Extraction Degrees of Zero Sequences of Finite Abelian Groups

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Abstract

Let G be a finite abelian group and let $ZS(G)$ and $MZS(G)$ be the set of zero sequences and minimal zero sequences of G, respectively. One way of relating two zero sequences of a finite abelian group, is by using the extraction degree. In this paper, we attempt to classify the sets of possible extraction degrees for all combinations of two zero sequences. These results sometimes vary depending on whether G is cyclic or not. To begin with, we address some general properties of the extraction degree when there are no restrictions on the zero sequences. Then, we explore the sets of possible extraction degrees for all combinations of two zero sequences. Ultimately, we have been able to determine most of the sets of possible extraction degrees, when restricting one or both of the zero sequences, as minimal zero sequences.

1 Introduction

Let G be a finite abelian group. Then, by the Fundamental Theorem of Finite Abelian Groups, we can represent G canonically as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ such that $n_i | n_{i+1}$ for all $1 \leq i < s$ [4]. Then, the *rank* of G is s, and when $s = 1$ we simply write $G = \mathbb{Z}_n$. For convenience, we denote X, a sequence in G, as $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ where each $g_i \in G$ and $g_i = g_j$ if and only if $i = j$ for all $i, j \in \{1, 2, \ldots, k\}$. Each α_i is the multiplicity of g_i, that is, α_i is the number of times g_i appears in X. The multiplicity of g_i is also notated as $(g_i)_X$, so that $(g_i)_X = \alpha_i$.

We say that X is a *zero sequence* of G, that is $X \in ZS(G)$, if

$$\sum_{i=1}^{k} \alpha_i \cdot g_i = \mathbf{0}.$$

The bold zero represents the zero element in G, namely $(0, 0, \ldots, 0)$. In addition, we say X is a *minimal zero sequence* of G, that is $X \in MZS(G)$, if there is no proper subset of X whose sum is zero[1, 2, 3, 5].

We must introduce two other notations we use, which we adopt since they are helpful in many of our results. First, we define the floor of a zero sequence, and second we define the restriction of one zero sequence to another.

Definition 1.1. Given any zero sequence, X, the *floor* of X, denoted $\lfloor X \rfloor$, is the set of all distinct elements of X, that occur at least once, repetition not allowed.

Definition 1.2. Given any two zero sequences, X and Y, the *restriction* of Y to X, denoted $Y|_X$, is the set of all $g \in Y$ such that $g \in X$, and $(g)_Y = (g)_{Y|_X}$.

1.1 The Extraction Degree

One avenue of study with zero sequences of finite abelian groups, that has not been explored much, is the extraction degree. The extraction degree was introduced in [6] and is defined as follows.

Definition 1.3. Given $X, Y \in ZS(G)$, the *extraction degree* is given by

$$\lambda(X, Y) = \sup \left\{ \frac{\beta}{\alpha} | \beta \in \mathbb{Z}^{\geq 0}, \alpha \in \mathbb{Z}^{>0}, \beta X \subseteq \alpha Y \right\}$$

where βX means β copies of X.

We use this definition in the following example to demonstrate how the extraction degree is found for two zero sequences.

Example 1.4. Let $G = \mathbb{Z}_{10}$. Let $X = 5^1 2^1 3^1$ and $Y = 1^1 2^2 3^1 5^1 7^1$. If we let $\beta = 1$ then in order for $\beta X \subseteq \alpha Y$, $\alpha \geq 1$ and the maximum $\frac{\beta}{\alpha} = \frac{1}{1}$. If we let $\beta = 2$, then in order for $\beta X \subseteq \alpha Y$, $\alpha \geq 2$ and the maximum $\frac{\beta}{\alpha} = \frac{2}{2} = 1$. Since this is true for all $\beta \geq 0$, $\lambda(X, Y) = 1$.

Throughout this paper we look at the set of all extraction degrees possible when X and Y are confined to certain subsets of $ZS(G)$. If X is confined to $A \subseteq ZS(G)$ and Y is confined to $B \subseteq ZS(G)$, then $\lambda(A, B)$ will represent the set of all possible extraction degrees where $X \in A$ and $Y \in B$. For example, if $X, Y \in MZS(G)$, then $\lambda(MZS(G), MZS(G))$ is the set of extraction degrees for $X, Y \in MZS(G)$.

