ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF f-INVARIANT δ-SCRAMBLED SETS

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Abstract. This paper is a study in the necessary and sufficient conditions of a map for the existence of f-invariant δ-scrambled sets. It has been shown [3] that all turbulent maps have such sets. Here, we give our own version of the proof that all strictly turbulent maps have such sets. We go on to show that any map with an f-invariant δ-scrambled set is chaotic, and give an example of a map with such a set that has no periodic points of odd period.

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1. Introduction

Definition 1. *A* δ-scrambled set *for a map* $f : I \to I$ *is an uncountable set* $S \subset I$ *such that given* $\delta > 0$*, fixed,*

(1) *for every* $s_1, s_2 \in S$ *with* $s_1 \neq s_2$*,*

$$\limsup_{n\to\infty} d[f^n(s_1), f^n(s_2)] \geq \delta, \tag{1}$$

$$\liminf_{n\to\infty} d[f^n(s_1), f^n(s_2)] = 0, \tag{2}$$

(2) *for any* $s \in S$ *and any periodic point* z*,*

$$\limsup_{n\to\infty} d[f^n(s), f^n(z)] \geq \delta. \tag{3}$$

Such a set S *is* f-invariant *if for any* $s \in S$ *and any* $n \geq 0$*,* $f^n(s) \in S$*.*

Definition 2. *A map* $f : I \to I$ *is* turbulent *if there exist compact subintervals* J, K *with at most one common point such that*

$$J \cup K \subseteq f(J) \cap f(K).$$

f *is* strictly turbulent *if* J *and* K *can be chosen disjoint.*

Definition 3. *A map* $f : I \to I$ *is* chaotic *if the following equivalent conditions hold for* f*:*

(1) f *has a periodic point whose period is not a power of 2,*
(2) f^m *is strictly turbulent for some positive integer* m*,*
(3) f^n *is turbulent for some positive integer* n*.*

Block and Coppel term the following a *turbulence stratification*:

$$\mathbb{S}_1 \subset \mathbb{T}_1 \subset \mathbb{P}_3 \subset \mathbb{P}_5 \subset ... \subset \mathbb{S}_2 \subset \mathbb{T}_2 \subset \mathbb{P}_6 \subset \mathbb{P}_{10} \subset ... \subset \mathbb{S}_4 \subset \mathbb{T}_4 \subset \mathbb{P}_{12} \subset \mathbb{P}_{20} \subset ... \subset \mathbb{K} \subset ... \subset \mathbb{P}_8 \subset \mathbb{P}_4 \subset \mathbb{P}_2 \subset \mathbb{P}_1$$

where $\mathbb{S}_k$ designates the set of maps f such that f^k is strictly turbulent, $\mathbb{T}_k$ the set of maps f such that f^k is turbulent, $\mathbb{P}_k$ the set of maps with periodic points of period k, and $\mathbb{K}$ the set of all chaotic maps. The goals of this project are twofold:

(1) to determine whether containment in any one of these sets implies the existence of an f-invariant δ-scrambled set for that map.
(2) to determine whether all maps with f-invariant δ-scrambled sets are contained in one of the sets listed in the above stratification.

Before we begin dealing with f*-invariant* δ-scrambled sets, we'll answer the above questions for general δ-scrambled sets. In order to do that, we must recognize a certain subset of the non-chaotic maps:

Definition 4. *A point* $x \in I$ *is* approximately periodic *if for every* $\epsilon > 0$*, there exists a periodic point* y *and a positive integer* N *such that*

$$d[f^n(x), f^n(y)] < \epsilon \text{ for all } n > N.$$

Definition 5. *A map* $f : I \to I$ *is* uniformly non-chaotic *if every point* $x \in I$ *is approximately periodic.*

Block and Coppel show that a uniformly non-chaotic map is non-chaotic.

It follows from the definitions that no uniformly non-chaotic map can have a δ-scrambled set, for if $f : I \to I$ is uniformly non-chaotic, then given any δ we can choose an $\epsilon < \delta$ and for any $x \in I$ we can find a periodic point y of f such that

$$\limsup_{n\to\infty} d[f^n(x), f^n(y)] < \epsilon < \delta.$$

So *no* $x \in I$ could satisfy (2). Furthermore, it is known that all chaotic maps and non-chaotic maps which are *not* uniformly non-chaotic do have δ-scrambled sets [Propositions VI.26 and VI.27 of [1]]. Thus being either chaotic or non-uniformly non-chaotic is both a necessary and a sufficient condition for a map to have a δ-scrambled set.

2. Results

2.1. **Sufficient Conditions for the Existence of f-Invariant δ-Scrambled Sets.** As we have complete knowledge about the maps in which δ-scrambled sets can and do exist, we now turn our attention to f-invariant δ-scrambled sets. It has been shown [3] that all turbulent maps have f-invariant δ-scrambled sets. We have a version of the weaker result that all strictly turbulent maps have f-invariant δ-scrambled sets.

Proposition 1. *For a strictly turbulent map $f : I \to I$, there exists a subset S of I such that for some fixed $\delta \geq 0$, S is both δ-scrambled and f-invariant. i.e., S satisfies (1), (2), and (3), and given any $s \in S$ and any $n \geq 0$, $f^n(s) \in S$.*

Proof. Suppose $f : I \to I$ is strictly turbulent. Then there are disjoint, compact intervals $I_0, I_1 \subseteq I$ such that $I_0 \cup I_1 \subseteq f(I_0) \cap f(I_1)$. Since I_0 and I_1 are disjoint, $d[I_0, I_1] > 0$. Let $\delta = d[I_0, I_1]/2$. For a_i=0 or 1, let $I_{a_1 a_2}$ be the subinterval of minimum length of I_{a_1} that is mapped to I_{a_2} (i.e., $f(I_{a_1 a_2}) = I_{a_2}$). Similarly, let $I_{a_1 \dots a_k}$ be the subinterval of minimum length of $I_{a_1 \dots a_{k-1}}$ that is mapped to $I_{a_2 \dots a_k}$. Let Σ be the set of all infinite sequences of 0's and 1's, and for $\alpha \in \Sigma$, define I_α as above. Let σ be the shift operator, where $\sigma : \Sigma \to \Sigma$ is given by $\sigma(a_1, a_2, a_3, ...) = (a_2, a_3, ...)$. Let $X \subseteq I$ be the set of all endpoints of the I_α, and define $h : X \to \Sigma$ by $h(x) = \alpha$ if $x \in \alpha$. Since I_α has at most two endpoints (and only one if it is a point rather than an interval), each α is the image of at most two $x \in X$. By Proposition II.15 of [1], h is continuous and $h \circ f(x) = \sigma \circ h(x)$

