A NOTE ON f-INVARIANT δ-SCRAMBLED SETS

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Abstract. This paper adjoins to the paper, *On the Necessary and Sufficient Conditions for the Existence of f-Invariant δ-Scrambled Sets*, submitted by Gwyneth Harrison-Shermoen and Omar Zeid.

1. Alternatingly f-Invariant δ-Scrambled Sets

So, we know that for any map $f : I \mapsto I$, which contains an f-invariant δ -scrambled set S there exists infinitely many $n > 1$ and $x_0 \in S$ such that either

$$f^n(x_0) \leq x_0 < f(x_0) \tag{1}$$

or

$$f(x_0) < x_0 \leq f^n(x_0). \tag{2}$$

By BC's Theorem 2.9, if f has no alternating trajectory for an arbitrary $x \in I$ then f has a periodic point of odd period k where $1 < k \leq n$ if n is odd or $1 < k \leq n/2 + 1$ if n is even. Therefore, it suffices to show that a map with an f-invariant δ -scrambled set has no alternating trajectories in order to prove that f is in $\mathbb{P}_k$ for some odd k but not necessarily in $\mathbb{T}_1$. This question is motivated by Du's result that dealt with turbulent maps and proved that they must have f-invariant δ -scrambled sets. However, first we must use a weaker example of a δ -scrambled set.

We define an *alternatingly* f-invariant δ -scrambled set as an f-invariant δ -scrambled set for which the even terms of the trajectory of given $x_0, y_0 \in S$ follow the same rules as the whole trajectories. Also, since the odd iterates of x_0 are the even iterates of $x_1 = f(x_0)$, and both are in S, an alternately f-invariant δ -scrambled set's odd *and* even iterates follow the same rules as a general δ -scrambled set.

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Proposition 1. *A map $f : I \mapsto I$ with an alternatingly f-invariant δ -scrambled set S has no alternating trajectories and hence is in $\mathbb{P}_k$ for some k.*

Proof. First, we start with a map f with an alternatingly f-invariant δ -scrambled set S and try to show that trajectories of points in S are not alternating. This means that an arbitrary point $x_0 \in S$ would have a trajectory with at least one odd iteration of x_0 on each side of x_0. In other words, there would exists some positive integers $n, m > 1$ such that either

$$f^{2m+1}(x_0) < x_0 < f(x_0) \tag{3}$$

or its reverse. Without loss of generality, we will work with this inequality as shown.

First we note that

$$f^{2n}(x_0) < x_0 < f^2(x_0) \tag{4}$$

holds true for an alternatingly f-invariant δ -scrambled set.

Now we set $F(x) = f^2(x)$. Now we can re-write inequality 4 as

$$F^n(x_0) < x_0 < F(x_0) \tag{5}$$

.

since x_0 is in S, we know this statement is true for infinitely many n by S's alternatingly f-invariance.

Now we can show that inequality 3 is true. To simplify notation, we will define x_i as $f^i(x_0)$. This makes inequality 3 look like this ...

$$x_{2m+1} < x_0 < x_1. \tag{6}$$

Now, since x_1 is in S just as x_0 is, we know a positive integer m exists such that

$$x_{2m+1} < x_1 < x_3 \tag{7}$$

or it's reverse is true. When x_0's position is also considered, four possibilities arise,

$$x_0 < x_{2m+1} < x_1 < x_3, \tag{8}$$

$$x_{2m+1} < x_0 < x_1 < x_3, \tag{9}$$

$$x_0 < x_3 < x_1 < x_{2m+1}, \tag{10}$$

and

$$x_3 < x_0 < x_1 < x_{2m+1}. \tag{11}$$

Now, if 9 or 11 are the case for some $m > 1$, then x_0's trajectory is clearly non-alternating.

For 8, if this configuration is indeed alternating then there is some odd iterate of x_0 (we'll call it γ), which is the least odd iterate greater than x_0. This causes a contradiction because $f^2(\gamma)$ and $f^{2k+1}(\gamma)$ have to be on different sides of γ and this means either $f^2(\gamma)$ or $f^{2k+1}(\gamma)$ must either be between γ and x_0 or less than x_0. Since γ is the least odd iterate greater than x_0, there can be no other iterates in between it and x_0, and if it's on the other side of x_0, then this trajectory isn't alternating because there's an odd iterate on each side of x_0.

