A CHARACTERIZATION OF THE HÉNON MAP FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$ FOR $a > 0$ AND $\frac{-3(1-a)^2}{4} < c < \frac{(1-a)^2}{4}$

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Abstract. In this study we seek to characterize the recurrent dynamics of the map

$$F(x, y) = (y, y^2 + ax + c)$$

from $\mathbb{R}^2$to $\mathbb{R}^2$, for $a \geq 0$, and $c < \frac{(1-a)^2}{4}$. In §3 and §4 we define two regions in the parameter space $\Lambda = \{(a, c)|a \geq 0\}$ that have no chain recurrent points other than the fixed points. In §5 and §6 we explore the dynamics of the rest of Λ to demonstrate that the dynamics in the regions defined in §3 and §4 are unique in Λ. In §7, we conduct a survey of the dynamics of F for $0 < a < 1$ and $c = 0$.

1. Introduction

The map

$$T : (x, y) \mapsto (1 + y - ax^2, bx) \; a = 1.4, b = 0.3 \tag{1.1}$$

presented by Hénon in 1978 as the simplest, smooth map that contains a strange attractor sparked interest in the study of polynomial plane automorphisms. In recent years, there has been a flurry of research activity concerning generalized Hénon maps, which are defined as maps of the form

$$\begin{aligned} &g : (x, y) \mapsto (y, z) = (y, p(y) - \delta x); \\ &\delta = \det Dg \\ &[5] \end{aligned} \tag{1.2}$$

whose importance is confirmed by Friedland and Milnor's discovery that every planar polynomial automorphism is conjugate to a composition of generalized Hénon transformations [5].

This study seeks to utilize the tools developed for planar polynomial automorphisms in $\mathbb{C}^2$by Bedford and Smilie [2][3], Friedland and Milnor[5], and Shafikov and Wolf[9] to characterize the recurrent dynamics of a specific family of Hénon maps in $\mathbb{R}^2$, given by the equation:

$$F_{(a,c)}(x, y) = (y, y^2 + ax + c), \tag{1.3}$$

for $a > 0$. We pay particular attention to the families of functions $F_{(a,c)}$ for which $a > 0$, $a \neq 1$ and $\frac{-3(1-a)^2}{4} < c < \frac{(1-a)^2}{4}$, where we prove that the only chain recurrent (and hence nonwandering and periodic points) are the two fixed points, and then show that these are the only regions that exhibit this behavior for $a \geq 0$.

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In order to punctuate the surprisingly regular behavior of maps in this parameter space, we characterize the recurrent dynamics of the case where $c = 0$ and $0 < a < 1$.

2. Background Information

In this section we list the definitions and theorems necessary for our proofs.

2.1. Definitions.

Definition 2.1. [6] *A map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is* $\mathbf{C^1}$ *if all of its first partial derivatives exist and are continuous. F is* $\mathbf{C}^\infty$ *if all of its mixed k^{th} partial derivatives exist and are continuous for all k.*

Definition 2.2. [8] *A function $f : X \to X$ from a metric space X to itself is called a* ***diffeomorphism*** *provided it is*

(1) *one-to-one*

(2) *onto*

(3) *continuous*

(4) *its inverse $F^{-1} : X \to X$ is continuous.*

Definition 2.3. [6] *Let $F^n(p) = p$*

(1) *p is a* ***sink or attracting periodic point*** *if all of the eigenvalues of $DF^n(p)$ are less than one in absolute value.*
(2) *p is a* ***source or repelling periodic point*** *if all of the eigenvalues of $DF^n(p)$ are greater than one in absolute value.*
(3) *p is a* ***saddle point*** *otherwise, i.e., if some of the eigenvalues of $DF^n(p)$ are larger and some are less than one in absolute value.*

Definition 2.4. [8] *For a map $f : X \to X$ a point p is called* ***nonwandering*** *provided for every neighborhood U of p there is an integer $n > 0$ such that $F^n(U) \cap U \neq \emptyset$. Thus, there is a point $q \in U$ with $F^n(q) \in U$. The set of all nonwandering points for f is called the* ***nonwandering set*** *and is denoted by $\Omega(f)$.*

Definition 2.5. [6]*A closed region $N \subset \mathbb{R}^2$ is a* ***trapping region*** *for F if $F(N)$ is contained in the interior of N.*

Definition 2.6. [3] *The* ***stable and unstable sets*** *of a point* $\mathbf{p}$ *are defined as*

$$W^s(p) = \{q : \lim_{n\to\infty} d(f^n(p), f^n(q)) = 0\},$$
$$W^u(p) = \{q : \lim_{n\to-\infty} d(f^n(p), f^n(q)) = 0\}$$

Definition 2.7. [3] *We can construct a conjugacy invariant, which we call the* ***dynamical degree****, as follows:*

$$d = d(f) = \lim_{n\to\infty} (deg\ f^n)^{\frac{1}{n}} = \lim_{n\to\infty} (deg\ f \circ \cdots \circ f)^{\frac{1}{n}}$$

Definition 2.8. [3]

$$K^{\pm} = \{q \in \mathbb{C}^2 : \{f^{\pm n}(q) : n = 1, 2, 3, \ldots\} \text{ is bounded}\},$$
$$J^{\pm} = \partial K^{\pm}$$
$$K = K^+ \cap K^-$$
$$J = J^+ \cap J^-$$

Definition 2.9. *Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are said to be **topologically conjugate** if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy.*

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics. [6]

Definition 2.10. [8] *Let $r \geq 0$ be an integer. Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^r functions and $J \subset R$ be an interval (usually closed and bounded. Define the C^r **distance from f to g** by*

$$d_{r,J}(f, g) = sup\{|f(x) - g(x)|, |f'(x) - g'(x)|, \cdots, |f^{(r)}(x) - g^{(r)}(x)| : x \in J\}.$$

Definition 2.11. [6] *Assume $r \geq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^r function. A function f is C^r **structurally stable** provided there exists an $\epsilon > 0$ such that f is conjugate to g on all of $\mathbb{R}$ whenever $g : \mathbb{R} \to \mathbb{R}$ is a C^r-function with $dr, \mathbb{R}(f, g) < \epsilon$. A function f is said to be* structurally stable *provided it is C^1 structurally stable.*

Definition 2.12. [6] *Let F be a diffeomorphism. A point c is **chain recurrent** for F, if, for any $\epsilon > 0$, there are points $x = x_0, x_1, x_2, \ldots x_k = x$ and positive integers $n_1, \ldots n_k$ such that*

$$|F^{n_i}(x_{i-1}) - x_i| < \epsilon$$

for each i.

Definition 2.13. [1] *Let f be a C^r diffeomorphism on a C^∞ manifold M, with the uniform C^r topology, $1 \leq r \leq \infty$. Then f satisfies **Axiom A** if and only if:*

1. *$\Omega(f)$ has hyperbolic structure, and*
2. *The periodic points are dense in $\Omega(f)$.*

2.2. Theorems.

Theorem 2.1. [6] *Let F be a diffeomorphism in $\mathbb{R}^2$ Suppose F has a saddle point at p, Then there exists $\epsilon > 0$ and a smooth curve, i.e., a C^1 curve*

$$\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$$

such that

1. *$\gamma(0) = p$.*
2. *$\gamma'(t) \neq 0$*
3. *$\gamma'(0)$ is an unstable eigenvector for $DF(p)$.*
4. *$\gamma$is F^{-1}invariant.*
5. *$F^{-n}\gamma(t) \to p$ as $n \to \infty$.*
6. *If $|F^{-n}(q) - p| < \epsilon$ for all $n \geq 0$ then $q = \gamma(t)$ for some t.*

The curve γ is called the *local unstable manifold at p.* This implies that the stable and unstable manifolds are smooth curves that emanate from the fixed or periodic point.

Theorem 2.2. [3] *Let f be a polynomial diffeomorphism of $\mathbb{C}^2$satisfying $d(f) > 1$. Let p be a saddle point of f. Then J^+ is the closure of the stable manifold $W^s(p)$ and J^- is the closure of the stable manifold $W^s(p)$ and J^- is the closure of $W^u(p)$.*

Theorem 2.3. [3] *when $|det\ Df| < 1$ the chain recurrent set of f is equal to the set of bounded orbits (in forward and backwards time) not in punctured basins.*

Theorem 2.4. [5] *If g is hyperbolic, then the periodic points are dense in J, i.e.*

$$\overline{Per(g|_J)} = J.$$

Theorem 2.5. [2] *If g is a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$, then it satisfies Axiom A.*

Theorem 2.6. [9] *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$ and let p be a point in C^n. Then one of the following exclusive properties hold*

1. *There exists $q \in J$ such that $|f^k(p) - f^k(q)| \to 0$ as $k \to \infty$;*
2. *There exists an attracting periodic point α of f such that $|f^k(p) - f^k(\alpha)| \to 0$ as $k \to \infty$*
3. *$\{f^k(p) : k \in \mathbb{N}\}$ converges to ∞ as $k \to \infty$.*

Theorem 2.7. [9] *Let f be a hyperbolic regular polynomial automorphism of $\mathbb{C}^n$ with $|det\ Df| \leq 1$. Then*

1. *$W^s(J) = J^+$;*
2. *$W^u(J) = J^- \setminus \{\alpha_1, \ldots, \alpha_m\}$, where the α_i are the attracting periodic points of f;*
3. *$\Omega(f) = J \cup \{\alpha_1, \ldots, \alpha_m\}$.*

Theorem 2.8. [9] *Let f be a hyperbolic regular polynomial automorphism of C^n. Then f is Axiom A.*

Theorem 2.9. [2] *If $|detDg| = 1$ then $int\ K^+ = int\ K^- = int\ K$. If $|detDg| < 1$ then $int\ K^- = \emptyset$,If $|detDg| > 1$ then $int\ K^+ = \emptyset$.*

Theorem 2.10. [2] $g : (x, y) \mapsto (y, z) = (y, p(y) - ax)$
there exists a constant $R > 0$ so that $|y| > R$ implies that either $|z| > |y|$ or $|x| > |y|$ or both.