Let $\beta \in \Sigma$ be given by $(b_1, b_2, b_3, ...)$ and consider the sequences of the form $\gamma_\beta = (0, b_1, 1, 1, b_1, b_2, 0, 0, 0, b_1, b_2, b_3, ...)$. Let x_β be the smaller $x \in X$ such that $h(x_\beta) = \gamma_\beta$ (if I_β is a point, then there is only one such x). Consider the set $S = \{f^n(x_\beta) | n \geq 0\}$.

This set certainly is certainly f-invariant, for if $x \in S$, then $x = f^m(x_\beta)$ for some $m \geq 0$ and some x_β. Then for any $n \geq 0$, $f^n(x) = f^{n+m}(x_\beta)$, and since $n, m \geq 0$, $n + m \geq 0$, and $f^{n+m}(x_\beta) \in S$ as well.

Now consider $s_1, s_2 \in S, s_1 \neq s_2$. Show that (1) holds.

(1) Suppose $s_1 = f^m(x_\beta), s_2 = f^m(x_{\beta'})$
Since $s_1 \neq s_2$, $x_\beta \neq x_{\beta'}$, and since we have chosen one $x \in X$ such that $h(x) = \alpha$, for every $\alpha \in \Sigma$, $\beta \neq \beta'$, i.e., there exists some k such that $b_k \neq b'_k$. So for all $n \geq k$, $d[f^{n^2-m+k-1}(s_1), f^{n^2-m+k-1}(s_2)] \geq 2\delta$. So we can take a sequence of iterations of f that are $n^2 - m + k - 1$ for $n \geq k$ (with k and m fixed as above), and the distance between the images of s_1 and s_2 for these iterations is greater than or equal to 2δ, because the images will be in different intervals. So $\limsup_{n\to\infty} d[f^n(s_1), f^n(s_2)] \geq 2\delta > \delta$.

(2) Suppose $s_1 = f^\ell(x_\beta), s_2 = f^m(x_{\beta'})$, and suppose, without loss of generality, that $m > \ell$

$f^{n^2-m}(s_1) \in I_{b_1}$, for all n such that $n^2 > m$. $b_1 = 0$ *or* 1 (b_1 is fixed).
$f^{n^2-m}(s_2), n > m - \ell$, is alternately contained in I_0 and I_1.
So there is an infinite sequence of n's, $n > m-\ell$, such that $d[f^{n^2-m}(s_1), f^{n^2-m}(s_2)] \geq 2\delta$. So $\limsup_{n\to\infty} d[f^n(s_1), f^n(s_2)] \geq 2\delta > \delta$

These two cases cover every possible pairing of elements in S, since any two elements are shifted from the original form x_β either by the same amount or by different amounts.

Consider again $s_1, s_2 \in S, s_1 \neq s_2$. Show that (2) holds.

(1) Suppose $s_1 = f^m(x_\beta), s_2 = f^m(x_{\beta'})$
Then, for n such that $n^2 - n > m$,
$$\lim_{n\to\infty} d[f^{n^2-n-m}(s_1), f^{n^2-n-m}(s_2)] = 0.$$
So $\liminf_{n\to\infty} d[f^n(s_1), f^n(s_2)] = 0$, as required.

(2) Suppose $s_1 = f^\ell(x_\beta), s_2 = f^m(x_{\beta'})$, and suppose, without loss of generality, that $m > \ell$.
Then, for n such that $n^2 - n > \ell$, $n > m - \ell$,
$$\lim_{n\to\infty} d[f^{n^2-n-\ell}(s_1), f^{n^2-n-\ell}(s_2)] = 0.$$
So $\liminf_{n\to\infty} d[f^n(s_1), f^n(s_2)] = 0$.

As before, these two cases cover every possible pairing of elements in S.

Now consider $s \in S$ and periodic point z. Show that (3) holds. Suppose $s = f^m(x_\beta)$, z is periodic of period p. Then for $n > p + 1$, $n^2 - n > m$, there is some multiple of p contained in the interval of integers $[n^2 - m - n, n^2 - m - 1]$. Moreover, for n such that $n^2 - n > m$, $f^{n^2-m-n}(s) \in I_\alpha$, where α is a sequence whose first n entries are either all 0 or all 1 (this alternates with n). So given a sequence of k such that k is a multiple of p and $k \in [n^2 - m - n, n^2 - m - 1]$ for some $n > p + 1, n^2 - n > m$, $f^k(s)$ alternates between being contained in I_0 and I_1. For such k, however, $f^k(z) = z$, so the image of z is fixed under this sequence of iterations. z must either be closer to one of I_0, I_1, or halfway between them. Either way, there is a sequence of n (choose alternating values of the k mentioned above: those such that $f^k(s) \in I_0$ if z is closer to I_1, or those such that $f^k(s) \in I_1$! if z is closer to I_0. If z is halfway between I_0 and I_1, then we can take the entire sequence of multiples of p contained in intervals $[n^2 - m - n, n^2 - m - 1]$ for $n > p+1, n^2 - n > m$.) such that $d[f^n(s), f^n(z)] \geq \delta$ for all n in that sequence. Thus $\limsup_{n\to\infty} d[f^n(s), f^n(z)] \geq \delta$, as required. □

2.2. **Necessary Conditions for the Existence of an f-Invariant δ-Scrambled Set.** We now explore the necessary conditions of a map for the existence of an f-invariant δ-scrambled set.