Since little to nothing is known about the extraction degree of zero sequences of finite abelian groups, we first introduce some properties of the extraction degree, many of which are useful throughout this paper. Then, we look at what sets of extraction degrees are possible for all combinations of two zero sequences, including when either X or Y is restricted as a minimal zero sequence or fixed.

2 Properties of the Extraction Degree

2.1 General Properties

We begin with five very general properties of the extraction degree; some of these general properties are extremely helpful when proving later results; the properties we do not directly use in this paper may warrant further study. This first result is used often throughout our paper. As we began studying the extraction degree of finite abelian groups, we noticed that finding the extraction degree using the definition was very cumbersome. Then, by discovering the following theorem, we now have a way to calculate the extraction degree of any two zero sequences that is much easier.

Theorem 2.1. *Let* $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ *and* $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$ *where* $\alpha_i > 0$, $\beta_i \geq 0$, *and* $1 \leq i \leq k \leq j$. *Then*

$$\lambda(X, Y) = \min \left\{ \frac{\beta_i}{\alpha_i} | 1 \leq i \leq k \right\}.$$

Proof. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$ where $\alpha_i > 0$, $\beta_i \geq 0$, and $1 \leq i \leq k \leq j$. Let $\frac{\beta}{\alpha} = \min\left\{\frac{\beta_i}{\alpha_i} | 1 \leq i \leq k\right\}$ and let $\frac{\gamma}{\delta} = \lambda(X,Y)$. So, $\gamma X \subseteq \delta Y$. Then for all $i \in \{1, 2, \ldots, k\}$, $(g_i)_{\gamma X} \leq (g_i)_{\delta Y}$. Note that, $(g_i)_X = \alpha_i$ and $(g_i)_Y = \beta_i$ so $(g_i)_{\gamma X} = \gamma\alpha_i$ and $(g_i)_{\delta Y} = \delta\beta_i$. Then, we have that $\gamma\alpha_i \leq \delta\beta_i \Rightarrow \frac{\gamma}{\delta} \leq \frac{\beta_i}{\alpha_i} \Rightarrow \lambda(X,Y) \leq \frac{\beta}{\alpha}$.

Now, we show that $\beta X \subseteq \alpha Y$. Assume not. Then, there exists $m \in \{1, 2, \ldots, k\}$ such that $(g_m)_{\beta X} > (g_m)_{\alpha Y}$. Since $(g_m)_{\beta X} = \beta\alpha_m$ and $(g_m)_{\alpha Y} = \alpha\beta_m$ we have that $\beta\alpha_m > \alpha\beta_m \Rightarrow \frac{\beta}{\alpha} > \frac{\beta_m}{\alpha_m}$ for some $m \in \{1, 2, \ldots, k\}$, which contradicts that $\frac{\beta}{\alpha} = \min\left\{\frac{\beta_i}{\alpha_i} | 1 \leq i \leq k\right\}$. Thus, $\beta X \subseteq \alpha Y$. Then, by the definition of $\lambda(X,Y)$, $\frac{\beta}{\alpha} \leq \lambda(X,Y)$.

So, $\frac{\beta}{\alpha} \leq \lambda(X,Y) \leq \frac{\beta}{\alpha}$. Therefore $\lambda(X,Y) = \frac{\beta}{\alpha} = \min\left\{\frac{\beta_i}{\alpha_i} | 1 \leq i \leq k\right\}$.

■

This theorem implies that there is at least one element, g, in X such that $\frac{(g)_Y}{(g)_X} = \lambda(X,Y)$. We call this element the *critical element*. The existence of a *critical element* proves extremely helpful in many of our results.

Our next two general properties look specifically at factoring one zero sequence into the union of finitely many zero sequences. First, we will examine what happens when $X \in ZS(G)$ is factored into a finite number of zero sequences. Then, we will examine what happens when $Y \in ZS(G)$ is factored into a finite number of zero sequences.