3. Results for $0 < a < 1, \frac{-(1-a)^2}{4} < c < \frac{(1-a)^2}{4}$

Let $P = \{(a, c)|0 < a < 1, \frac{-3(1-a)^2}{4} < c < \frac{(1-a)^2}{4}\}$ and let $F_P = \{F_{(a,c)}(x, y)|(a, c) \in P\}$. From here on we shall simply refer to $F_{(a,c)}$ as F.
Note that in F_P,

$$\mathbf{DF}(x, y) = \begin{pmatrix} 0 & 1 \\ a & 2y \end{pmatrix} \tag{3.1}$$

So $|\det DF(x, y)| = |a| = a < 1$.
In order to apply the theorems in §2, we must first understand the behavior of fixed points of the maps in F_P.
The eigenvalues at a point(x, y) are given by

$$\begin{aligned} \lambda_1 &= y + \sqrt{(y^2 + a)} \\ \lambda_2 &= y - \sqrt{(y^2 + a)} \end{aligned} \tag{3.2}$$

And if (x_f, y_f) is a fixed point then

$$\begin{aligned} x_f &= y_f \\ y_f &= y_f^2 + ax_f + c \end{aligned} \tag{3.3}$$

So

$$\begin{aligned} &y_f = y_f^2 + ay_f + c \\ &0 = y_f^2 + (a-1)y_f + c \\ &\Rightarrow x_f = y_f = \tfrac{(1-a)\pm\sqrt{(1-a)^2-4c}}{2} \end{aligned}$$

Denote the fixed points by (x_+, y_+), (x_-, y_-).

$$\begin{aligned} x_+ = y_+ &= \tfrac{(1-a)+\sqrt{(1-a)^2-4c}}{2} \\ x_- = y_- &= \tfrac{(1-a)-\sqrt{(1-a)^2-4c}}{2} \end{aligned} \tag{3.4}$$

Through computation, we show that the character of the fixed points is similar for all maps in F_P, which allows us apply the theorems in §2 in a systematic manner.

Lemma 3.1. *If* $\frac{(1-a)^2}{4} > c > 0$ *then* F *has an attracting fixed point,* t*, and a fixed saddle point* p*.*

Proof. Since $c > 0$, $\sqrt{(1-a)^2 - 4c} < 1 - a$. Hence
$0 < y_+ = \frac{(1-a)+\sqrt{(1-a)^2-4c}}{2} < \frac{2(1-a}{2} < 1$ and
$y_- = \frac{(1-a)-\sqrt{(1-a)^2-4c}}{2} > 0$ yet
$y_- = \frac{(1-a)-\sqrt{(1-a)^2-4c}}{2} < \frac{1-a}{2}$.
Suppose $0 < y \le 1$. Then

$$\begin{aligned} &y + \sqrt{y^2 + a} \le 1 \\ &\Rightarrow \sqrt{y^2 + a} \le 1 - y \\ &\Rightarrow y^2 + a \le 1 - 2y + y^2 \\ &\Rightarrow y \le \tfrac{1-a}{2} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &y - \sqrt{y^2 + a} \ge 1 \\ &-\sqrt{y^2 + a} \ge 1 - y > 0 \\ &\Rightarrow\Leftarrow \end{aligned} \tag{3.6}$$

It follows from (3.6) that $|\lambda_2| < 1$ for y_+ and y_-. If follows from (3.5) that $|\lambda_1| > 1$ for y_+ and $|\lambda_1| < 1$ for y_-. So (x_+, y_+) is a sink, and (x_-, y_-) is a saddle point.

□

Lemma 3.2. *If* $c < 0$ *then* F *has an attracting fixed point,* t*, and a fixed saddle point,* p*, if and only if* $c > \frac{-3(1-a)^2}{4}$.

Proof. Let $y > 1$
Then clearly $|y + \sqrt{y^2 + a}| = y + \sqrt{y^2 + a} > y > 1$. Consider $|y - \sqrt{y^2 + a}| \ge 1$. Then

$$\begin{aligned} &|y - \sqrt{y^2 + a}| = \sqrt{y^2 + a} - y > 1 \\ &\Rightarrow y^2 + a > 1 + 2y + y^2 \\ &\Rightarrow a - 1 > 2y \\ &\Rightarrow 0 > \tfrac{a-1}{2} > y \\ &\Rightarrow\Leftarrow \end{aligned} \tag{3.7}$$

Consider $y < 0$.
Suppose $|y + \sqrt{y^2 + a}| \geq 1$
Then

$$\begin{aligned} &|y + \sqrt{y^2+a}| = \sqrt{y^2+a} - |y| < 1 \\ &\Rightarrow \sqrt{y^2+a} < 1 + |y| \\ &\Rightarrow y^2 + a < 1 + 2|y| + |y|^2 \\ &\Rightarrow \tfrac{a-1}{2} < |y| \\ &\Rightarrow y > \tfrac{1-a}{2} \end{aligned} \tag{3.8}$$

Hence $|y + \sqrt{y^2 + a}| \geq 1$ for all $y < 0$. Suppose $|y - \sqrt{y^2 + a}| < 1$ and $1 \leq y < 0$
Then

$$\begin{aligned} &|y - \sqrt{y^2+a}| < 1 \\ &\sqrt{y^2+a} + |y| < 1 \\ &\sqrt{y^2+a} < 1 - |y| \\ &y^2 + ax < 1 - 2|y| + |y|^2 \\ &a - 1 < -2|y| \\ &\tfrac{a-1}{2} < y \end{aligned} \tag{3.9}$$

It is clear that if c is negative, then $y_+ = \frac{(1-a)+\sqrt{(1-a)^2-4c}}{2} > \frac{1-a}{2}$, which fulfils the criteria of either (3.7) or (3.5) in Lemma 3.1.So (y_+, y_+) is a saddle point.
So we have to consider the behavior of $y_- = \frac{(1-a)-\sqrt{(1-a)^2-4c}}{2}$. From (3.8) and (3.9) we see that, in order for our parameter space to be connected, the following must hold:

$$\begin{aligned} &y \geq \tfrac{a-1}{2} \\ &\frac{(1-a)-\sqrt{(1-a)^2-4c}}{2} \geq \frac{a-1}{2} \\ &-\sqrt{(1-a)^2-4c} \geq 2(a-1) \\ &\sqrt{(1-a)^2-4c} \leq 2(1-a) \\ &(1-a)^2 - 4c \leq 4(1-a)^2 \\ &c \leq \tfrac{-3(1-a)^2}{4} \end{aligned} \tag{3.10}$$

So (y_-, y_-) is an attracting fixed point. □

Now that we've established the character of the fixed points and the non-existence of points of period 2, we can consider how the map in $\mathbb{C}^2 \setminus \mathbb{R}^2$ interacts with the map in $\mathbb{R}^2$.
Define $\Delta = \{(x, y) \in \mathbb{C}^2 | f^n(x, y) \in \mathbb{R}^2$ for some $n \in \mathbb{N}$.

Proposition 3.1. *$\Delta \cap \mathbb{R}^2 = \emptyset$.*

Proof. Suppose $\Delta \cap \mathbb{R}^2 \neq \emptyset$. Then there exists $(x_0, y_0) \in U$ such that $F(x_0, y_0) \in \mathbb{R}^2$. Then $F^2(x, y) \in \mathbb{R}^2$ since $F(x_r, y_r) \in \mathbb{R}^2\ \forall (x_r, y_r) \in \mathbb{R}^2$.

(3.11)
$$F^2(x_0, y_0) = (x_2, y_2) = (y_1, y_1^2 + x_1 + c) = (y_0^2 + ax_0 + c, (y_0^2 + ax_0 + c)^2 + ay_0 + c)$$

Since $y_0^2 + ax_0 + c$ is real, and $(y_0^2 + ax_0 + c)^2 + ay_0 + c$ is real, it follows that $(y_0^2 + ax_0 + c)^2 + c$ is real, ay_0 is real, which implies that y_0 is real.
But since y_0 is real, $y_0^2 + c$ is also real, so x_0 must be real. So $(x_0, y_0) \notin U$. $\Rightarrow\Leftarrow$ □

This permits us to consider only points initially in $\mathbb{R}^2$when studying the dynamics of the system, which is particularly useful when considering the stable and unstable manifolds of a saddle point.

Lemma 3.3. *There are no points of period 2 for $F(x,y) \in F_P$.*

Proof. If there are periodic points of period 2 then they must satisfy the following system of equations:

$$\begin{aligned} x &= y^2 + ax + c \\ y &= (y^2 + ax + c)^2 + ay + c \end{aligned} \tag{3.12}$$

So,

$$\begin{aligned} y &= x^2 + ay + c \\ y &= \tfrac{x^2+c}{1-a} \end{aligned} \tag{3.13}$$

Substituting for y,

$$\begin{aligned} &x = \left(\tfrac{x^2+c}{1-a}\right)^2 + ax + c \\ &(1-a)^2 x = (x^2+c)^2 + (1-a)^2 ax + c \\ &0 = x^4 + 2cx^2 - (1-a)^3 x + c((1-a)^2 + c) \\ &0 = (x^2 + (1-a)^x + ((1-a)^2 + c)(x^2 - (1-a)x + c) \end{aligned} \tag{3.14}$$

Since the equation $0 = x^2 - (1-a)x + c$ corresponds to the fixed points, if there are points of period 2 they must satisfy the equation

$$x^2 + (1-a)x + ((1-a)^2 + c) = 0 \tag{3.15}$$

But consider the discriminant to (3.15):

$$\begin{aligned} &(1-a)^2 - 4(1)[(1-a)^2 + c] \geq 0 \\ &-3(1-a)^2 - 4c \geq 0 \\ &\tfrac{-3(1-a)^2}{4} \geq c \end{aligned} \tag{3.16}$$

Which implies that $F(x,y) \notin F_P$. $\Rightarrow\Leftarrow$

□

Theorem 3.1. *The two fixed points, t, an attracting point, and p, a saddle point, are the only periodic points for F in $\mathbb{R}^2$.*

Proof. $d(f) = \underset{n\to\infty}{lim} (2^n)^{\frac{1}{n}} = 2$.
By Theorem 2.2, $J^+ = \overline{W^s_{\mathbb{C}^2}(p)}$ and $J^- = \overline{W^u_{\mathbb{C}^2}(p)}$. By definition, if a point is in $W^u_{\mathbb{C}^2}(p) \cap \mathbb{R}^2$ then it must attract to p under backwards iteration, so $W^u_{\mathbb{C}^2}(p) \cap \mathbb{R}^2 = W^u_{\mathbb{R}^2}(p)$.
Assume ν is a periodic point. By Theorem 2.9, since $|\det Df| = a < 1$, int $K^- = \emptyset$. Because the orbit of a periodic point must be bounded under backwards iteration, $\nu \in K^- \subset \partial K^- = J^- = \overline{W^u(p)}$.
Since, by the unstable manifold theorem, $W^u(p)$ can be parameterized by smooth curves, if $q \in J \setminus \{t\} \setminus \{p\}$ then

Case 1: $q \in W^u(p)$
$f^n(\nu) \to p$ as $n \to \infty$.
$\therefore \nu$ is not periodic. $\Rightarrow\Leftarrow$

Case 2: $q \in J^- \setminus W^u(p)$

Since $W^u(p)$ is a smooth curve, q is one of at most two limit points[1] of $W^u(p)$ that is not in $W^u(p)$.

