6}\sqrt{3-\frac{6}{A}}\right) \supseteq \left(\frac{1}{2}-\frac{1}{6}\sqrt{3-\frac{6}{A}}, \frac{1}{2}\right)$.

Thus $\frac{1}{9}(2-A)\sqrt{3-\frac{6}{A}}$ is the absolute minimum of q on $(0,\frac{1}{2})$ and is negative for $A > 2$.

Thus to ensure $F'(x_0) > -1$, by [4h]

$\frac{-1}{AB} < \frac{1}{9}(2-A)\sqrt{3-\frac{6}{A}}$

$\frac{1}{B} > \frac{-1}{9}A(2-A)\sqrt{3-\frac{6}{A}}$ which is positive for $A > 2$

$B < \frac{9}{A(A-2)\sqrt{3-\frac{6}{A}}}$
$= \frac{9\sqrt{3-\frac{6}{A}}}{A(A-2)(3-\frac{6}{A})}$
$= \frac{9\sqrt{3-\frac{6}{A}}}{3A^2-6A-6A+12}$
$= \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$

Claim 5: For $A \in (3,4)$, when $B < \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$, F automatically has only one fixed point $x_0 \in (0,1)$ and $F'(x_0) > -1$.

By Claim 4, $B < \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ for $A \in (3,4)$ guarantees $F'(x_0) > -1$.
By Claim 1, F has only one fixed point $x_0 \in (0,1)$ when $B < \left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)/(4-A)$. Define $g(A) = \frac{9-2A+(6-2A)\sqrt{1-\frac{3}{A}}}{4-A}$ and $h(A) = \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$.

Facts about g and h:

[5a] $g(3) = \frac{9-2(3)+\left(6-2(3)\right)\sqrt{1-\frac{3}{3}}}{4-3} = 3$

$h(3) = \frac{3\sqrt{3-\frac{6}{3}}}{(3-2)^2} = 3$

Thus, $g(3) = h(3)$.

[5b] $g'(A) > 0$ for all $A \in (3,4)$:

$$g'(A) = \frac{(4-A)\left(-2+(6-2A)(\frac{1}{2})\left(\frac{1}{\sqrt{1-\frac{3}{A}}}\right)(\frac{3}{A^2})-2\sqrt{1-\frac{3}{A}}\right)-(-1)\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{(4-A)^2}$$

$$= \frac{(4-A)\left(-2+\frac{3(3-A)}{A^2\sqrt{1-\frac{3}{A}}}-2\sqrt{1-\frac{3}{A}}\right)+\left(9-2A+(6-2A)\sqrt{1-\frac{3}{A}}\right)}{(4-A)^2}$$

$$= \frac{-8+\frac{12(3-A)}{A^2\sqrt{1-\frac{3}{A}}}-8\sqrt{1-\frac{3}{A}}+2A-\frac{3A(3-A)}{A^2\sqrt{1-\frac{3}{A}}}+2A\sqrt{1-\frac{3}{A}}+9-2A+6\sqrt{1-\frac{3}{A}}-2A\sqrt{1-\frac{3}{A}}}{(4-A)^2}$$

$$= \frac{1-2\sqrt{1-\frac{3}{A}}+\frac{12(3-A)}{A^2\sqrt{1-\frac{3}{A}}}-\frac{3(3-A)}{A^2\sqrt{1-\frac{3}{A}}}}{(4-A)^2}\cdot\frac{A^2\sqrt{1-\frac{3}{A}}}{A^2\sqrt{1-\frac{3}{A}}}$$

$$= \frac{A^2\sqrt{1-\frac{3}{A}}-2A^2\left(1-\frac{3}{A}\right)+12(3-A)-3A(3-A)}{A^2(4-A)^2\sqrt{1-\frac{3}{A}}}$$

$$= \frac{A^2\sqrt{1-\frac{3}{A}}-2A^2+6A+36-12A-9A+3A^2}{A^2(4-A)^2\sqrt{1-\frac{3}{A}}}$$

$$= \frac{A^2\sqrt{1-\frac{3}{A}}+A^2-15A+36}{A^2(4-A)^2\sqrt{1-\frac{3}{A}}}$$

Define $c = \sqrt{1-\frac{3}{A}} > 0$ for $A \in (3,4)$. Note

$c_{max} = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$ and $c_{min} = \sqrt{1-\frac{3}{3}} = 0$

on $A \in (3,4)$. Then

$g'(A) = \frac{(c+1)A^2-15A+36}{cA^2(A-4)^2}$

$cA^2(A-4)^2$ is obviously positive for all $A \in (3,4)$.