Proposition 2. *A map $f : I \to I$ that has an f-invariant δ-scrambled set has periodic points of periods 2 and 2r for all positive, even r.*

Proof. Suppose that $f : I \to I$ has an f-invariant δ-scrambled set S, i.e., $S \subset I$ is uncountable, f-invariant, and satisfies (1), (2) and (3) for some fixed $\delta > 0$. Suppose, further, that for some $r \geq 1$, f^r does not have a periodic point of period 2. Then, by Proposition VI.1 of [1], for every $c \in I$, $\{f^{kr}(c)\}$ converges to some fixed point of f^r. Choose $s \in S \subset I$, and let w be the fixed point of f^r such that $\{f^{kr}(s)\} \to w$.

Since f is uniformly continuous, $f(f^{kr}(s)) \to f(w)$ as $f^{kr}(s) \to w$. Similarly (by repeated composition of the continuous f), for any $n \geq 0$, $f^n(f^{kr}(s)) \to f^n(w)$ as $f^{kr}(s) \to w$. Therefore, for any $n \geq 0$, $\limsup_{k\to\infty} d[f^n(f^{kr}(s)), f^n(w)] = \underset{k\to\infty}{lim} d[f^n(f^{kr}(s)), f^n(w)] = 0$.

There are r distinct "sub-trajectories" of s: $\{f^{kr}(s)\}$, $\{f^{kr+1}(s)\}$, ..., $\{f^{kr+(r-1)}(s)\}$. For any $n > 0$, $f^n(s)$ falls into exactly one of these sub-trajectories, since n can be written as $kr + \ell$ with $\ell \in \{0, 1, ..., r-1\}$ in exactly one way. As seen above, by the uniform continuity of f, for any $\ell \in \{0, 1, ..., r-1\}$,

$$\{f^{kr+\ell}(s)\} \to f^\ell(w).$$

So for each $\{f^{kr+\ell}(s)\}$ and any $\epsilon > 0$, there is a $K_{\epsilon,\ell}$ such that for all $k > K_{\epsilon,\ell}$, $d[f^{kr+\ell}(s), f^\ell(w)] < \epsilon$. Take the set of all such $K_{\epsilon,\ell}$, and choose its maximum: call it $K_{\epsilon,max}$. Then, for all $k > K_{\epsilon,max}$ and *all* ℓ, $d[f^{kr+\ell}(s), f^\ell(w)] < \epsilon$. Moreover, for any n such that $\frac{n-\ell}{r} > K_{\epsilon,max}$ (where ℓ depends on n: $\ell = n \ (mod\ r)$), $d[f^n(s), f^n(w)] < \epsilon$. This can be done for any $\epsilon > 0$, so

$$\limsup_{n\to\infty} d[f^n(s), f^n(w)] = \lim_{n\to\infty} d[f^n(s), f^n(w)] = 0.$$

However, $s \in S$ and since w is fixed for f^r, it is periodic for f, so by (2) above, $\limsup_{n\to\infty} d[f^n(s), f^n(w)] \geq \delta > 0$. Thus we have a contradiction, and so f^r *does* have a periodic point of period 2. This point will have a prime period for f that divides $2r$. If its prime period divides r, then it would be fixed by f^r, which it is not. Thus its prime period for f can only be 2 or $2r$. If r is even, $2|r$, so the point cannot have prime period 2, and must instead have prime period $2r$. If r is odd, it is possible that the point have prime period 2 for f. So we know that f has periodic points of periods 2 and $2r$ for every positive, even r. $\square$

Corollary 1. *If $f : I \to I$ has an f-invariant δ-scrambled set, f is chaotic.*

We now know that any $f \in \mathbb{T}_1$ has an f-invariant δ-scrambled set, and that if f has an f-invariant δ-scrambled set, then $f \in \mathbb{P}_{12} \subset \mathbb{K}$. We continue to look for further restrictions on the type of map that can have an f-invariant δ-scrambled set - i.e., is there a proper subset of $\mathbb{P}_{12}$ that contains all such maps?
The following theorem appears in [1]:

Theorem 1. *Suppose that, for some $c \in I$ and some $n > 1$,*

$$f^n(c) \leq c < f(c)$$

1. *If n is odd, then f has a periodic point of period q, for some odd q satisfying $1 < q \leq n$.*
2. *If n is even, then at least one of the following alternatives holds:*
   (a) *f has a periodic point of period q, for some odd q: $1 < q \leq \frac{n}{2} + 1$*
   (b) *$f^k(c) < f^j(c)$ for all even k and all odd j with $0 \leq j, k \leq n$.*

So if we can show that any $f : I \to I$ with an f-invariant δ-scrambled set has points $x \in I$ such that the above inequality holds for some odd n, or for some even n for which (2a) is satisfied, then we will know that these maps have periodic points for some odd period.

Lemma 1. *If $f : I \to I$ has an f-invariant δ-scrambled set, then there are uncountably many $x \in I$ for which there is some $n > 1$ such that either*

$$f^n(x) < x < f(x) \tag{4}$$

or

$$f(x) < x < f^n(x) \tag{5}$$

Proof. Suppose $f : I \to I$ has an f-invariant δ-scrambled set S, and that there is *no* $x \in I$ such that, for some $n > 1$, either (1) or (2) holds. Then, in particular, for $s \in S$ [s not fixed] if $f(s) > s$, then there is no $n > 0$ such that $f^n(s) \leq s$, and if $s > f(s)$, then there is no $n > 0$ such that $f^n(s) \geq s$. So if $s < f(s)$, then for all $n > 0$, $s < f^n(s)$, and if $s > f(s)$, then for all $n > 0$, $s > f^n(s)$. The same applies for every $f^k(s)$, $k > 0$, since these are all members of S. That is, if $f^k(s) < f^{k+1}(s)$, then for all $\ell > k$, $f^k(s) < f^\ell(s)$, and if $f^k(s) > f^{k+1}(s)$, then for all $\ell > k$, $f^k(s) > f^\ell(s)$. Then, by definition, $\{f^k(s)\}$ is *bimonotonic.*

Definition 6. [1] *A sequence $\{x_k\}$ of real numbers is* bimonotonic *if for every $m \geq 0$, either $x_k > x_m$ for all $k > m$, or $x_k = x_m$ for all $k > m$, or $x_k < x_m$ for all $k > m$.*

Here we take $x_k = f^k(s)$. Following the statement of the definition of bimonotonic, Block and Coppel assert: "A sequence $\{x_k\}$ is bimonotonic if and only if, for some $c \in [-\infty, \infty]$, the terms $x_k < c$ form an increasing sequence, and the terms $x_k > c$ form a decreasing sequence, and $x_k = c$ implies $x_{k+1} = c$." Note that here, because s is a member of an f-invariant δ-scrambled set and is not fixed, it is certainly not eventually fixed, so for no k_1, k_2 do we have $f^{k_1}(s) = f^{k_2}(s)$. We are dealing just with strict inequalities within the trajectory of s.