Case A: Let q be a fixed point. Then there are three fixed points. But under $F^{-1}(x,y)$,

$$\begin{cases} x_{-1} = \frac{1}{a}(y_0 - x_0^2 - c) \\ y_{-1} = x_0 \end{cases} \tag{3.17}$$

Since $x_{-1} = x_0$ and $y_{-1} = y_0$ for a fixed point, $x_0 = \frac{1}{a}(x_0 - x_0^2 - c)$. Hence the existence of three fixed points violates the Fundamental Theorem of Algebra.

Case B: Let q be a periodic point.

Denote the other limit point of $W^u(p)$ not in $W^u(p)$ by q'. Then if q is periodic, it must be of period 2, or else there will be a point in $W^u(p)$ in its orbit. But there are no points of period 2 by Lemma 3.3. $\Rightarrow\Leftarrow$

So there are exactly two periodic points, which are the fixed points.

□

Lemma 3.4. *$J \cap \mathbb{R}^2 \subset \{t, p\}$, where t is the attracting point and p is the saddle point.*

Proof. By Theorem 2.7, $\Omega(f) = J \cup \{\alpha_1, \ldots, \alpha_m\}$, so $J \cap \mathbb{R}^2 \subset \Omega(f) \cap \mathbb{R}^2 = \{t, p\}$. □

Lemma 3.5. *Let $r \in \mathbb{R}^2$ and $m \in \mathbb{C}^2 \setminus \mathbb{R}^2$. Then $|f^k(m) - f^k(r)| \to 0$ as $k \to \infty$ implies that $F^k(r) \to p$ or $F^k(r) \to t$.*

Proof. Consider Theorem 2.6. When $r \in J$ or $r \in Per(F)$, r is a fixed point. So consider following cases:

Case 1: $m \in J \setminus Per(F)$

Since J is F-invariant, $F^k(m) \to \mathbb{R}^2 \Rightarrow F^k(m) \to J \cap \mathbb{R}^2 \subset \{t, p\}$.

Case 2: $m \in Per(F)$

Suppose m is periodic of prime period j. Since $F^k(m) \in \mathbb{C}^2 \setminus \mathbb{R}^2 \forall k \in \mathbb{N}$,

$$\begin{aligned} &\min\{d(F^i(m), \mathbb{R}^2)\}_{i=0}^{\infty} = \min\{d(F^i(m), \mathbb{R}^2)\}_{i=0}^{j} = \delta > 0 \\ &\Rightarrow d(F^i(m), F^i(r)) \geq d(F^i(m), \mathbb{R}^2) \geq \delta > 0 \\ &\Rightarrow |F^k(r) - F^k(m)| \nrightarrow 0 \text{ as } k \to \infty \\ &\Rightarrow\Leftarrow \end{aligned}$$

By (iii) in Theorem 2.6, if neither of these cases hold, then $F^k(r) \to \infty$ as $k \to \infty$. □

Theorem 3.2. *There are no chain recurrent points of F in $\mathbb{R}^2$.*

Proof. Suppose q is a chain recurrent point. Then its orbit is bounded under forwards and backwards iteration by Theorem 2.3. Since it is chain recurrent, it cannot be in the basin of attraction of the attracting point, so it must lie in $W^s_{\mathbb{R}^2}(p)$. Since int $K^- = \emptyset$, $K^- \cap \mathbb{R}^2 \subset J^- \cap \mathbb{R}^2 \subset \overline{W^u_{\mathbb{C}^2}(p)}$. Note that we can disregard points in $[(J^+ \cup J^-) \cap \mathbb{R}^2] \setminus (W^s_{\mathbb{R}^2}(p) \cup W^u_{\mathbb{R}^2}(p))$ because from Theorem 2.6 and Lemma 3.5, they must converge to ∞ under iteration. So a point can be chain recurrent if

[1]This fact remains to be verified, but we are confident that it is true because the stable and unstable manifolds are Lipschitz [8].

and only if it is in $W^s_{\mathbb{R}^2}(p) \cap W^s_{\mathbb{R}^2}(p)$. If $q \neq p$, then q is a homoclinic point, which implies the existence of periodic points of infinitely many periods. But there are no periodic points other than the fixed points. $\Rightarrow\Leftarrow$ $\square$

4. Results for $a > 1$, $\frac{-(1-a)^2}{4} < c < \frac{(1-a)^2}{4}$

Let $M = \{(a,c) | a > 1, \frac{-3(1-a)^2}{4} < c < \frac{(1-a)^2}{4}\}$.

In our treatment of this region, we realize that the same ideas that apply to the map under forward iteration also apply to it under backwards iteration.

Theorem 4.1. *If $F \in F_M$ then F has a repelling fixed point, r, and a saddle point, p.*

Proof. Recall that the fixed points are given by the equations

$$\begin{aligned} y_+ &= \tfrac{(1-a)+\sqrt{(1-a)^2-4c}}{2} \\ y_- &= \tfrac{(1-a)-\sqrt{(1-a)^2-4c}}{2} \end{aligned} \tag{4.1}$$

and the eigenvalues of a point are given by

$$\begin{aligned} \lambda_1 &= y + \sqrt{y^2+a} \\ \lambda_2 &= y - \sqrt{y^2+a} \end{aligned} \tag{4.2}$$

Consider $a > 1, y < 0$. Then

$$\begin{aligned} &|y + \sqrt{y^2+a}| > 1 \\ &\Rightarrow -|y| + \sqrt{y^2+a} > 1 \\ &\Rightarrow \sqrt{y^2+a} > 1 + |y| \\ &\Rightarrow y^2 + a > 1 + 2|y| + y^2 \\ &\Rightarrow \tfrac{a-1}{2} > |y| \\ &\Rightarrow \tfrac{1-a}{2} < y \end{aligned} \tag{4.3}$$

Clearly, $|y - \sqrt{y^2+a}| > |-\sqrt{1}| = 1$.

For $\frac{(1-a)^2}{4} > c > 0$ it follows that $y_+ > \frac{1-a}{2}$ and $y_- < \frac{1-a}{2}$, so y_+ is an attracting point and y_- is a saddle point.

Now consider $y > 0, a > 1$.
Clearly $y + \sqrt{y^2+a} > \sqrt{1} > 1$.
Suppose

$$\begin{aligned} &|y - \sqrt{y^2+a}| \leq 1 \\ &\Rightarrow y - \sqrt{y^2+a} \geq -1 \\ &\Rightarrow y + 1 \geq \sqrt{y^2+a} \\ &\Rightarrow y^2 + 2y + 1 \geq y^2 + a \\ &\Rightarrow y \geq \tfrac{a-1}{2} \end{aligned} \tag{4.4}$$

Consider the case where $(a,c) \in U, c < 0$. Suppose $y_+ \geq \frac{a-1}{2}$. Then

$$\begin{aligned} &(1-a) + \sqrt{(1-a)^2 - 4c} \geq a - 1 \\ &\sqrt{(1-a)^2 - 4c} \geq 2(a-1) \\ &(1-a)^2 - 4c \geq 4(1-a)^2 \\ &-4c \geq 3(1-a)^2 \\ &c \leq \tfrac{-3(1-a)^2}{4} \Rightarrow\Leftarrow \end{aligned} \tag{4.5}$$

So when $(a,c) \in U, c < 0$, (y_+, y_+) is a repelling fixed point and (y_-, y_-) is a saddle point.

□

It follows that under backwards iteration, (y_+, y_+) is an attracting fixed point and (y_-, y_-) is a saddle fixed point. Since

$$F^{-1}(x,y) = (\frac{1}{a}(y - x^2 - c), x) \tag{4.6}$$

The Jacobian of which is given by

$$DF^{-1}(x,y) = \begin{pmatrix} \frac{-2x}{a} & \frac{1}{a} \\ 1 & 0 \end{pmatrix} \tag{4.7}$$

and

$$|\det DF^{-1}(x,y)| = \left|\frac{-1}{a}\right| = \frac{1}{a} < 1$$

and

$$d(F^{-1}) = 2$$

.

Using arguments analogous to those in §3, we can conclude the following about U:

Theorem 4.2. *The chain recurrent set (and hence $\Omega(f)$ and the set of periodic points) for $F \in U$ consists exclusively of $\{r, p\}$.*

5. Periodic Points for $a > 0$, $c < \frac{-3(1-a)^2}{4}$

This section demonstrates how the maps in F_N, where $N = \{(a,c) | a > 0, a \neq 1, c < \frac{-3(1-a)^2}{4}\}$, have infinitely many periodic points.

Proof. In the following proof we are only considering $a > 0$.