Assume $(c+1)A^2 - 15A + 36 < 0$.

By the quadratic formula,

$$0 > \left[x - \left(\frac{15+3\sqrt{25-16(c+1)}}{2(c+1)}\right)\right]\left[x - \left(\frac{15-3\sqrt{25-16(c+1)}}{2(c+1)}\right)\right]$$

These are distinct roots only when

$25 - 16(c+1) > 0$

$9 > 16c$

$\frac{9}{16} > c = \sqrt{1-\frac{3}{A}}$

$\frac{81}{256} > 1 - \frac{3}{A}$ since $\sqrt{1-\frac{3}{A}} > 0$ for $A \in (3,4)$

$\frac{3}{A} > \frac{175}{256}$

$175A < 768$

$A < \frac{768}{175} = 4 + \frac{68}{175}$,

so they are distinct for $A \in (3,4)$. For the product to be negative with $A \in (3,4)$, then either

(i) $A > \frac{15+3\sqrt{25-16(c+1)}}{2(c+1)} > \frac{15+3\sqrt{25-16(c_{min}+1)}}{2(c_{max}+1)} = 8$

and

$A < \frac{15-3\sqrt{25-16(c+1)}}{2(c+1)} < \frac{15-3\sqrt{25-16(c_{max}+1)}}{2(c_{min}+1)} = 6$

Thus, $A > 8$ and $A < 6 \Rightarrow\Leftarrow$

or

(ii) $A < \frac{15+3\sqrt{25-16(c+1)}}{2(c+1)}$

and

$A > \frac{15-3\sqrt{25-16(c+1)}}{2(c+1)}$

By (i),

$\frac{15+3\sqrt{25-16(c+1)}}{2(c+1)} > 8$ and $\frac{15-3\sqrt{25-16(c+1)}}{2(c+1)} < 6$

so at the very least, $A \in (6, 8)$.

Thus, consider $A = 7$:

$$\begin{aligned}
&(c+1)(7)^2 - 15(7) + 36 \\
&= \left(\sqrt{1-\tfrac{3}{7}}+1\right)(49) - 69 \\
&= \left(\tfrac{2\sqrt{7}+7}{7}\right)(49) - 69 \\
&= 14\sqrt{7} + 49 - 69 \\
&= 14\sqrt{7} - 20 > 0
\end{aligned}$$

$\Rightarrow\Leftarrow$ Assuming $(c+1)A^2 - 15A + 36 < 0$.

Therefore, $(c+1)A^2 - 15A + 36 > 0$ for all $A \in (3, 4)$, and $g'(A) > 0$ for all $A \in (3, 4)$.

[5c] $h'(A) < 0$ for all $A \in (3, 4)$:

$$\begin{aligned}
h'(A) &= \frac{(A-2)^2(3)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{3-\frac{6}{A}}}\right)\left(\frac{6}{A^2}\right) - 3\sqrt{3-\frac{6}{A}}(2)(A-2)}{(A-2)^4} \\
&= \frac{\frac{9(A-2)^2}{A^2\sqrt{3-\frac{6}{A}}} - 6(A-2)\sqrt{3-\frac{6}{A}}}{(A-2)^4} \\
&= \frac{9(A-2)^2 - 6A^2(A-2)\left(3-\frac{6}{A}\right)}{A^2\sqrt{3-\frac{6}{A}}(A-2)^4} \\
&= \frac{9(A-2) - 6A^2\left(3-\frac{6}{A}\right)}{A^2\sqrt{3-\frac{6}{A}}(A-2)^3} \\
&= \frac{9A-18-18A^2+36A}{A^2(A-2)^3\sqrt{3-\frac{6}{A}}} \\
&= \frac{(-18A^2+45A-18)\sqrt{3-\frac{6}{A}}}{A^2(A-2)^3\left(3-\frac{6}{A}\right)} \\
&= \frac{-9(2A^2-5A+2)\sqrt{3-\frac{6}{A}}}{(3A^2-6A)(A-2)^3} \\
&= \frac{-9(2A-1)(A-2)\sqrt{3-\frac{6}{A}}}{3A(A-2)(A-2)^3} \\
&= \frac{-3(2A-1)\sqrt{3-\frac{6}{A}}}{A(A-2)^3}
\end{aligned}$$

Both $A(A-2)^3$ and $3(2A-1)\sqrt{3-\frac{6}{A}}$ are clearly positive for $A \in (3, 4)$. Thus, $h'(A) = \frac{-3(2A-1)\sqrt{3-\frac{6}{A}}}{A(A-2)^3} < 0$ for $A \in (3, 4)$.