Since we have some $c \in I$ (since all the x_k in question are members of I) as described above, the orbit of s is divided into at most two bounded, monotonic sequences. In fact, we know that the orbit cannot be a single monotonic sequence (note that monotonicity is a stronger condition than bimonotonicity, and in the case of a sequence being monotonic, we could take c to be some value either entirely below or entirely above the sequence, depending on whether it was increasing or decreasing. c could be one of the endpoints of I, for example.) because then the entire orbit would converge (all bounded monotonic sequences converge), and we would have points $x, y \in S$, for example, s and $f(s)$, such that $\limsup_{n\to\infty} d[f^n(x), f^n(y)] = 0$, contrary to the definition of a δ-scrambled set. For the same reason, once we divide the orbit into two separate bounded, monotonic sequences, each of which must converge, we know that they must have different limits, else the entire orbit would converge.

Let α be the limit of the lower, increasing sequence (call it $\{f^k(s)\}$), and let β be the limit of the upper, decreasing sequence (call it $\{f^\ell(s)\}$. Since we know $\alpha \neq \beta$, $d[\alpha, \beta] = \gamma > 0$. Moreover, since all the points in the orbit of s are members of the same f-invariant δ-scrambled set, we must have $\gamma \geq \delta$. (That is, the points in the orbit will get arbitrarily close to α and β, and we need subsequences of iterations of f such that the distances between the images of any two points in the orbit of s converge to something $\geq \delta$. As we look farther and farther along in the sequence of iterations (for both points), however, the greatest possible distance will approach

γ (from above), so if $\gamma < \delta$, we will not be able to satisfy the first condition of δ-scrambledness.)

Because f is continuous, for any $\epsilon > 0$ there is a $\xi > 0$ such that for $x, y \in I$ such that $d[x,y] < \xi$, $d[f(x), f(y)] < \epsilon$. Choose $\epsilon_0 < \gamma$, and let ξ_0 be the appropriate corresponding ξ.

Because $\{f^k(s)\} \to \alpha$, there is a K_0 such that for $n_1, n_2 > K_0$, $d[f^{n_1}(s), f^{n_2}(s)] < \xi_0$ (since the sub-trajectory converges, it is Cauchy), where n_1 and n_2 are iterates of f such that $f^{n_1}(s), f^{n_2}(s) \in \{f^k(s)\}$. Now suppose that there is a $n > K_0$ such that both $f^n(s), f^{n+1}(s) \in \{f^k(s)\}$, but $f^{n+2}(s) \in \{f^\ell(s)\}$. (It is certainly not the case that for all n, $f^n(s)$ and $f^{n+1}(s)$ lie in the same sub-trajectory, because then s's entire trajectory would converge. So if there are indeed n such that $f^n(s)$ and $f^{n+1}(s)$ lie in the same sub-trajectory, then there must be such a pair with $f^{n+2}(s)$ lying in the other sub-trajectory). Since $n > K_0$, $d[f^n(s), f^{n+1}(s)] < \xi_0$. So we know also that $d[f(f^n(s)), f(f^{n+1}(s))] = d[f^{n+1}(s), f^{n+2}(s)] < \epsilon_0 < \gamma$. But $f^{n+1}(s) \in \{f^k(s)\}$, which is an increasing sequence converging to α, and so $f^{n+1}(!s) < \alpha$. Similarly, $f^{n+2}(s) \in \{f^\ell(s)\}$, which is a decreasing sequence converging to β, and so $f^{n+2}(s) > \beta$. Since $d[\alpha, \beta] = \gamma$, $d[f^{n+1}(s), f^{n+2}(s)] > \gamma$. Thus we have a contradiction when we assume that there are $n > K_0$ such that $f^n(s)$ and $f^{n+1}(s)$ lie in the same sub-trajectory.

The only other alternative is to have the orbit of s be alternating for $n > K_0$. That is, either for all $j, k > K_0$, with j even and k odd, $f^k(s) < f^j(s)$, or for all such j, k, $f^j(s) < f^k(s)$. But then for every $n > K_0$, $d[f^n(s), f^{n+1}(s)] > \gamma \geq \delta$, so $\liminf_{n\to\infty} d[f^n(s), f^n(f(s))] \geq \delta > 0$, which contradicts the fact that s and $f(s)$ are in the same δ-scrambled set.

We have considered all the possibilities for a bimonotonic trajectory of a non-fixed $s \in S$, and in each case have ended up with a contradiction. Therefore, it must not be the case that any (non-fixed) member of the f-invariant δ-scrambled set has a bimonotonic trajcectory. So there are points $x \in I$ for which either $f(x) > x$ and for some $n > 1$, $f^n(x) < x$, or $f(x) < x$, and for some $n > 1$, $f^n(x) > x$. □

We have not shown, however, that it is impossible for every n that works to be even with only the second condition satisfied, so there is still no guarantee that these maps have periodic points of odd period. In fact, the following example shows that there *are* maps with no periodic points of odd period greater than 1 that *do* have f-invariant δ-scrambled sets.

Proposition 3. *The map given by*

$$T(x) = \begin{cases} 2x+1, \text{ if } 0 \leq x < \frac{1}{2} \\ -2x+3, \text{ if } \frac{1}{2} \leq x < 1 \\ -x+2, \text{ if } 1 \leq x \leq 2 \end{cases} \tag{6}$$

has no periodic points of odd period and has a T-invariant δ-scrambled set.

Proof. First note that $T : [0,1] \to [1,2]$, and $T : [1,2] \to [0,1]$. So even iterations of T map these intervals onto themselves, and odd iterations of T switch the intervals. The only point in either interval that could be mapped onto itself by an odd iteration of T is their intersection, 1, and 1 is fixed by T. So T has no periodic points of odd period > 1.