Case 1: $y > 0$

If $y \geq 1$, then clearly $y + \sqrt{y^2 + a} > 1$. If $y < 1$, then

$$\begin{aligned} &y + \sqrt{y^2 + a} > 1 \\ &\sqrt{y^2 + a} > 1 - y \\ &y^2 + a > 1 - 2y + y^2 \\ &y > \tfrac{1-a}{2} \end{aligned} \tag{5.1}$$

and for all $y > 0$

$$\begin{aligned} &|y - \sqrt{y^2 + a}| \geq 1 \\ &\sqrt{y^2 + a} - y \geq 1 \\ &\sqrt{y^2 + a} \geq 1 + y \\ &y^2 + a \geq 1 + 2y + y^2 \\ &\tfrac{a-1}{2} \geq y \end{aligned} \tag{5.2}$$

Case 2: $y < 0$

$$\begin{aligned} &|y + \sqrt{y^2 + a}| \geq 1 \\ &\sqrt{y^2 + a} - |y| \geq 1 \\ &y^2 + a \geq 1 + 2|y| + y^2 \\ &\tfrac{a-1}{2} \geq |y| \end{aligned} \tag{5.3}$$

If $|y| \geq 1$, clearly $|y - \sqrt{y^2+a}| = |y| + \sqrt{y^2+a} > 1$. If $|y| < 1$,

$$\begin{aligned}
&|y - \sqrt{y^2+a}| \leq 1\\
&|y| + \sqrt{y^2+a} \leq 1\\
&\sqrt{y^2+a} \leq 1 - |y|\\
&y^2 + a \leq 1 - 2|y| + y^2\\
&|y| \leq \tfrac{1-a}{2}\\
&y \geq \tfrac{a-1}{2}
\end{aligned} \tag{5.4}$$

Case A: $0 < a < 1$

Suppose $y_- = \frac{(1-a)-\sqrt{(1-a)^2-4c}}{2} \geq \frac{a-1}{2}$ Then

$$\begin{aligned}
&(1-a) - \sqrt{(1-a)^2-4c} \geq a-1\\
&2(1-a) \geq \sqrt{(1-a)^2-4c}\\
&4(1-a)^2 \geq (1-a)^2 - 4c\\
&3(1-a)^2 \geq -4c\\
&\tfrac{-3(1-a)^2}{4} \leq c \Rightarrow\Leftarrow
\end{aligned} \tag{5.5}$$

The fact that $a < 1$ makes (5.3) a contradiction, so $|\lambda_1| < 1$, and (5.5) and (5.4) implies that $|\lambda_2| > 1$, so (y_-, y_-) is a saddle point.
Since $a < 1$, $1 - a > 0$, so $y_+ > 0$, and by (5.1) and (5.2), (y_+, y_+) is a saddle point.

Case B: $a > 1$ Since $y_+ > 0$, $y_+ + \sqrt{y_+^2 + a} > \sqrt{a} > 1$, so $\lambda_1 > 1$.

Suppose $y_+ \leq \frac{a-1}{2}$. Then

$$\begin{aligned}
&(1-a) + \sqrt{(1-a)^2-4c} \leq (a-1)\\
&\sqrt{(1-a)^2-4c} \leq 2(a-1)\\
&(1-a)^2 - 4c \leq 4(a-1)^2\\
&-4c \leq 3(1-a)^2\\
&c \geq \tfrac{-3(1-a)^2}{4}
\end{aligned} \tag{5.6}$$

So $y_+ > \frac{a-1}{2}$. It follows from (5.2) that $\lambda_2 < 1$, so y_+ is a saddle point.
Consider y_-. Since $a > 1$,

(5.7)

$$|(1-a) - \sqrt{(1-a)^2-4c}| = |1-a| + \sqrt{(1-a)^2-4c} > |1-a| + |1-a| = 2(a-1)$$
$$\Rightarrow |y_-| > \tfrac{a-1}{2}$$

It follows from (5.3) that $|\lambda_1| < 1$. Since $a > 1$,(5.4) becomes a contradiction, so $|\lambda_2| > 1$, and we have a saddle point for y_-.

□

Lemma 5.1. *For $F \in F_N$, F has two points of period two.*

Proof. From Lemma 3.3, we know that if a point (x, y) of period 2 must satisfy the following conditions:

$$\begin{aligned}
x &= y^2 + ax + c\\
y &= (y^2 + ax + c)^2 + ay + c
\end{aligned} \tag{5.8}$$

Substituting for $(y^2 + ax + c)$ we see that

$$\begin{aligned}
y &= x^2 + y + c\\
y &= \tfrac{x^2+c}{1-a}
\end{aligned} \tag{5.9}$$

Hence if x is real, since $a \neq 1$, it is guaranteed that y is real.
From Lemma 3.3, if there is a point of period 2, then

$$x^2 + (1-a)x + ((1-a)+c) = 0 \tag{5.10}$$

Clearly, it is necessary that there are two real solutions, or else the orbit of the periodic point does not exist. So the discriminant must be greater than zero.

$$\begin{aligned} &(1-a)^2 - 4(1)((1-a)^2 + c) > 0 \\ &-4c > 3(1-a)^2 \\ &c < \tfrac{-3(1-a)^2}{4} \end{aligned}$$

So for all $F \in F_N$ there are periodic points of period 2. □

Let $\{p_1, p_2\}$ be the fixed saddle points.

Theorem 5.1. *For $F \in F_M$, there are infinitely many periodic points.*

Proof. Consider v_1, v_2, the points of period 2.

Claim: v_1 is a saddle point. In calculating eigenvalues, we can treat the x value that we solve for in the equation $x + (1-a)x + ((1-a)+c) = 0$ as a y value, since it should be the y value of the other point of period 2. Solving for x:

$$\begin{aligned} x_+ &= \tfrac{(a-1)+\sqrt{(1-a)^2-4((1-a)+c)}}{2} \\ x_- &= \tfrac{(a-1)-\sqrt{(1-a)^2-4((1-a)+c)}}{2} \end{aligned} \tag{5.11}$$

Case 1: $a > 1$
Consider x_+. Then $x_+ > \frac{a-1}{2}$, so $\lambda_1 > 1$ and $\lambda_2 < 1$.
Case 2: $a < 1$
Consider x_-. Then $x_- < \frac{a-1}{2}$. By (5.3) and (5.4), x_- is a saddle point.

Recall from 2.2 that if $d(F) > 1$ then $J^+ = \overline{W^s_{\mathbb{C}^2}(p)}$ and $J^- = \overline{W^u_{\mathbb{C}^2}(p)}$. It follows that $J^+ \cap \mathbb{R}^2 = \overline{W^s(p_1)} = \overline{W^s(p_2)}$ and $J^- \cap \mathbb{R}^2 = \overline{W^u(p_1)} = \overline{W^u(p_2)}$. We will hitherto refer to $J^+ \cap \mathbb{R}^2$, $J^- \cap \mathbb{R}^2$, and $J \cap \mathbb{R}^2$as J^+,J^-, and J for simplicity.
Let $v_1 = (x_+, x_-)$ for $a < 1$. Suppose v_2 is an attracting point. Then any point that has an orbit that attracts to $\mathcal{O}(v_2)$ also attracts to $\mathcal{O}(v_1)$, which implies that v_1 is also attracting. But v_1 is a saddle point by (5) $\Rightarrow\Leftarrow$. From Theorem 2.7 we know that $v_1 \in \Omega(F) = J \cup \{\alpha_1, \ldots, \alpha_m\}$, so it must be in J. Since $W^s(p_1)$ is a smooth, continuous curve in $\mathbb{R}^2$, there are at most two points in $J^+ \setminus W^s(p_1)$, and one which is p_2. Similarly, $p_1 \in J^+ \setminus W^s(p_2)$. But then these must be the endpoints of J, which implies that $v_1 \in W^s(p_1)$. Similarly, $v_1 \in W^u(p_1)$. So v_1 is a homoclinic point, which implies the existence of infinitely many periodic points. An analogous argument follows for (x_0, x_+) for $a > 1$, if we consider backwards iteration in order to invoke Theorem 2.7.

□

This result leads us to conclude that no map in F_N can have the same recurrent dynamics (and hence cannot be topologically conjugate) to any map in F_U or F_P.

6. Classifying the Parameter Space

This section confirms that the dynamics of maps in F_P and F_U are unique for all $a \geq 0$, by expounding on the differences between maps in these sets and all other maps on in the parameter space.
Let $\Theta = \{(a, c)|c > \frac{(1-a)^2}{4}\}$.

Fact 6.1. *If $F \in F_\Theta$ then F has no fixed points.*

Proof. In order for a point to be a fixed point, the equation

$$0 = y^2 + (a-1)y + c \tag{6.1}$$

must have a real solution. But in this case the discriminant is

$$(1-a)^2 - 4c < 0 \tag{6.2}$$

□

Let $\mathcal{T} = \{(a, c)|c = \frac{(1-a)^2}{4}\}$.

Fact 6.2. *For $F \in F_\mathcal{T}$ there is only one fixed point.*

Proof. The fixed points are given by $(y_+, y_+), (y_-, y_-)$. Furthermore, $y_\pm = \frac{(1-a)\pm\sqrt{(1-a)^2-4c}}{2}$. But $(1-a)^2 - 4c = 0$, so $y_+ = y_-$, so on this curve the fixed points collapse to one fixed point. □

Let $\mathcal{U} = \{(a, c)|a = 1, c < 0\}$.

Fact 6.3. *For $F \in F_\mathcal{U}$, the fixed two fixed points are saddle points.*

Proof. For $a = 1$, the fixed points are given by

$$y_\pm = \frac{(1-a) \pm \sqrt{(1-a)^2 - 4c}}{2} = \frac{\sqrt{-4c}}{2} = \sqrt{-c} \tag{6.3}$$

Consider $\lambda_\pm$. For $\sqrt{-c}$

$$\lambda_+ = \sqrt{-c} + \sqrt{|-c| + 1} > 1$$

Suppose $|\lambda_-| \geq 1$. Then

$$\begin{aligned}
&|\sqrt{-c} - \sqrt{|-c|+1}| \geq 1 \\
&\sqrt{|-c|+1} - \sqrt{-c} \geq 1 \\
&\sqrt{|-c|+1} \geq 1 + \sqrt{-c} \\
&|-c| + 1 \geq 1 + 2\sqrt{-c} + |-c| \\
&0 \geq \sqrt{-c} \Rightarrow\Leftarrow
\end{aligned}$$

So $(\sqrt{-c}, \sqrt{-c})$ is a saddle point. Similarly, $|\lambda_-| = |-\sqrt{-c} - \sqrt{|-c|+1}| = \sqrt{-c} + \sqrt{|-c|+1} > 1$.
Suppose $|\lambda_+| \geq 1$

$$\begin{aligned}
&|-\sqrt{-c} + \sqrt{|-c|+1}| = -\sqrt{-c} + \sqrt{|-c|+1} \geq 1 \\
&\sqrt{|-c|+1} \geq \sqrt{-c} + 1 \\
&0 \geq \sqrt{-c} \Rightarrow\Leftarrow
\end{aligned} \tag{6.4}$$

So $(-\sqrt{-c}, -\sqrt{-c})$ is also a saddle point. □

Fact 6.4. *For $F \in F_\mathcal{U}$, there are no points of period two.*

Proof. From the proof of 3.3, in order for a point $(x, y)$to be of period two, it must satisfy the equation

$$y = \frac{x^2 + c}{1 - a} \tag{6.5}$$

whose solution is undefined for $a = 1$. □

Fact 6.5. *For $a = 0$, F is not a diffeomorphism.*

Proof. If $a = 0$, $F(x, y) = (y, y^2 + ax + c) = (y, y^2 + c)$. Since $F(1, 0) = (0, c)$ and $F(153642, 0) = (0, c)$, F is not one-to-one, and hence is not a diffeomorphism. □

7. A Survey of $c = 0$, $0 < a < 1$

The following section serves an an example of the behavior of the maps in F_P, and classifies the dynamics of the set $F_{(a,0)}$, $0 < a < 1$. Our approach is to tile the plane with regions and to explain the behavior of the points within these regions under iteration as well as the relationships between the regions, in order to create a global picture of the dynamics of the system.