[5d] Combining [5b] and [5c] yields $h'(A) < g'(A)$ for all $A \in (3, 4)$.

Applying the Racehorse Theorem to facts [5a] and [5d] yields $h(A) < g(A)$ for all $A \in (3,4)$. Thus when $B < \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ we are guaranteed exactly one fixed point $x_0 \in (0,1)$ and $F'(x_0) > -1$.

Claim 6: For all $A \in (3,4)$, $\frac{8}{4A-A^2} > 2$.

$0 < (A-2)^2$
$-2(A^2 - 4A + 4) < 0$
$-2A^2 + 8A < 8$
$2 < \frac{8}{4A-A^2}$ since $4A - A^2 > 0$ for all $A \in (3,4)$
$\therefore \frac{8}{4A-A^2} > 2$ for all $A \in (3,4)$.

Claim 7: There exists a point $A' \in (3.2, 3.3)$ [$A' \approx 3.2728898$] such that for $A \in (3, A')$, $2 < \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ and for $A \in (A', 4)$, $2 > \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$.

Define $h(A) = \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ and $i(A) = 2$.
Define $V(A) = h(A) - i(A)$.
Facts about $V(A)$:

[7a] $V(3.2) > 0$:

$h(3.2) = \frac{3\sqrt{3-\frac{6}{3.2}}}{(3.2-2)^2} = \frac{3}{1.2^2}\sqrt{\frac{3.6}{3.2}} \approx 2.2097$
$i(3.2) = 2$
$\therefore V(3.2) > 0$.

[7b] $V(3.3) < 0$:

$h(3.3) = \frac{3\sqrt{3-\frac{6}{3.3}}}{(3.3-2)^2} = \frac{3}{1.3^2}\sqrt{\frac{3.9}{3.3}} \approx 1.95971$
$i(3.3) = 2$
$\therefore V(3.3) < 0$.

Thus, by the Intermediate Value Theorem, $\exists\, A' \in (3.2, 3.3)$ such that $V(A') = h(A') - i(A') = 0$ and hence, $h(A') = i(A')$. Simplification and the quartic equation yields $A' \approx 3.2728898$. By Claim 5 [5c], $h'(A) < 0$ for all $A \in (3,4)$, thus h is monotone decreasing on $(3,4)$. $i'(A) = 0$ for all $A \in (3,4)$. It is therefore evident that A' is the only intersection of h and i in $(3,4)$, and by [7a] and [7b]

$2 < \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ for $A \in (3, A')$ and
$2 > \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2}$ for $A \in (A', 4)$.

Claim 8: For $B < 2$, F has a maximum value $\frac{B}{4} < \frac{1}{2}$.

By Lemma 5, F has a maximum value of $\frac{B}{4}$.
For $B < 2$, $\frac{B}{4} < \frac{2}{4} = \frac{1}{2}$.
$\therefore F$ has a maximum value less than $\frac{1}{2}$.