To show that T has a T-invariant δ-scrambled set, first note that every point in the interval [0,1] can be written as an infinite sum

$$\frac{a_1}{2^1}+\frac{a_2}{2^2}+\frac{a_3}{2^3}+...$$

where for all i, $a_i = 0$ or 1
We take $\Omega = \{0.a_1a_2a_3...|(\forall i)\ a_i = 0 \text{ or } 1\}$. Then we let $h : [0,1] \to \Omega$ be given by:

$$h(x) = .a_1a_2a_3... \tag{7}$$

for $x = \frac{a_1}{2^1}+\frac{a_2}{2^2}+\frac{a_3}{2^3}+....$

Next, note that $T^2(x) : [0,1] \to [0,1]$, and is given by

$$T^2(x) = \begin{cases} -2x+1, \text{if } 0 \le x < \frac{1}{2} \\ 2x-1, \text{if } \frac{1}{2} \le x \le 1 \end{cases} \tag{8}$$

We now construct the following topologically conjugate map, $f : \Omega \to \Omega$:

$$f(\alpha) = \begin{cases} .\bar{a}_2\bar{a}_3\bar{a}_4..., \text{if } a_1 = 0 \\ .a_2a_3a_4..., \text{if } a_1 = 1 \end{cases} \tag{9}$$

where $\alpha = 0.a_1a_2a_3...$, and $\bar{a}_i = 0$ if $a_i = 1$, and $\bar{a}_i = 1$ if $a_i = 0$. (Note that the first case corresponds to $h^{-1}(\alpha) \in [0, \frac{1}{2}]$, and the second case corresponds to $h^{-1}(\alpha) \in [\frac{1}{2}, 1]$.) Consider the set,

$$S = \{.01^{n_1}01^{n_2}01^{n_3}...\} \cup \{.1^{n_1}01^{n_2}01^{n_3}...\} \cup \{.10^{n_1}10^{n_2}10^{n_3}...\} \cup \{.0^{n_1}10^{n_2}10^{n_3}...\}$$

where $n_1 < n_2 < n_3 < ...$, and a^{n_i} represents a string of a's that is n_k long, for $a = 0$ or 1. Furthermore, for any two points $\alpha, \beta \in S$, such that α and β are not contained in the same orbit, there are infinitely many k for which $(n_k)_\alpha \ne (n_k)_\beta$.

Note that S is f-invariant: Suppose $\alpha \in S$. Then there are four possible cases:

1. $\alpha = .01^{n_1}01^{n_2}01^{n_3}...$, for some $n_1 < n_2 < n_3 < ....$ Then $f(\alpha) = .0^{n_1}10^{n_2}10^{n_3}... \in S$.
2. $\alpha = .1^{n_1}01^{n_2}01^{n_3}...$, for some $n_1 < n_2 < n_3 < ....$ Then $f(\alpha) = .1^{n_1-1}01^{n_2}01^{n_3}... \in S$.
3. $\alpha = .10^{n_1}10^{n_2}10^{n_3}...$, for some $n_1 < n_2 < n_3 < ....$ Then $f(\alpha) = .0^{n_1}10^{n_2}10^{n_3}... \in S$.
4. $\alpha = .0^{n_1}10^{n_2}10^{n_3}...$, for some $n_1 < n_2 < n_3 < ....$ Then $f(\alpha) = .1^{n_1-1}01^{n_2}01^{n_3}... \in S$.

Note also that in each of the above cases the set of string lengths for $f(\alpha)$ is the same - at least after the first string - as the set of string lengths for α. So any point in S not in the same orbit will have infinitely many string lengths that differ from the corresponding ones for $f(\alpha)$, as required for membership in S.

We will now show that S is δ-scrambled for f. First, we will show that given any points $\alpha, \beta \in S$, $\alpha \ne \beta$, (1) holds.

First suppose that α and β are not contained in the same orbit (there is no k such that either $\alpha = f^k(\beta)$ or $\beta = f^k(\alpha)$). Then, by construction of S, there are infinitely many k for which $(n_k)_\alpha \ne (n_k)_\beta$. It follows that there are infinitely many positions where one of α, β starts a new string of 0's or 1's where the other does not. So we can pick infinitely many k such that f^k shifts the binary expansions so that one of them is at a new "starting point" while the other is not. Without loss of generality, suppose f^k shifts α to a new "starting point", and β to the middle of a string. Then we have $f^k(\alpha) = .01^{n_\ell}...$ or $f^k(\alpha) = .10^{n_\ell}$, and $f^k(\beta) = .00...$ or $f^k(\beta) = .11....$

1. Suppose $f^k(\alpha) = .01^{n_\ell}...$

(a) Suppose $f^k(\beta) = .00....$ Then
$$d[f^k(\alpha), f^k(\beta)] \geq \tfrac{1}{8}$$
(b) Suppose $f^k(\beta) = .11....$ Then
$$d[f^k(\alpha), f^k(\beta)] \geq \tfrac{1}{4}$$
(2) Suppose $f^k(\alpha) = .10^{n_\ell}...$

(a) Suppose $f^k(\beta) = .00....$ Then
$$d[f^k(\alpha), f^k(\beta)] \geq \tfrac{1}{4}$$
(b) Suppose $f^k(\beta) = .11....$ Then
$$d[f^k(\alpha), f^k(\beta)] \geq \tfrac{1}{8}.$$

Since there are infinitely many k for which we can do this, and in each case, the images of α and β under f^k are at least $\frac{1}{8}$ apart,

$$\limsup_{n\to\infty} d[f^n(\alpha), f^n(\beta)] \geq \tfrac{1}{8}.$$

Now suppose that $\beta = f^k(\alpha)$ for some k. Choose $n_k \geq k$, and let j_{k+1} be the number such that $f^{j_{k+1}}(\alpha) = .01^{n_{k+1}}01^{n_{k+2}}...$ or $.10^{n_{k+1}}10^{n_{k+2}}....$ Then $f^{j_{k+1}-k}(\alpha) = .1^k01^{n_{k+1}}01^{n_{k+1}}...$ or $.0^k10^{n_{k+1}}10^{n_{k+2}}...$, and $f^{j_{k+1}-k}(\beta) = .01^{n_{k+1}}01^{n_{k+2}}...$ or $.10^{n_{k+1}}10^{n_{k+2}}....$