In the remainder of the paper the following notation will be used:
$f(x, y) = (y, y^2 + ax)$.
Let $f^{-1}(x, y) = g(x, y) = (b(y - x^2), x)$, where $b = \frac{1}{a}$. Clearly, $b > 1$.

$f^n(x_0, y_0) = (x_n, y_n)$ denotes the n^{th} iteration of an initial value (x_0, y_0).
$f^{-1}(x, y) = g(x, y) = (b(y - x^2), x)$, where $b = \frac{1}{a}$. Clearly, $b > 1$.
Define the following regions:
$Q_1 = \{(x, y)|x \geq 0, y \geq 0\}$
$Q_2 = \{(x, y)|x < 0, y \geq 0\}$
$Q_3 = \{(x, y)|x < 0, y < 0\}$
$Q_4 = \{(x, y)|x \geq 0, y < 0\}$

Define the following regions in Q_1:
$S_1 = \{(x, y)|x > 0, y > 0, y < x^2\}$
$S_2 = \{(x, y)|x > 0, y > 0, y > x^2 + a\sqrt{x}\}$
$S_3 = \{(x, y)|x > 0, y > 0, x^2 + a\sqrt{x} > y > x^2 + ax, x > \frac{y-y^2}{a}\}$
$S_4 = \{(x, y)|x > 0, y > 0, x < \frac{y-y^2}{a}, x^2 + a\sqrt{x} > y > x^2 + ax, x < 1 - a, y > 1 - a\}$
$S_5 = \{(x, y)|x > 0, y > 0, x > \frac{y-y^2}{a}, x^2 + ax > y > x^2, y > x\}$
$S_6 = \{(x, y)|x > 0, y > 0, x > \frac{y-y^2}{a}, x^2 + ax > y > x^2, y < x\}$
$S_7 = \{(x, y)|x > 1 - a, y > 1 - a, x < \frac{y-y^2}{a}\}$
$S_8 = \{(x, y)|x > 0, y > 0, x > y, \frac{y-y^2}{a} > x\}$
As well as the following regions in Q_1 :
$T_1 = \{(x, y)|x > 1 - a, y > 1 - a\}$
$T_2 = \{(x, y)|x < 1 - a, y > 1 - a\}$
$T_3 = \{(x, y)|x < 1 - a, y < 1 - a\}$
$T_4 = \{(x, y)|x > 1 - a, y < 1 - a\}$

The reader may verify that these regions cover the plane.

Proposition 7.1. *f has a fixed attracting point at $(0,0)$ and a fixed saddle point at $(1-a, 1-a)$.*

Proof. By computation, we find that the fixed points are $(0,0)$ and $(1-a, 1-a)$.
The Jacobian of $f(x,y)$ is given by
$\mathbf{Df}((x,y)) = \begin{pmatrix} 0 & 1 \\ a & 2y \end{pmatrix}$
So the eigenvalues are given by $y \pm \sqrt{y^2+a}$.
At $(0,0)$, $\lambda_1 = \sqrt{a}, \lambda_2 = -\sqrt{a}$.
Since $0 < a < 1$, $|\sqrt{a}| < 1$. So $(0,0)$ is an attracting point.
At $(1-a, 1-a)$,
$\lambda_1 = (1-a) + \sqrt{(1-a)^2 + a}$
$\sqrt{(1-a)^2+a} = \sqrt{1-2a+a^2+a} = \sqrt{(1-a)+a^2}$.
$a < 1 \Rightarrow (1-a) > 0 \Rightarrow (1-a)+a^2 > a^2 \Rightarrow \sqrt{(1-a)^2+a^2} > a$
$\Rightarrow |(1-a) + \sqrt{(1-a)^2+a}| > |(1-a)+a| > 1$.
$\therefore |\lambda_1| > 1$. $\lambda_2 = (1-a) - \sqrt{(1-a)^2+a}$
Suppose $|\lambda_2| \geq 1$. Then $a + \sqrt{(1-a)^2+a} \geq 2$
$\Rightarrow \sqrt{(1-a)^2+a} \geq 2-a$
$\Rightarrow 1-2a+a^2 \geq 4-4a+a^2$
$\Rightarrow 3a \geq 3 \Rightarrow\Leftarrow$
$\therefore |\lambda_2| < 1$.
So $(1-a, 1-a)$ is a saddle point. □

Fact 7.1. *Q_1 is a trapping region under forward iteration.*

Proof. $x_1 = y_0 \geq 0$
$x_0 > 0 \Rightarrow ax_0 > 0 \Rightarrow y_1 = x_0^2 + ax_0 > 0$.
$\therefore (x_1, y_1) \in Q_1$. □

Proposition 7.2. *Let $R = \{(x,y) | x, y \in \mathbb{R}, |x| < (1-a), |y| < (1-a)\}$. Then R is the largest open square such that $\lim_{n\to\infty} f^n(R) = (0,0)$.*

Proof. **Claim 1:** The $\{k_i\}$, defined by the y-values of $f^i(k_0, k_0)$, where $k_0 < 1-a$, are strictly decreasing.

Since $k_0 < 1-a$, it is geometrically evident that $(k_0, k_0) \in \{(x,y) | \frac{|y|-|y|^2}{a} < |x|\}$. It follows that $k_1 = k_0^2 + ak_0 < k_0$.
Assume that $k_m < k_{m-1}$ for all integers less than or equal to m. Then $k_m^2 < k_{m-1}^2$ and $k_{m-1} < k_{m-2}$.
From a Fact 7.1, we know that $k_i \geq 0$ for all $i \in \mathbb{N}$. So $k_{m+1} < k_m^2 + ak_{m-1} < k_{m-1}^2 + ak_m = k_m$.
Let $W_0 = \{(x,y) || x| < k_0, |y| < k_0, 0 < k_0 < 1-a\}$.
For every $(x_0, y_0) \in W_0$,
$|y_1| = |y_0^2 + ax_0| < k_0^2 + ak_0 = k_1 \Rightarrow |x_2| = |y_1| < k_1$
Furthermore, $|y_1| < |k_1| < |k_0|$ and $|y_0| < |k_0|$
$\Rightarrow |y_1|^2 < |k_1|^2$
$\Rightarrow |y_2| < |y_1|^2 + a|y_0| < k_1^2 + ak_0 = k_2 < k_1$.
It follows that $f(W_0) \subset W_1 = W_0$ and $f^2(W_1) \subsetneq W_2$, where $W_2 = \{(x,y) || x| < k_1, |y| < k_1\}$.

Define $W_n = \{(x,y)||x| < k_{n-1}, |y| < k_{n-1}\}$ where $k_i = k_{i-1}^2 + ak_{i-2}$.

Claim 2: $f^2(W_{n-1}) \subsetneq W_n$ for all $n \in \mathbb{N}, n \geq 2$.

Let $(x_0, y_0) \in W_2$. Then
$|x_0| < k_0$, $|y_0| < k_1$
$\Rightarrow |x_1| = |y_1| < k_0$ and $|x_2| = |y_1| = |y_0^2 + ax_0| < |y_0|^2 + a|x_0| < k_1^2 + k_0^2 = k_2$.
Similarly, $|y_2| = |y_1^2 + ay_0| < k_2^2 + k_1^2 = k_3 < k_2$. So $f(W_2) \subsetneq W_2$ and $f^2(W_2) \subsetneq W_3$.

By induction, assume that $f(W_{m-1}) \subsetneq W_m$ for $m = 2, 3, \ldots m$.
Let $(x_0, y_0) \in W_m$. Then $|x| < k_m < k_{m-1}$, $|y| < k_m$
$\Rightarrow |x_{m+1}| = |y_m| = |y_{m-1}^2 + ax_{m-1}| < |y_{m-1}|^2 + a|x_{m-1}| < k_m^2 + k_{m-1}^2 = k_{m+1}$ Similarly, $|y_{m+1}| = |y_m^2 + ay_{m-1}| < k_{m+1}^2 + k_m^2 = k_{m+2} < k_{m+1}$.
So $f(W_m) \subsetneq W_m$ and $f^2(W_m) \subsetneq W_{m+1}$.

Clearly, the absolute values of (x_i, y_i) are dominated by the values of (k_{i-1}, k_i) under iteration, so $f^j(W_0) \subsetneq W_j, j \geq 2$.

Claim 3: As $i \to \infty$, $k_i \to 0$.

From a previous result $L = \overline{i \to \infty}\lim k_i \geq 0$
$underseti \to \infty \lim k_i = \lim_{i\to\infty} k_{i-1}^2 + \lim_{i\to\infty} ak_{i-2} \Rightarrow L = L^2 + aL$
So L=(1-a) or L=0. Since the k_i are strictly decreasing and $k_0 < 1 - a$, $L \neq 1 - a$. So $L = 0$.

It follows that $\lim_{i\to\infty} W_i = (0,0)$.

Take W_0 to be R. Then it follows that $lim_{n\to\infty} f^n(R) = (0,0)$.