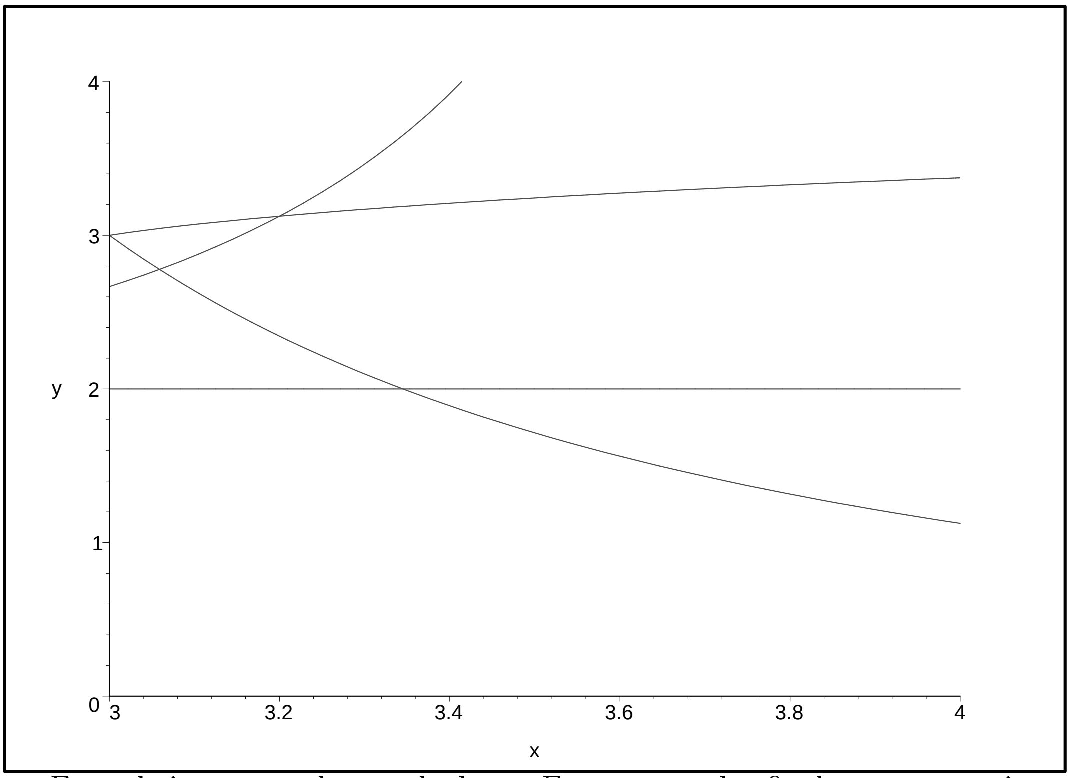
Combining Claims 2, 5, 6, 7, and 8, it is conclusive that for

$1 < B < \mathbf{P}_B$ with

$$\mathbf{P}_B(A) \begin{cases} 2 & A \in (3, A') \\ \frac{3\sqrt{3-\frac{6}{A}}}{(A-2)^2} & A \in (A', 4) \end{cases}$$

F is guaranteed the following properties:
(i) $F(x)$ has exactly one fixed point $x_0 \in (0, 1)$
(ii) $|F'(x_0)| < 1$
(iii) F has a maximum value $\frac{B}{4} < \frac{1}{2}$. □

Below is a graph displaying the various bounds found throughout the Lemma 6. Our function $\mathbf{P}_B(A)$ is then the piecewise function formed by taking, for each $A \in (3, 4)$, the bound with the lowest output value.



Enough is now understood about F to prove the final two properties required to include F in the family of functions that Lemma 1 covers.

Lemma 7. *For $1 < B < \mathbf{P}_B$ and $A \in (3, 4)$,*
(i) $x \in (0, x_0) \Rightarrow F(x) > x$
(ii) $x \in (x_0, 1) \Rightarrow F(x) < x$.

Proof. Proof by Contradiction.

By Lemma 6, F with $1 < B < \mathbf{P}_B$ has exactly one fixed point $x_0 \in (0,1)$.

(i) Assume that $\exists x' \in (0, x_0)$ such that $F'(x') < x'$. By Lemma 2, there exists a neighborhood about 0 such that for $x \in (0, \varepsilon)$, $F(x) > x$ for some $0 < \varepsilon < x_0$. Let $x'' \in (0, \varepsilon)$ and thus, $F(x'') > x''$. Define $S(x) = F(x) - x$. Then

$S(x') < 0$ and

$S(x'') > 0$.

By the Intermediate Value Theorem, there exists an x^* in between x' and x'', $x^* \in (0, x_0)$, such that

$S(x^*) = F(x^*) - x^* = 0$.

Thus, $F(x^*) = x^*$ and x^* is a fixed point, $x^* \neq x_0$.

$\Rightarrow\Leftarrow$ For $1 < B < \mathbf{P}_B$, F has exactly one fixed point in $(0,1)$.

$\therefore x \in (0, x_0) \Rightarrow F(x) > x$.

(ii) Assume that $\exists \bar{x} \in (x_0, 1)$ such that $F(\bar{x}) > \bar{x}$.