(1) Suppose $f^{j_{k+1}-k}(\alpha) = .1^k01^{n_{k+1}}01^{n_{k+2}}....$

(a) Suppose $f^{j_{k+1}-k}(\beta) = .01^{n_{k+1}}01^{n_{k+2}}....$ Since $k \geq 1$ and $n_{k+1} > k$,
$$d[f^{j_{k+1}-k}(\alpha), f^{j_{k+1}-k}(\beta)] \geq \tfrac{3}{16}$$
(b) Suppose $f^{j_{k+1}-k}(\beta) = .10^{n_{k+1}}10^{n_{k+2}}....$ Since $k \geq 1$, $n_{k+1} > k$,
$$d[f^{j_{k+1}-k}(\alpha), f^{j_{k+1}-k}(\beta)] \geq \tfrac{1}{16}$$
(2) Suppose $f^{j_{k+1}-k}(\alpha) = .0^k10^{n_{k+1}}10^{n_{k+2}}....$

(a) Suppose $f^{j_{k+1}-k}(\beta) = .01^{n_{k+1}}01^{n_{k+2}}....$ Since $k \geq 1$, $n_{k+1} > k$,
$$d[f^{j_{k+1}-k}(\alpha), f^{j_{k+1}-k}(\beta)] \geq \tfrac{1}{16}$$
(b) Suppose $f^{j_{k+1}-k}(\beta) = .10^{n_{k+1}}10^{n_{k+2}}....$ Since $k \geq 1$, $n_{k+1} > k$,
$$d[f^{j_{k+1}-k}(\alpha), f^{j_{k+1}-k}(\beta)] \geq \tfrac{3}{16}$$

Since the above can be done for any j_k for which the corresponding n_k is greater than or equal to k, there are infinitely many iterations of f for which one of the above cases holds. Thus,

$$\limsup_{n\to\infty} d[f^n(\alpha), f^n(\beta)] \geq \tfrac{1}{16}.$$

Two points in S are either in the same orbit or not, so we can generalize to say that *any* two points, $\alpha, \beta \in S$ satisfy

$$\limsup_{n\to\infty} d[f^n(\alpha), f^n(\beta)] \geq \tfrac{1}{16}.$$

Next, we show that for any points $\alpha, \beta \in S$, $\alpha \neq \beta$, (2) holds. Suppose a_{k_0} is the first entry of the first string of either 1's or 0's in α that is at least j long, and b_{ℓ_0} is the first entry of the first string of either 1's or 0's in β that is at least j long. Let $j_0 = max\{k_0, \ell_0\}$. $f^{j_0-1}(\alpha) = .a_{j_0}a_{j_0+1}a_{j_0+2}...$ or $.\bar{a}_{j_0}\bar{a}_{j_0+1}\bar{a}_{j_0+2}...$, and $f^{j_0-1}(\beta) = .b_{j_0}b_{j_0+1}b_{j_0+2}...$ or $.\bar{b}_{j_0}\bar{b}_{j_0+1}\bar{b}_{j_0+2}....$ Without loss of generality, suppose $k_0 \geq \ell_0$, so that $f^{j_0-1}(\alpha)$ starts with a string of either 0's or 1's that is j long. Then $f^{j_0-1}(\beta)$ either starts with a string of 0's or 1's, or starts with a single 0 or 1 that is followed by a string of 1's or 0's. Furthermore, since $k_0 \geq \ell_0$, the first string in $f^{j_0-1}(\beta)$ is at least j long (and if $f^{j_0-1}(\beta)$ starts with a single entry, the string that follows is longer than j). Consider the following cases:

(1) $f^{j_0-1}(\beta)$ starts with a string.
   (a) $f^{j_0-1}(\alpha)$ and $f^{j_0-1}(\beta)$ either both start with strings of 0's or both start with strings of 1's. Then at least the first j entries of $f^{j_0-1}(\alpha)$ and $f^{j_0-1}(\beta)$ agree. Moreover, in either case, $f^{j_0}(\alpha)$ and $f^{j_0}(\beta)$ both start with a string of 1's, and therefore agree with each other and with $.\bar{1}$ for at least the first $j-1$ entries.
   (b) One of $f^{j_0-1}(\alpha)$ and $f^{j_0-1}(\beta)$ starts with a string of 0's, and the other starts with a string of 1's. Then $f^{j_0}(\alpha)$ and $f^{j_0}(\beta)$ both start with a string of 1's that is at least $j-1$ long. Thus $f^{j_0}(\alpha)$, $f^{j_0}(\beta)$, and $.\bar{1}$ agree for at least the first $j-1$ entries.

(2) $f^{j_0-1}(\beta)$ starts with a single entry.
   (a) $f^{j_0-1}(\beta) = .10^{n_r}10^{n_{r+1}}1...$ where $n_{r+1} > n_r > j$. Then $f^{j_0}(\beta) = .0^{n_r}10^{n_{r+1}}1...$, and $f^{j_0+1}(\beta) = .1^{n_r-1}01^{n_{r+1}}0....$
      (i) $f^{j_0-1}(\alpha) = .0^{n_s}10^{n_{s+1}}1...$, $n_s \geq j$. Then at least the first $j-1$ entries of $f^{j_0}(\alpha)$ and $f^{j_0}(\beta)$ agree. Moreover, at least the first $j-2$ entries of $f^{j_0+1}(\alpha)$, $f^{j_0+1}(\beta)$, and $.\bar{1}$ agree.
      (ii) $f^{j_0-1}(\alpha) = .1^{n_s}01^{n_{s+1}}0...$, $n_s \geq j$. Then $f^{j_0+1}(\alpha) = .1^{n_s-2}01^{n_{s+1}}0...$, so at least the first $j-2$ entries of $f^{j_0+1}(\alpha)$, $f^{j_0+1}(\beta)$, and $.\bar{1}$ agree.
   (b) $f^{j_0-1}(\beta) = .01^{n_r}01^{n_{r+1}}0...$ where $n_{r+1} > n_r > j$. Then $f^{j_0}(\beta) = .0^{n_r}10^{n_{r+1}}1...$, and $f^{j_0+1}(\beta) = .1^{n_r-1}01^{n_{r+1}}0....$
      (i) $f^{j_0-1}(\alpha) = .0^{n_s}10^{n_{s+1}}1...$, $n_s \geq j$. Then $f^{j_0}(\alpha) = .1^{n_s-1}01^{n_{s+1}}0...$, and $f^{j_0+1}(\alpha) = .1^{n_s-2}01^{n_{s+1}}0....$ So $f^{j_0+1}(\alpha)$ and $f^{j_0+1}(\beta)$ agree for at least the first $j-2$ entries.
      (ii) $f^{j_0-1}(\alpha) = .1^{n_s}01^{n_{s+1}}0...$, $n_s \geq j$. Then $f^{j_0}(\alpha) = .1^{n_s-1}01^{n_{s+1}}0...$, and $f^{j_0+1}(\alpha) = .1^{n_s-2}01^{n_{s+1}}0....$ So $f^{j_0+1}(\alpha)$ and $f^{j_0+1}(\beta)$ agree for at least the first $j-2$ entries.