R is the largest such square because $(1-a, 1-a)$ is a limit point of R, so any open square that contains R will also contain $(1-a, 1-a)$. Since $(1-a, 1-a)$ is a fixed point of f, this implies that any such square will not converge to $(0,0)$. ☐

Proposition 7.3. *For $f(x,y) = (y, y^2 + ax)$, $R = 1 + a$, where R is the value defined in Theorem 2.10.*

Proof. If $|x| > |y|$ the conditions of Lemma 2.10 are fulfilled.
Consider $|x| \leq |y|$. If y is fixed, then
$inf|z| = inf|y^2 + ax| = inf(|y^2 - a|x||) = |y|^2 - a|y|$
$\Rightarrow R^2 - aR \geq R$
$\Rightarrow R(R - (1+a)) \geq 0$
Since $R > 0$, $(R - (1+a)) > 0 \Rightarrow R > 1 + a$.

Let $m(|y|) = (|y|^2 - a|y|) - |y|$. Then $m'(x) = 2|y| - (1+a)$, so if $|y| > 1+a$, then $m(x) \leq (|z| - |y|)$ is increasing, and hence is always greater than 0. So $R = 1+a$. ☐

Propositions 7.4 to 7.10 consider $f^{-1}(x,y) = g(x,y)$.

Proposition 7.4. *If $(x,y) \in Q_4$ then the x_{2i} are strictly decreasing.*

Proof. Let $(x_0, y_0) \in Q_4$.
$g^2(x_0, y_0) = (b(x_0 - b(y_0 - x_0^2))^2, b(y_0 - x_0^2))$. Since $(b(y_0 - x_0^2))^2 \geq 0$, $x_0 - (b(y_0 - x_0^2))^2 \leq x_0$,
$\Rightarrow x_2 = b(x_0 - (b(y_0 - x_0^2))^2) \leq bx_0 < x_0$. ☐

Proposition 7.5. *There are no periodic points in Q_3.*

Proof. $\|(x_i, y_i)\|$ is strictly increasing.
Consider $k \geq 1$.

$g^k(x_0, y_0) = (x_k, y_{k-1})$
$g^{k+1}(x_0, y_0) = (x_{k+1}, x_k)$
For k even, by Proposition 7.4, $x_{k+1} < x_k < 0$. For k odd, since (x_1, y_1) may be re-indexed as (x'_0, y'_0) as the sequence of (x'_i, y'_i), by Proposition 7.4, $x_{k+1} < x_{k-1} < 0$.
$\|g^k(x_0, y_0)\| = \|(x_k, x_{k-1})\| = \sqrt{(x_k)^2 + (x_{k-1})^2}$,
and$\|g^{k+1}(x_0, y_0)\| = \|(x_{k+1}, x_k)\| = \sqrt{(x_k)^2 + (x_{k+1})^2}$.
$x_{k+1} < x_{k-1} < 0 \Rightarrow x_{k+1}^2 > x_{k-1}^2$
$\Rightarrow \|g^k(x_0, y_0)\| < \|g^{k+1}(x_0, y_0)\|$.
For $k = 0$,
$g^0(x_0, y_0) = (x_0, y_0)$
$g^1(x_0, y_0) = (x_1, x_0)$
Since $b > 1, y_0 < 0$, $by_0 < y_0 \Rightarrow by_0 - bx_0^2 < y_0 < 0$.
Similar to the argument above, $\|g^0(x_0, y_0)\| < \|(g^1(x_0, y_0)\|$
Since the norms of the points in the orbit are strictly increasing, there are no periodic points in the region. □

Proposition 7.6. *If $(x, y) \in Q_2$ and $y \leq x^2$ then $g(x, y)$ maps (x, y) to Q_3.*

Proof. Consider (x_0, y_0) in the region defined in the proposition.
$g(x_0, y_0) = (b(y_0 - x_0^2), x_0) = (x_1, y_1)$.
Clearly, $y_1 = x_0 < 0$.
$y_0 < x_0^2 \Rightarrow y_0 - x_0^2 < 0 \Rightarrow x_1 = b(y_0 - x_0^2) < 0$.
$\therefore (x_1, y_1) \in Q_3$.
□

Proposition 7.7. *If $(x, y) \in Q_4$ then $g(x, y) \in Q_2$.*

Proof. Let $(x_0, y_0) \in Q_4$. Then $y_1 = x_0 > 0$. Since $y_0 < 0$ and $x_0^2 > 0$,$y_0 - x_0^2 < 0 \Rightarrow x_1 = b(y_0 - x_0^2) < 0$.
$\therefore (x_1, y_1) \in Q_2$. □

Proposition 7.8. *If $(x, y) \in Q_2$ and $y \geq x^2$ then $g(x, y)$ maps (x, y) to Q_4.*

Proof. Let (x_0, y_0) in the region defined in the proposition.
$g(x_0, y_0) = (b(y_0 - x_0^2), x_0) = (x_1, y_1)$.
Clearly, $y_1 = x_0 < 0$.
$y_0 > x_0^2 \Rightarrow y_0 - x_0^2 > 0 \Rightarrow x_1 = b(y_0 - x_0^2) > 0$.
$\therefore (x_1, y_1) \in Q_4$. □

Corollary 7.1. *There are no periodic points of odd period in Q_2 or Q_4.*

Proof. Suppose (x_0, y_0) is a periodic point in Q_4. Then its orbit must oscillate between $Q_2 \cap \{(x, y)|y \geq x^2\}$ and Q_4. Then for every $n \in \mathbb{N}$, its $2n$ iteration is in Q_4. But then its $(2n + 1)$ iteration must be in $Q_2 \cap \{(x, y)|y \geq x^2\}$. $\Rightarrow\Leftarrow$.
An analogous argument follows for $(x_0, y_0) \in Q_2 \cap \{(x, y)|y \geq x^2\}$. □

Proposition 7.9. *If $(x_0, y_0) \in Q_4$ then the odd x_i's are strictly decreasing.*

Proof. Let $(x_0, y_0) \in Q_4$.
$f(x_0, y_0) = (b(y_0 - x_0^2), x_0)$
Since $y_0 < 0$ and $x_0^2 > 0$, $x_1 = b(y_0 - x_0^2) < 0$.
$f^2(x_0, y_0) = f(x_1, y_1) = (b(y_1 - x_1^2), x_1) = (x_2, y_2)$
$f^3(x_0, y_0) = f(x_2, y_2) = (b(y_2 - x_2^2), x_2) = (b(y_2 - x_2^2), x_2) = (x_3, y_3) = (b(x_1 -$

$x_2^2, y_3)$ Since $x_1 < 0$ and $x^2 > 0$, $x_3 = y_2 - x_2^2 = x_1 - x_2^2 < x_1 < 0$.
Remark that the either $g^{2n+2}(x_0, y_0)$ maps into Q_3 for some $n \in \mathbb{N}$,otherwise $g^{2n+2}(x_0, y_0)$ maps back into Q_4, so for all odd i, $x_i < 0$.
By induction, assume that $x_1 > x_3 > \cdots < x_{2n-1}$.
$g^{2n-1}(x_0, y_0) = (x_{2n-1}, x_{2n-2})$
$g^{2n}(x_0, y_0) = (x_{2n}, x_{2n-1})$
$g^{2n+1}(x_0, y_0) = (x_{2n+1}, x_{2n})$
$x_{2n+1} = b(x_{2n-1} - x_{2n}^2 < x_{2n-1} < 0$. □

Proposition 7.10. *There are no periodic points of even period in Q_2 or Q_4.*

Proof. Suppose (x_0, y_0) is a periodic point in Q_2.

> **Claim:** $x_2 < x_0$
> $f^2(x_0, y_0) = (b(y_1 - x_1^2), x_1) = (b(x_0 - x_1^2), x_1)$
> $x_0 < 0 \Rightarrow x_0 - x_1^2 < x_0 \Rightarrow x_2 = b(x_0 - x_1^2) < x_0 - x_1^2 < x_0$.

We can take the even x_{2i} for $i > 0$ to be the odd x_i' of the orbit of $(x_0', y_0') = (x_1, y_1)$.
By Proposition 7.9, the $x_2 i$ are strictly decreasing. So there can be no points of even period in Q_2.
Since every periodic point in Q_4 must oscillate between Q_2 and Q_4, every orbit of a periodic point in Q_4 must contain a periodic point in Q_2. But there are no such points in Q_2, so there are no periodic points in Q_4. □

Proposition 7.11. *S_8 is a trapping region, and there are no periodic points in S_8.*

Proof. Consider $(x_0, y_0) \in S_8$.
$x_0 < \frac{y_0 - y_0^2}{a}$
$\Rightarrow y_1 = y_0^2 + ax_0 < y_0 = x_1$.
Furthermore,
$y_1 < y_0 \Rightarrow y_1^2 < y_0^2$
$\Rightarrow y_1^2 + ay_0 < y_0^2 + ax_0 = y_1$
$\Rightarrow \frac{y_1 - y_1^2}{a} > x_1$.
$\therefore S_8$ is a trapping region.
It is evident that they y_i are strictly decreasing in S_8, which implies that there are no periodic points. □

Proposition 7.12. *S_8 is in the basin of attraction of $(0, 0)$.*

Proof. S_8 lies below the line $y = 1 - a$.
For all $(x, y) \in S_8$, $\frac{y - y^2}{a} - y > 0$.
So $(1 - a)y > y^2$, and since $y > 0, y > 1 - a$. Suppose that $\lim_{i \to \infty} y_i = L$
Then $L^2 + aL = L$
$\Rightarrow L = 1 - a, 0$.
So $L = 0$. □

Proposition 7.13. *There are no periodic points in S_1.*

Proof. Let $(x_0, y_0) \in S_1$.
Then $y_0 - x_0^2 < 0 \Rightarrow x_1 = b(y_0 - x_0^2) < 0$.
Since $y_1 = x_1 > 0$, $g(x_0, y_0) \in Q_2$.

We know from a previous result that there are no periodic points in Q_2, so no periodic orbit can contain a point in S_1.

□

Proposition 7.14. *There are no periodic points in* S_2.