$F(1) = 0 < 1$.

Define $S(x) = F(x) - x$. Then

$S(\bar{x}) > 0$ and

$S(1) < 0$.

By the Intermediate Value Theorem, there exists an $x^* \in (\bar{x}, 1)$ such that $S(x^*) = F(x^*) - x^* = 0$. Thus, $F(x^*) = x^*$ and x^* is a fixed point, $x^* \neq x_0$.

$\Rightarrow\Leftarrow$ For $1 < B < \mathbf{P}_B$, F has exactly one fixed point in $(0,1)$.

$\therefore x \in (x_0, 1) \Rightarrow F(x) < x$. □

With the knowledge that F is of the correct type for Lemma 1, it is simply necessary to prove that F has no other periodic points of period $n \in \mathbf{N}$, $n > 1$. This argument is laid out in the next three lemmas.

Lemma 8. *For F with $1 < B < \mathbf{P}_B$,*

(i) $F'(x) \in (-1, 0)$ for all $x \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$

(ii) $F'(x) \in (-1, 0)$ for all $x \in \left(x_0, \frac{1}{2}\right)$

(iii) $F'(x) > 0$ for all $x \in \left(0, \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right)$

Proof. By Lemma 5,

$$F'(x) = -4A^2B\left(x - \tfrac{1}{2}\right)\left(x - \left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{2}{A}}\right)\right)\left(x - \left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{2}{A}}\right)\right)$$

(i) For $x \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$,

$\left(x - \frac{1}{2}\right) < 0$

$\left(x - \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right]\right) < 0$

$\left(x - \left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right]\right) > 0$

$-4A^2B < 0$

Thus, $F'(x) \in (-1, 0)$ for $x \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$ when combined with Lemma 6, Claim 4 [4h].

(ii) Claim: $x_0 \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$

Let $I(x) = F(x) - x$.

$$
\begin{aligned}
B &> 1 \\
&> 2 - 2\sqrt{\tfrac{1}{2}} \\
&> 2 - 2\sqrt{1 - \tfrac{2}{A}} \qquad \text{since } A \in (3, 4) \\
&> 4\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{2}{A}}\right)
\end{aligned}
$$

$\frac{B}{4} - \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2}{A}} > 0$

$F\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right) - \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right) > 0$

$I\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right) > 0$

And $I(\frac{1}{2}) < 0$ by Lemma 6, Claim 2. Thus, by the Intermediate Value Theorem, there exists a point $\bar{x} \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$ such that $I(\bar{x}) = F(\bar{x}) - \bar{x} = 0$ making $\bar{x}$ a fixed point of F. Yet by Lemma 6, F with $1 < B < \mathbf{P}_B$ has exactly one fixed point $x_0 \in (0, 1)$.

$\therefore \bar{x} = x_0$ and $x_0 \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$.

Thus, $\left(x_0, \frac{1}{2}\right) \subset \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$ and $F'(x) \in (-1, 0)$ for $x \in (x_0, \frac{1}{2})$.

(iii) For $x \in \left(0, \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right)$

$(x - \frac{1}{2}) < 0$

$\left(x - \left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right]\right) < 0$

$\left(x - \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right]\right) < 0$

$-4A^2B < 0$

Thus, $F'(x) > 0$ for $x \in \left(0, \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}\right)$. □

Lemma 9. *For all $1 < B < \mathbf{P}_B$ with $A \in (3, 4)$, F has no periodic 2-cycle.*

Proof. F will have a distinct periodic 2-cycle, $\{x_1, x_2\}$, only if $F^2(x) = x$ at $x = x_1, x_2$ with $x_1, x_2 \neq x_0$.

Claim 1: For F to have a periodic 2-cycle, exactly one of $x_1, x_2 \in (0, x_0)$.

Proof by Contradiction:

Assume to the contrary:

Case 1: Both $x_1, x_2 \in (0, x_0)$.

Without loss of generality, let $0 < x_1 < x_2 < x_0$. By Lemma 7, since $x_1, x_2 \in (0, x_0)$, $x_1 < F(x_1) = x_2 < f(x_2) = x_1$

$\Rightarrow\Leftarrow x_1 \not< x_1$

$\therefore x_1$ and x_2 are not both in $(0, x_0)$.