Since, in each case, we have been able to find for any j an iterate i of f such that $f^i(\alpha)$ and $f^i(\beta)$ agree for at least the first $j-2$ entries, we can find iterates of f such that the images under that iterate agree for arbitrarily many entries. We can also choose these iterates such that the images are at the same time converging to $.\bar{1}$. Since this "agreement" translates to "closeness" of the points, we know that for *any* $\alpha, \beta \in S$,

$$\liminf_{n\to\infty} d[f^n(\alpha), f^n(\beta)] = 0$$

Next, we suppose that γ is periodic of period k for f, and we will show that for any $\alpha \in S$, (3) holds.

First suppose that $\gamma = .\bar{1}$. Since α has infinitely many 0's, we can choose n_1, n_2, n_3, ... such that for all i, $f^{n_i}(\alpha) = .0...$ (note that it may not be the case that a 0 occurs at position k in α, and $f^{k-1}(\alpha) = .0...$, but since there are infinitely many 0's and infinitely many 1's in α, and applying f to α either leaves the 0's as 0's and the 1's as 1's, or switches all of them, we will always be able to find more k such that $f^k(\alpha) = .0...$). For each i, $d[f^{n_i}(\alpha), f^{n_i}(\gamma)] \geq \frac{1}{2}$.

Now suppose that $\gamma \neq .\bar{1}$. γ consists of either one block of length k that repeats, or such a block that alternates with its "complement" block. Furthermore, since $\gamma \neq 1$, γ has infinitely many 0's. Suppose the first 0 occurs at position $m < k$. Then $f^{nk+m}(\gamma) = .0...$, $n \geq 0$. Suppose r_1, r_2, r_3, ... are the positions of α where strings of lengths s_1, s_2, s_3, ... (respectively) start (for all i, $s_i \geq 3$). Then

$f^{r_i}(\alpha) = .1^{s_i-1}01^{s_{i+1}}01^{s_{i+2}}....$ For $q \leq s_i - 3$, $f^{r_i+q}(\alpha) = .11...$ (the image binary expansion starts with at least two 1's). Since we can choose s_i to be as long as we want, we can choose $s_i > k+3$, so that for some $q \leq s_i - 3$, $r_i + q = nk + m$ for some n. Then, for that i and that q, $f^{r_i+q}(\gamma) = .0...$ and $f^{r_i+q}(\alpha) = .11...$, so $d[f^{r_i+q}(\gamma), f^{r_i+q}(\alpha)] \geq \frac{1}{4}$. Since there are infinitely many ! such i,

$$\limsup_{n\to\infty} d[f^n(\alpha), f^n(\gamma)] \geq \tfrac{1}{4}$$

We have shown that S is f-invariant and δ-scrambled for f. Since f is topologically conjugate to T^2, it follows that $h^{-1}(S)$ is T^2-invariant and δ-scrambled for T^2. It remains to find a T-invariant, δ-scrambled set for T. We will show that $S^* = h^{-1}(S) \cup T(h^{-1}(S))$ is T-invariant and δ-scrambled for T.

It is easy to see that S^* is T-invariant: if $x \in S^*$, then $x = h^{-1}(s)$ or $T(h^{-1}(s))$ for some $s \in S$, and $T(x) = T(h^{-1}(s))$ or $T(T(h^{-1}(s))) = T^2(h^{-1}(s))$. In the first case, $T(x) \in T(h^{-1}(S))$. In the second case, $T(x) \in h^{-1}(S)$, since $h^{-1}(S)$ is T^2-invariant.

We already know that the points in $h^{-1}(S)$ satisfy the δ-scrambled conditions with respect to one another and to periodic points in [0,1]. We need to show that they also satisfy (3) with respect to periodic points in (1,2], and that the elements of $T(h^{-1}(S))$ satisfy these same conditions with respect to one another and to $h^{-1}(S)$.

First, take $x \in h^{-1}(S)$ and $y \in (1,2]$. Since the binary expansion of x, $h(x)$, has infinitely many 0's, there are infinitely many k for which $T^{2k}(x) \in [0, \frac{1}{2}]$. For each of these k, $T^{2k}(y) \in (1,2]$, so $d[T^{2k}(x), T^{2k}(y)] \geq \frac{1}{2}$. Thus,

$$\limsup_{n\to\infty} d[T^n(x), T^n(y)] \geq \tfrac{1}{2}.$$

By the same argument, given $x_1 \in h^{-1}(S) \subset [0,1]$ and $x_2 \in T(h^{-1}(S)) \subset [1,2]$, there are infinitely many k for which $T^{2k}(x_1) \in [0, \frac{1}{2}]$, and thus for which $d[T^{2k}(x_1), T^{2k}(x_2)] \geq \frac{1}{2}$. Thus,