Proof. Consider $(x_0, y_0) \in S_2$.
$g(x_0, y_0) = (b(y_0 - x_0^2), x_0) = (x_0, y_1)$.
Then $y > x^2 + a\sqrt{x}$
$\Rightarrow y_0 - x_0^2 > a\sqrt{x_0}$
$\Rightarrow x_1 = b(y_0 - x_0^2) > \sqrt{x_0} = \sqrt{y_0}$
$\Rightarrow x_1^2 > y_1$.
Hence $g(x_0, y_0) \in S_1 \cup \ Q_2 \cup Q_3 \cup Q_4$, so there are no periodic points in S_2.

□

It is clear that S_3, S_4, S_5, S_6, and S_7 cannot map into S_1 or S_2 under forward iteration, because this would imply that they are in S_1 or $Q_2 \cup Q_3 \cup Q_4$, which is a contradiction since these regions are disjoint.

Proposition 7.15. S_5 *maps into* S_5*, and there are no periodic points in* S_5.

Proof. Let $(x_0, y_0) \in S_5$. Then $x_0 < y_0 = x_1$ and $y_0 < y_0^2 + ax_0 = y_1$ implies that $x_1 < y_1$.
Suppose

$$y_1 \geq x_1^2 + ax_1 = y_0^2 + ay_0 > y_0^2 + ax_0 = y_1 \Rightarrow\Leftarrow \tag{7.1}$$

So $y_1 < x_1^2 + x_1$. Clearly, from $y_1 > x_1^2$.
It follows from (7.1) that $y_1 < y_0^2 + ay_0 < y_1^2 + ay_0 = y_1^2 + x_1$.
$\therefore (x_1, y_1) \in S_5$.
Since the y_i are monotonically increasing, there are no periodic points in S_5. □

Proposition 7.16. S_3 *maps into* S_5.

Proof. Let $(x_0, y_0) \in S_3$.
$y_1 > y_0 \Rightarrow y_1 > x_1$ and $x_1 > x_0$.
Suppose $y_1 > x_1^2 + ax_1$
$y_1 \geq y_0^2 + y_0 > y_0^2 + ax_0 = y_1 \ \Rightarrow\Leftarrow$
So $y_1 < x_1^2 + ax_1$. As in Proposition 7.15, $y_1 < y_0^2 + ay_0 < y_1^2 + ay_0$.
$\therefore (x_1, y_1) \in S_5$. □

Corollary 7.2. *There are no periodic points in* S_3.

Proposition 7.17. *If* $(x_0, y_0) \in S_6 \cap T_1$ *then* $(x_1, y_1) \in S_3 \cup S_5$.

Proof. Since $\frac{y_0 - y_0^2}{a} < x_0$, $y_1 > y_0$, so $y_1 > x_1$.
Suppose $y_1 < x_1^2 + ax_1$.
$\Rightarrow y_1 < y_0^2 + ay_0$. But $x_0 > y_0 \Rightarrow y_1 > y_0^2 + ay_0 \Rightarrow\Leftarrow$. So if $y_1 < 1$ then $(x_1, y_1) \in S_3$.
Otherwise, $(x_1, y_1) \in S_5$. □

Proposition 7.18. T_1 *is a trapping region.*

Proof. Let $(x_0, y_0) \in T_1$.
Then $y_1 = y_0^2 + ax_0 > (1-a)^2 + a(1-a) = 1-a$.
Also, $x_1 = y_0 > 1-a$.
$\therefore (x_1, y_1) \in T_1$ □

Proposition 7.19. $f(S_7) \subset S_6 \cap T_1$.

Proof. Let $(x_0, y_0) \in S_7$.
Then $y_0 > x_0$ by geometry. Hence $x_1 > x_0$.
$\frac{y_0 - y_0^2}{a} > x_0 \Rightarrow y_0 > y_1 \Rightarrow x_1 > y_1$.
Furthermore, $y_1 = y_0^2 + ax_0 > x_1^2$. Since $x_0 > 1-a, y_0 > 1-a$, $(x_1, y_1) \in T_1 \cap S_6$. □

Corollary 7.3. *There are no periodic points in* T_1.

Proposition 7.20. *There are no periodic points in* $S_4 \cup (S_6 \cap T_4)$.

Proof. Let $(x_0, y_0) \in S_4$. Since $S_4 \subset T_2$, $x_1 = y_0 > 1-a$. So $(x_1, y_1) \in T_1 \cup T_4$. If $(x_1, y_1) \in T_1$, then (x_0, y_0) or $(x_0, y_0) \in T_4 \cup S_1$ then (x_0, y_0) is not a periodic point. Otherwise, $(x_1, y_1) \in S_6 \cap T_4$. But if $(x_1, y_1) \in T_4$ then $x_2 = y_1 < 1-a$, so $(x_2, y_2) \in T_2 \cup T_3$. If $(x_2, y_2) \in T_3$ then it is in the basin of attraction of $(0, 0)$ and is not a periodic point, as shown in a previous result. An analogous argument can be made with (x_0, y_0) initially in $S_6 \cap T_4$. It follows that if there are periodic points in $S_4 \cup (S_6 \cap T4)$, then they must alternate between T_2 and T_4.
Let $(x_0, y_0) \in S_4$, and suppose $x_0 > x_2$.
Then
$x_0 > y_0^2 + ax_0$
$\Rightarrow (1-a)x_0 > y_0^2$
But since $x_0 < 1-a$ and $y_0 > 1-a$, this is a contradiction.
So the x_{2i} are strictly increasing in S_4, and as a result the y_{2i+1} are strictly decreasing in $S_6 \cap T_4$. □

Theorem 7.1. *There are no periodic points of* f *in* $\mathbb{R}^2$ *other than the fixed points.*

Proof. This follows from all the previous propositions and corollaries, since a periodic point under forward iteration implies one under backwards iteration, and vice versa.

□

Proposition 7.21. $S_7 \cap W^s(p) = \emptyset$.

Proof. This follows from Propositions 7.19 and 7.17, and the strictly increasing y_i's established in the proof of 7.15. □

Proposition 7.22. $(S_2 \cap T_1) \cap W^s(p) = \emptyset$.

Proof. Let $(x_0, y_0) \in S_2 \cap T_1$. If $(x_0, y_0) \in M_1$ its orbit does not intersect the stable set, by Proposition 7.19.
If $(x_0, y_0) \in S_2 \cap T_1 \setminus M_1$, then $y_1 > y_0$, and $y_1 = y_0^2 + ax_0 > y_0^2 = x_1^2$, so $(x_1, y_1) \in T_1 \cap (S_2 \cup S_3 \cup S_5 \cup S_6 \cup S_7) = T_1 \setminus (S_1 \cap T_1)$. As long as the $(x_i, y_i) \in S_2 \cap (T_1 \setminus S_7)$, the y_i are strictly increasing. In all cases, the y_i eventually increase strictly, so $S_2 \cap T_1 = \emptyset$. □

Proposition 7.23. $(S_1 \cap T_1) \cap W^s(p) = \emptyset$.

Proof. $y_1 = y_0^2 + ax_0 > y_0^2 = x_1^2$. From Propositions 7.21,7.22, 7.17, 7.15, and 7.16, $(S_1 \cap T_1) \cap W^s(p) = \emptyset$. □

Corollary 7.4. *$T_1 \cap W^s(p) = \emptyset$.*

Corollary 7.5. *The orbits of the points in $W^s(p) \cap Q_1$ must alternate between T_2 and T_4.*

Proof. By Corollary 7.4 $T_1 \cap W^s(p) = \emptyset$. By Proposition 7.2, all points in T_3 are attracted to $(0,0)$ under forward iteration, so $T_3 \cap W^s(p) = \emptyset$.
So $W^s(p) \in S_2 \cup S_4$. Suppose $(x_0, y_0) \in S_4 \cap W^s(p)$. Since $x_1 = y_0 < 1 - a$, $(x_1, y_1) \in T_2$. Similarly, if $(x_0, y_0) \in S_2 \cap W^s(p)$, $x_1 = y_0 > 1 - a$, so $(x_1, y_1) \in T_4$. Clearly, this holds for all $(y_i, y_i) \in W^s(p) \cap Q_1$. □

This following theorem implies that the stable or unstable manifold cannot be self-intersecting.

Theorem 7.2. *Let Φ be a region in $\mathbb{R}^2$. Then $f^n(\Phi) \subset \overline{int\ f^n(\partial\Phi)}$, and $f^{-n}(\Phi) \subset \overline{int\ f^{-n}(\partial\Phi)}$.*

Proof. Let β be a rectangle in $\mathbb{R}^2$ with sides parallel to the axes;
$\beta = \{(x, y)|c \le x \le d, s \le y \le t\}$.
Label $A = (c, s), B = (c, t), C = (d, t), D = (d, s)$.
Then
$f(\overline{AB}) = (y_0, y_0^2 + ac) = (x_1, x_1^2 + ac), s < x_1 < t)$
$f(\overline{CD}) = (y_0, y_0^2 + ad) = (x_1, x_1^2 + ad), s < x_1 < t)$
$f(\overline{AD}) = (s, s^2 + ax_0), c < x_0 < d)$
$f(\overline{BC}) = (t, t^2 + ax_0), c < x_0 < d)$
Let $(x_\beta, y_\beta) \in \beta$. Then $f(x_\beta, y_\beta) = (y_\beta, y_\beta^2 + ax_\beta) = (x_{\beta 1}, y_{\beta 1}$.
$x_\beta > c \Rightarrow y_{\beta 1} = y_\beta^2 + ax_\beta > y_\beta^2 + ac$
$x_\beta < d \Rightarrow y_{\beta 1} = y_\beta^2 + ax_\beta < y_\beta^2 + ac$
And clearly, $s < y_\beta = x_{\beta 1} < t$. So $f(\beta) \subset \overline{\text{int } f(\partial\beta)}$

This argument can be extended to show that $f^n(\Phi) \subset \overline{\text{int } f^n(\partial\Phi)}$
Φ can be covered by a finite number of (not necessarily equal) β_i whose interiors do not intersect. As $n \to \infty$, $\bigcup_{i=1}^{n} \beta_i \to \Phi$, and the parts of the perimeters of the βi that are not common between two β_i approximate the boundary of Φ.