Case 2: Both $x_1, x_2 \in (x_0, 1)$.

Without loss of generality, let $x_0 < x_1 < x_2 < 1$. By Lemma 7, since $x_1, x_2 \in (x_0, 1)$, $x_2 > F(x_2) = x_1 > F(x_1) = x_2$

$\Rightarrow\Leftarrow x_2 \not< x_2$

$\therefore x_1$ and x_2 are not both in $(x_0, 1)$.

$\therefore$ Exactly one of x_1, x_2 are in $(0, x_0)$.

Claim 2: For all $1 < B < \mathbf{P}_B$ with $A \in (3, 4)$, $F^2(x) > x$ for all $x \in (0, x_0)$.

Let $x \in (0, x_0)$.

Case 1: $0 < F(x) \leq x_0$.

By Lemma 6, Claims 1 [1d] and 2,

$$p(F(x)) \geq p(x_0)$$

since p is monotone decreasing on $(0, 2/3)$ and $x_0 \in (0, 1/2)$.

But by Lemma 6, Claim 3, $p(x_0) = \frac{1}{AB}$. Thus,

$$p(F(x)) \geq \frac{1}{AB}$$

$$ABp(F(x)) \geq 1$$

$$ABF(x)p(F(x)) > x \qquad \text{since } F(x) > x \text{ for all } x \in (0, x_0) \text{ by Lemma 7.}$$

$$F(F(x)) > x$$

$$F^2(x) > x$$

Case 2: $x < x_0 < F(x)$.

Since $F(x) \in (x_0, 1)$, by Lemma 7, $F^2(x) < F(x)$.

Subcase 1: $x < F^2(x) < F(x)$

$\therefore F^2(x) > x$

Subcase 2: $F^2(x) < x < x_0 < F(x)$.

$F^2(x)$ is a continuous differentiable function on (x, x_0) by virtue of F. Thus, by the Mean Value Theorem, $\exists \hat{x} \in (x, x_0)$ such that

$$(F^2)'(\hat{x}) = \frac{F^2(x_0) - F^2(x)}{x_0 - x} = \frac{x_0 - F^2(x)}{x_0 - x} > 1$$

by our case assumptions. Yet, $(F^2)'(\hat{x}) = F'(f(\hat{x}))F'(\hat{x})$ by the chain rule.

Subcase a: $\tilde{x} < \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}$

$F'(\hat{x}) > 0$ by Lemma 8 *(iii)*. By case assumptions and Lemma 5, $F(\hat{x}) \in \left(x_0, \frac{B}{4}\right)$. Applying Lemma 6, Claim 7, $\frac{B}{4} \le \frac{1}{2}$. Thus, $F(\hat{x}) \in \left(x_0, \frac{B}{4}\right) \subseteq \left(x_0, \frac{1}{2}\right)$. Thus, by Lemma 8 *(ii)*, $F'(F(\hat{x})) \in (-1, 0)$.

$\therefore (F^2)'(\hat{x}) = F'(F(\hat{x}))F'(\hat{x}) < 0$

$\Rightarrow\Leftarrow (F^2)'(\hat{x}) > 1$.

Subcase b: $\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}} < \hat{x} < x < x_0$.

Since $\hat{x} \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, x_0\right) \subset \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{A}}, \frac{1}{2}\right)$ by Lemma 6, Claim 3, $F'(\hat{x}) \in (-1, 0)$ by Lemma 8. By case assumptions and Lemma 5, $F(\hat{x}) \in \left(x_0, \frac{B}{4}\right)$. Applying Lemma 6, Claim 7, $\frac{B}{4} \le \frac{1}{2}$, thus $F(\hat{x}) \in \left(x_0, \frac{B}{4}\right) \subseteq \left(x_0, \frac{1}{2}\right)$. Thus, by Lemma 8 *(ii)*, $F'(F(\hat{x})) \in (-1, 0)$.

$\therefore (F^2)'(\hat{x}) = F'(F(\hat{x}))F'(\hat{x}) \in (0, 1)$

$\Rightarrow\Leftarrow (F^2)'(\hat{x}) > 1$.

$\therefore F^2(x) \not< x$

$\therefore$ In both cases, $F^2(x) > x$ for all $x \in (0, x_0)$.