For 10, the argument is the same, keeping in mind that γ might be x_3 or it might be some odd iterate that isn't our x_{2m+1} that is between x_3 and x_0. Either way, the same contradiction holds for cases which could be alternating, so any map with an alternatingly f-invariant δ -scrambled set *must* have non-alternating trajectories and is therefore in $\mathbb{P}_k$ for some k without necessarily being turbulent.

□

2. Eventually Alternating Orbits in f-Invariant δ-Scrambled Sets

Now, we wish to tighten the boundary on where f-invariant δ -scrambled sets can occur. Since they must have periodic points of all periods which are multiples of 4, we already know they must at least be in $\mathbb{P}_{12}$ However, we may be able to lay a boundary at $\mathbb{P}_6$ or somewhere else earlier than $\mathbb{P}_{12}$. Here is one question we want to look at first which may lead us in the right direction.

Lemma 1. *For a continuous map $f : I \mapsto I$ with an f-invariant δ -scrambled set S and* no *periodic orbits of odd periods, all trajectories with a seed in S are eventually alternating.*

Proof. First, we start with a seed $x_0 \in S$. Since x_0 is in S, we know it has a trajectory of the form

$$f^n(x_0) \leq x_0 < f(x_0) \tag{12}$$

or its reverse. we also know that this is true for infinitely many n's, and since we don't want any periodic points with odd periods, all of these infinitely many n's must be even. That means that all odd iterations must be on the"f-side" of x_0 and not on the"f^n-side." However, even iterates can also exist on the f-side of x_0 (if this wasn't the case then this lemma would be a trivial result).

Now we will look at an arbitrary odd iterate of x_0 on the f-side ...

$$f^n(x_0) \leq x_0 < f^{2k+1}(x_0) \tag{13}$$

Since there are infinitely many even n's on the f^n-side of x_0, there will be infinitely many n's that are greater than $2k+1$. These will be odd iterates of $f^{2k+1}(x_0)$, so there cannot be any odd iterates of $f^{2k+1}(x_0)$ on the side of $f^{2k+1}(x_0)$ which is strictly on x_0's f-side. Hence, there are no even iterations of x_0 after the $2k+1_{st}$ iteration of x_0 on the far side of $f^{2k+1}(x_0)$.

This limits the area where both even and odd iterates past $2k+1$ can exist to the space between x_0 and $f^{2k+1}(x_0)$. Now, we can also look at an arbitrary *even* iterate on x_0's f-side...

$$f^n(x_0) \leq x_0 < f^{2j}(x_0) \tag{14}$$

Once again, there will be even iterates of $f^{2j}(x_0)$ on the f^n-side of x_0, so there can be no odd iterates after $2j$ in between x_0 and $f^{2j}(x_0)$. This limits where both odd and even iterates greater than 2j can exist to the side of $f^{2j}(x_0)$ that is away from x_0 (the right-hand side in our example, but it could be the left-hand side if we started with the f-side on the left).

Now, we have a method of systematically diminishing that troublesome area where there are both odd and even iterations of x_0. First, we start with expression 12 from above. Note that there can be no even iterates at all to the right of $f(x_0)$ since all even nonzero iterates are after 1. Now, we either find the odd iterate closest to x_0 or the even iterate closest to $f(x_0)$. Without loss of generality, suppose we do the former and label that point β. Now, if there are any even iterates between x_0 and β that are iterations of both x_0 and β, then we find the even iteration that is closest to β and call it σ. Now, there are no iterations in between β and σ, so from the point in x_0's orbit which is either β or σ (whichever came first), we definitely have an alternating trajectory.

The only way this could not work is if both the even and odd iterations of x_0 were dense in this region. In other words, there would have to be a string of infinitely many even iterations going away from x_0 by iteration and infinitely many odd iterations going from $f(x_0)$ to x_0 by iteration. If these two sequences pass each other, then we get odd periodic points, so these sequences would either have to converge to a single point or the odd iterations could converge to one point and the even iterations could converge to another. However, for this sequence to include only some even and odd iterations and not all of them would require infinitely many points of either nondifferentiability or discontinuity to bounce the trajectories somewhere else in S before bringing the orbit back to the 2 sequences aforementioned. Since we stipulate that f must be continuous and differentiable everywhere but countably many places, we know this cannot happen. Therefore, the only case where this won't work will still have all the even iterates on one side of all the odd iterates, which produces an alternating trajectory anyway. $\square$

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