From above, $f(\beta_i) \subset \overline{\text{int } f(\partial\beta_i)}$ for all i. Since f is C^∞, the image of a closed curve remains closed and the image of $\bigcup_{i=1}^{n} \beta_i$ remains connected, so the image of points on its boundary are boundary points in its image under f. So $f(\partial \bigcup_{i=1}^{n} \beta_i) \subset \overline{\text{int } f(\bigcup_{i=1}^{n} \beta_i)}$.

This implies that $f(\Phi) \subset \overline{\text{int } f(\partial\Phi)}$. This process can be repeated infinitely often, by partitioning the $f^n(\Phi)$ with rectangles, βi_n, as was done for the base case. It follows that $f^n(\Phi) \subset \overline{\text{int } f^n(\partial\Phi)}$.

An identical process can be followed for f^{-1}, since if we impose the restriction that either $c \ge 0, d > 0$, or $c < 0, d \le 0$, then
$f^{-1}(\overline{AB}) = (b(y - c^2), c), s < y < t$

$f^{-1}(\overline{CD}) = (b(y-d^2), d), s < y < t$
$f^{-1}(\overline{AD}) = (b(s-x^2), x), c < x < d$
$f^{-1}(\overline{BC}) = (b(t-x^2), x), c < x < d$
Let $(x_\beta, y_\beta) \in \beta$. $f^{-1}(x_\beta, y_\beta) = (b(y_\beta - x_\beta^2), x_\beta) = (x_{\beta 1}, y_{\beta 1})$, where $b = \frac{1}{a}$.
Clearly, $c < x_\beta = y_{\beta 1} < d$.
For $c \geq 0, d > 0$, $c < x_\beta < d \Rightarrow b(y_\beta - c^2) > b(y_\beta - x_\beta^2) > b(y_\beta - d^2)$, and for $c < 0, d \leq 0, c < x_\beta < d \Rightarrow b(y_\beta - c^2) < b(y_\beta - x_\beta^2) < b(y_\beta - d^2)$.
Hence $f^{-1}(\beta) \subset \overline{\text{int } f^{-1}(\partial\beta)}$

□

Corollary 7.6. $W^s(p) \cap W^u(p) = \{p\}$

Proof. Suppose $(x_0, y_0) \in (W^s(p) \cap W^u(p)) \setminus \{p\}$.
Then (x_0, y_0) is a homoclinic point. This implies the existence of periodic points of infinitely many periods. But there are no periodic points other than the fixed points. $\Rightarrow\Leftarrow$ □

Since for consecutive iterations of a point (x_0, y_0), $d(f(x_0, y_0), f(x_0, y_0)) = 0 < \epsilon$ for all $\epsilon > 0$, the definition of a chain recurrent point can be stated as follows:

Definition 7.1. *Let F be a diffeomorphism. A point x is chain recurrent for F,if, for any $\epsilon > 0$, there are points $x = x_0, x_1, x_2, \ldots, x_k = x$, such that*
$|F(x_{i-1}) - x_i| < \epsilon$
for each i.

Lemma 7.1. *Suppose Φ is a trapping region under f (f^{-1}) where the y_i (x_i) are strictly increasing(decreasing), and whose boundaries are not determined by lines parallel to the axes. Then there are no chain recurrent points in Φ.*

Proof. Suppose $(x_0, y_0) \in \Phi$ is chain recurrent. Let $f(x_0, y_0) = (x_0', y_0')$. Construct $\Upsilon = \{(x, y) | y_0 \leq y \leq y_0'\}$. Consider $\overline{\Upsilon \cap \Phi}$. Since $\overline{\Upsilon \cap \Phi}$ is a closed subset of $\mathbb{R}^2$, it is compact. By compactness, $\forall (\hat{x_0}, \hat{y_0}) \in \Upsilon \cap \Phi$, for some $\delta > 0$, $\hat{y_0}' > y_0 + \delta$. Choose $\epsilon < \frac{\delta}{2}$.
Consider (x_1, y_1), as in Definition 7.1. Then

Case 1: $(x_1, y_1) \in \Upsilon \cap \Phi$.
Then $y_1' > y_1 > y_0 + \delta$, so y_2 cannot be in an ϵ-neighborhood of y_0.
Case 2: $(x_1, y_1) \notin \Upsilon \cap \Phi$.
Then $y_1 > y_0'$. So $y_1' > y_1 > y_0' > y_0 + \delta$, so y_2 cannot be in an ϵ-neighborhood of y_0.

By induction, it follows that either
- $(x_i, y_i) \in \Upsilon \cap \Phi$
  or
- $y_i > y_0'$.

So $d[(x_0, y_0), (x_i, y_i)] \geq |y_i - y_0| > \delta$, so $(x_i, y_i) \notin N_\epsilon(x_0, y_0)$, so there are no chain recurrent points in Φ. □

Lemma 7.2. *Suppose Γ is a trapping region under f^{-1} (f) where the norms of the iterates are strictly increasing(decreasing). Then there are no chain recurrent points in Γ.*

Proof. Suppose $(x_0, y_0) \in \Gamma$ is chain recurrent. Let $f^{-1}(x_0, y_0) = (x_0', y_0')$. Construct $\Psi = \{(x,y) | \|(x_0, y_0)\| < \|(x,y)\| < \|(x_0', y_0')\|\}$. Consider $\overline{\Gamma \cap \Psi}$. Since $\overline{\Gamma \cap \Psi}$ is a closed subset of $\mathbb{R}^2$, it is compact. By compactness, $\forall (\hat{x_0}, \hat{y_0}) \in \Gamma \cap \Psi$, for some $\delta > 0$, $\|(\hat{x_0}, \hat{y_0})\| > \|(x_0, y_0)\| + \delta$. Choose $\epsilon < \frac{\delta}{2}$.
Consider (x_1, y_1), as in Definition 7.1. Then

Case 1: $(x_1, y_1) \in \Gamma \cap \Psi$.
Then $\|(x_1', y_1')\| > \|(x_1, y_1)\| > \|(x_0, y_0)\| + \delta$, so (x_2, y_2) cannot be in an ϵ-neighborhood of y_0.
Case 2: $(x_1, y_1) \notin \Gamma \cap \Psi$.
Then $\|(x_1, y_1)\| > \|(x_0', y_0')\|$. So $\|(x_1', y_1')\| > \|(x_1, y_1)\| > \|(x_0, y_0)\| + \delta$, so y_2 cannot be in an ϵ-neighborhood of y_0.

By induction, it follows that either

- $(x_i, y_i) \in \Gamma \cap \Psi$
  or
- $\|(x_1, y_1)\| > \|(x_0', y_0')\|$.

So $d[(x_0, y_0), (x_i, y_i)] \geq |y_i - y_0| > \delta$, so $(x_i, y_i) \notin N_\epsilon(x_0, y_0)$, so there are no chain recurrent points in Γ. □

Theorem 7.3. *There are no chain recurrent points of f in $\mathbb{R}^2$.*

Proof. If the orbit of $(x_0, y_0) \in A$ alternates between disjoint regions A and B with increasing (decreasing) y_i (x_i) in A, since ϵ can be chosen so that $\epsilon < \inf d[(x_0, y_0), \partial B]$, so all odd iteration will not be in an ϵ-neighborhood of (x_0, y_0), and Lemma 7.1 can be applied to even iterations. In light of the strictly increasing (decreasing) arguments made in the proof of Theorem (where we show there are no periodic points), and Lemma 7.1 and Lemma 7.2, there are no chain recurrent points of f in $\mathbb{R}^2$. □

8. Future Questions

Since this research was conducted over an eight week summer program, many questions remain. The project has furnished us with an overview of the recurrent dynamics in $\mathbb{R}^2$for the maps F in the parameter space $a > 0$ and $\frac{-3(1-a)^2}{4} < c < \frac{(1-a)^2}{4}$, but we seek to extend this result by proving,

Conjecture 8.1. *All maps in F_P are structurally stable.*

Conjecture 8.2. *All maps in F_U are structurally stable.*

Conjecture 8.3. *All maps in the F_P are topologically conjugate.*

Conjecture 8.4. *All maps in the F_U are topologically conjugate.*

Conjecture 8.5. *All maps in F_P under forward iteration are topologically conjugate to maps in F_U under backwards iteration.*

The approach we are currently pursuing to prove Conjectures 8.3 and 8.4 involves proving Conjectures 8.1 and 8.2 by compactifying $\mathbb{R}^2$in a manner that is consistent with the dynamics of F, and appealing to

Theorem 8.1. [7] *Let M be a compact C^∞ manifold without boundary, and for $r \geq 1$ let $Diff(M)$ be the set of C^r diffeomorphisms of M with the uniform C^r topology. Consider $F \in Diff(M)$ which satisfies*

(1) *$\Omega(f)$ is finite,*

(2) *$\Omega(f)$ is hyperbolic,*

(3) *Transversality Condition.*

Then f is structurally stable.

Furthermore, we seek to construct conjugacies between maps of the families $F_{(a,0)}$, $0 < a < 1$ and $F(a, 0)$, $a > 1$.

Other potential questions include classifying the dynamics for maps in regions of the parameter space where $a < 0$, on the line $x = 1$, and for regions where $c > \frac{(1-a)^2}{4}$.

We shall continue pursuing these questions through the remainder of the summer and into the academic year.

References

[1] R. Abraham and S. Smale, *Nongenericity of Ω-stability*, Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society, **XIV**(1968),5–8.

[2] Eric Beford and John Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure, and hyperbolicity*, Inventiones Mathematicae, **103** (1991),69–99.

[3] Eric Bedford and John Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$. II: stable manifolds and recurrence*, **4**, no. 4, (October 1991), 657–679.

[4] Michael Benedicks and Lennart Carleson. *The dynamics of the Hénon map*, The Annals of Mathematics, 2nd Series, **131**, no. 1 (January 1991), 73–169.

[5] Schmuel Friedland and John Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory and Dynamical Systems, **9** (1989),67–99.

[6] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, 2nd edition, 1989.

[7] J. Palis and S. Smale, *Structural stability theorems*, Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society, **XIV**(1968),223–231.

[8] Clark Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 2nd edition, 1999.

[9] Rasul Shafikov and Christian Wolf, *Filtrations, hyperbolicity, and dimension for a polynomial auutomorphism of $\mathbb{C}^n$*, Michigan Math Journal, **51**(2003), no. 3, 631–649.

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