$$\limsup_{n\to\infty} d[T^n(x_1), T^n(x_2)] \geq \tfrac{1}{2}.$$

Now suppose that $x_1, x_2 \in T(h^{-1}(S)) \subset [1,2]$, and show that (1) holds. Since $T(x_1), T(x_2) \in h^{-1}(S)$, there is a sequence $(k_1, k_2, k_3, ...)$ such that $\lim_{i\to\infty} d[T^{2k_i}(T(x_1)), T^{2k_i}(T(x_2))] \geq \delta$. But $\lim_{i\to\infty} d[T^{2k_i}(T(x_1)), T^{2k_i}(T(x_2))] = \lim_{i\to\infty} d[T^{2k_i+1}(x_1), T^{2k_i+1}(x_2)]$, so

$$\limsup_{n\to\infty} d[T^n(x_1), T^n(x_2)] \geq \delta$$

Next suppose that $x_1 \in h^{-1}(S) \subset [0,1]$, $x_2 \in T(h^{-1}(S)) \subset [1,2]$, and show that (2) holds. Then $x_2 = T(x_2^*)$, for $x_2^* \in h^{-1}(S)$. There is a sequence $(k_1, k_2, k_3, ...)$ such that $\lim_{i\to\infty} d[T^{2k_i}(x_1), T^{2k_i}(x_2^*)] = 0$. Furthermore, we can choose these k_i so that both orbits are converging to 1. i.e., $\lim_{i\to\infty} d[T^{2k_i}(x_1), 1] = 0 = \lim_{i\to\infty} d[T^{2k_i}(x_2^*), 1]$. Also note that for any $x \in [0,1]$, $d[x, 1] = d[T(x), 1]$. So $\lim_{i\to\infty} d[T^{2k_i}(T(x_2^*)), 1] = \lim_{i\to\infty} d[T(T^{2k_i}(x_2^*)), 1] = \lim_{i\to\infty} d[T^{2k_i}(x_2^*), 1] = 0$ (since $T^{2k_i}(x_2^*) \in [0,1]$). Thus, $\lim_{i\to\infty} d[T^{2k_i}(x_1), T^{2k_i}(x_2)] = \lim_{i\to\infty} d[T^{2k_i}(x_1), T^{2k_i}(x_2^*)] = 0$, and so,

$$\liminf_{n\to\infty} d[T^n(x_1), T^n(x_2)] = 0.$$

Now suppose that $x_1, x_2 \in T(h^{-1}(S)) \subset [1,2]$, so that $x_1 = T(x_1^*), x_2 = T(x_2^*)$, $x_1^*, x_2^* \in h^{-1}(S)$, and show that (2) holds. There is a sequence $(k_1, k_2, k_3, ...)$ such that $\lim_{i\to\infty} d[T^{k_i}(x_1^*), T^{k_i}(x_2^*)] = 0$. $\lim_{i\to\infty} d[T^{k_i-1}(T(x_1^*)), T^{k_i-1}(T(x_2^*))] = \lim_{i\to\infty} d[T^{k_i}(x_1^*), T^{k_i}(x_2^*)]$. So,

$$\liminf_{n\to\infty} d[T^n(x_1), T^n(x_2)] = 0.$$

Suppose that $x \in T(h^{-1}(S)) \subset [1,2]$, and $y \in [0,1]$ is periodic, and show that (3) holds. So $x = T(x^*)$, and, once again, $d[x^*, 1] = d[T(x^*), 1]$. Since $x^* \in h^{-1}(S)$, and 1 is fixed by T, there is a sequence $(k_1, k_2, k_3, ...)$ such that $\lim_{i\to\infty} d[T^{2k_i}(x_1^*), 1] \geq \delta$. $\lim_{i\to\infty} d[T^{2k_i}(T(x^*)), 1] = \lim_{i\to\infty} d[T^{2k_i}(x^*), 1] \geq \delta$. Also, for all i, $T^{2k_i}(T(x^*)) \in [1,2]$, and $T^{2k_i}(y) \in [0,1]$. So $\lim_{i\to\infty} d[T^{2k_i}(T(x^*)), T^{2k_i}(y)] \geq \delta$. Thus,

$$\limsup_{n\to\infty} d[T^n(x), T^n(y)] \geq \delta.$$

Lastly, suppose $x \in T(h^{-1}(S)) \subset [1,2]$, and $y \in [1,2]$ is periodic, and show that (3) holds. $T(x) \in h^{-1}(S)$, and $T(y) \in [0,1]$. Since y is periodic, $T(y)$ is periodic. So there is a sequence $(k_1, k_2, k_3, ...)$ such that $\lim_{i\to\infty} d[T^{k_i}(T(x)), T^{k_i}(T(y))] \geq \delta$. Since $\lim_{i\to\infty} d[T^{k_i}(T(x)), T^{k_i}(T(y))] = \lim_{i\to\infty} d[T^{k_i+1}(x), T^{k_i+1}(y)]$,

$$\limsup_{n\to\infty} d[T^n(x), T^n(y)] \geq \delta.$$

□

This concludes the proof that $h^{-1}(S) \cup T(h^{-1}(S))$ is T-invariant and δ-scrambled for T, and so we have shown that there do exist maps with no periodic points of odd period greater than 1 which *do* have f-invariant δ-scrambled sets.

3. Conclusion

We have established that f-invariant δ-scrambled sets occur only for maps in $\mathbb{P}_{12}$, but also that they can occur for maps outside of $\mathbb{P}_{odd}$. This adds to our knowledge from previous work that all maps in $\mathbb{T}_1$ have f-invariant δ-scrambled sets. Some avenues for further research are, firstly, to attempt to establish that some proper subset of $\mathbb{P}_{12}$ contains all maps with f-invariant δ-scrambled sets and, secondly, to search for more maps with f-invariant δ-scrambled sets that lie outside of certain sets named in the turbulence stratification. For example, the map T described above has no periodic points of odd period, but it is in $\mathbb{T}_2$. If we could find such a map that lies outside of $\mathbb{T}_2$, we would know that $\mathbb{T}_2$ does not contain all maps with f-invariant δ-scrambled sets, and that if there is some proper subset of $\mathbb{P}_{12}$ that does contain all these maps, that set must also properly contain $\mathbb{T}_2$. In short, we wish to discover which of the properties that lie between $\mathbb{P}_{odd}$ and $\mathbb{P}_{12}$ in the turbulence stratification are necessary for a map to have an f-invariant δ-scrambled set, and which are not.

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