Applying Claim 1, it is conclusive that for all $1 < B < \mathbf{P}_B$, F has no distinct periodic 2-cycle. □

Lemma 10. *For all $1 < B < \mathbf{P}_B$, F has no distinct periodic n-cycles for $n \in \mathbf{N}$, $n > 1$.*

Proof. Since, by Lemma 9, F has no distinct periodic 2-cycle for all $1 < B < \mathbf{P}_B$, the contrapositive of Sarkovskii's Theorem rules out all other periodic n-cycles. □

With that proven, we can now confirm that our fixed point x_0 of F is globally attractive on the open interval $(0, 1)$.

Lemma 11. *For all $A \in (3, 3.9)$, $\exists\ B \in (1, 3)$ such that F has a globally attractive fixed point $x_0 \in (0, 1)$.*

Proof. Let $A \in (3, 3.9)$.

Choose $B \in (1, \mathbf{P}_B)$.

Lemmas 6 and 7 combine to ensure F has the following properties:

(i) $x \in (0, x_0) \Rightarrow F(x) > x$

(ii) $x \in (x_0, 1) \Rightarrow F(x) < x$

(iii) $|F'(x_0)| < 1$
(iv) $F(0) = AB(0)(1-0)\big(1 - A(0)(1-0)\big) = 0$

Furthermore, by Lemma 10, F with $1 < B < \mathbf{P}_B$ has no other periodic orbits.

$\therefore$ By Lemma 1, the fixed point x_0 is globally attracting. $\square$

To fully disprove the aforementioned conjecture [3], it must be shown that the knowledge that F has a globally attractive fixed point $x_0 \in (0,1)$ guarantees that the infinite composition of f and g has the globally attractive 2-cycle. This, it turns out, relies mainly on the continuity of f.

Lemma 12. *For every $A \in (3, 3.9)$, $\exists B \in (1,3)$ such that the non-autonomous dynamical system $\{f, g\}$ has a globally attractive 2-cycle, $\{x_0, f(x_0)\}$, where x_0 is the fixed point of F, $x_0 \in (0,1)$.*

Proof. Let $A \in (3, 9)$.
Choose $B \in (1, \mathbf{P}_B)$.
By Lemma 6, F has exactly one fixed point $x_0 \in (0,1)$. Applying Lemma 4 yields that $f(x_0)$ is the only fixed point of G, $f(x_0) \in (0,1)$.
Lemma 10 guarantees that x_0 is globally attractive in F.
Thus, let $x \in (0,1)$.
Fix $\delta > 0$.
$\lim_{n\to\infty} F^n(x) = x_0$, so for every $0 < \varepsilon_1 < x_0$, $\exists k_0 \in \mathbf{N}$ such that for $k \geq k_0$, $F^k(x) \in N(x_0, \varepsilon_1)$.
By the continuity of f, choose $\varepsilon_2 > 0$ small enough such that for $F^m(x) \in N(x_0, \varepsilon_2)$, $f(F^m(x)) \in N(f(x_0), \delta)$.
Fix $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$.
Then for $k \geq k_0$, $f(F^k(x)) \in N(f(x_0), \delta)$ and $\lim_{n\to\infty} f(F^n(x)) = \lim_{n\to\infty} G^n(x) = f(x_0)$.
Thus $\{x_0, f(x_0)\}$ is a globally attractive 2-cycle for $\{f, g\}$.
$\therefore \forall A \in (3, 3.9)$, $\exists B \in (1, \mathbf{P}_B)$ such that the non-autonomous quadratic dynamical system $\{f, g\}$ has the globally attractive 2-cycle $\{x_0, f(x_0)\}$ where x_0 is the fixed point of F, $x_0 \in (0,1)$. $\square$

IV. Further Research

It is worth noting that $1 < B < \mathbf{P}_B$ is a sufficient, but not necessary, condition. Empirical evidence suggests that, in fact, both A and B can be larger than 3 and still have a globally attractive 2-cycle for the non-autonomous dynamical system $\{f, g\}$. For example,

$$\{f(x) = 3.05x(1-x), g(x) = 3.1x(1-x)\}.$$

It is our hope that further research will allow the bound $\mathbf{P}_B$ to be extended upward to take such cases into consideration. The ideal upper bound will, at the very least, include the possibility for $A = 3, B = 3$ since this reduces to $f = g$, and we would be back in the simple case detailed in the introduction.

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