Theorem 2.2. *Let G be a finite abelian group. Let $X, Y \in ZS(G)$. If X can be written as $X_1 \cup X_2 \cup \cdots \cup X_m$ where $X_i \in ZS(G)$ for all i such that $1 \leq i \leq m$ then*

$$\frac{1}{\lambda(X,Y)} \leq \sum_{i=1}^{m} \frac{1}{\lambda(X_i,Y)}.$$

Proof. Let $X, Y \in ZS(G)$. Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ where $X_i \in ZS(G)$ for all i such that $1 \leq i \leq m$, and let g be the critical element for X and Y. Then $\lambda(X,Y) = \frac{(g)_Y}{(g)_X}$ and and $(g)_X = \sum_{i=1}^{m} (g)_{X_i}$. From Theorem 2.1, $\lambda(X_i,Y) \leq \frac{(g)_Y}{(g)_{X_i}}$ for all i such that $1 \leq i \leq m$. Then we have that

$$\frac{1}{\lambda(X,Y)} = \frac{(g)_X}{(g)_Y} = \frac{\sum_{i=1}^{m} (g)_{X_i}}{(g)_Y} = \sum_{i=1}^{m} \frac{(g)_{X_i}}{(g)_Y} \leq \sum_{i=1}^{m} \frac{1}{\lambda(X_i,Y)}.$$

■

Note that if g is critical for all X_i and Y where $1 \leq i \leq m$ then the inequality becomes an equality because $\lambda(X_i, Y) = \frac{(g)_Y}{(g)_{X_i}}$ for all i such that $1 \leq i \leq m$.

Theorem 2.3. *Let G be a finite abelian group. Let $X, Y \in ZS(G)$. If Y can be written as $Y_1 \cup Y_2 \cup \cdots \cup Y_m$ where $Y_i \in ZS(G)$ for all i such that $1 \leq i \leq m$ then*

$$\lambda(X, Y) \geq \sum_{i=1}^{m} \lambda(X, Y_i).$$

Proof. Let $X, Y \in ZS(G)$. Let $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_m$ where $Y_i \in ZS(G)$ for all i such that $1 \leq i \leq m$, and let g be the critical element for X and Y. Then $\lambda(X, Y) = \frac{(g)_Y}{(g)_X}$ and and $(g)_Y = \sum_{i=1}^{m} (g)_{Y_i}$. From Theorem 2.1, $\lambda(X, Y_i) \leq \frac{(g)_{Y_i}}{(g)_X}$ for all i such that $1 \leq i \leq m$. Then we have that

$$\lambda(X, Y) = \frac{(g)_Y}{(g)_X} = \frac{\sum_{i=1}^{m} (g)_{Y_i}}{(g)_X} = \sum_{i=1}^{m} \frac{(g)_{Y_i}}{(g)_X} \geq \sum_{i=1}^{m} \lambda(X, Y_i).$$

■

Note that if g is critical for all X and Y_i where $1 \leq i \leq m$ then the inequality becomes an equality because $\lambda(X, Y_i) = \frac{(g)_{Y_i}}{(g)_X}$ for all i such that $1 \leq i \leq m$.

This next general property, although not explored further in this paper, may warrant further study as it examines the extraction degree of three zero sequences.

Theorem 2.4. *Let G be a finite abelian group, and let $X, Y, Z \in ZS(G)$. Then*

$$\lambda(X, Y) \cdot \lambda(Y, Z) \leq \lambda(X, Z).$$

Proof. Let g be the critical element for X and Z. Then $\lambda(X, Z) = \frac{(g)_Z}{(g)_X}$. From Theorem 2.1, $\lambda(X, Y) \leq \frac{(g)_Y}{(g)_X}$ and $\lambda(Y, Z) \leq \frac{(g)_Z}{(g)_Y}$. Then

$$\lambda(X, Y) \cdot \lambda(Y, Z) \leq \frac{(g)_Y}{(g)_X} \cdot \frac{(g)_Z}{(g)_Y} = \frac{(g)_Z}{(g)_X} = \lambda(X, Z).$$

■

This next general property is useful when working in higher ranks of finite abelian groups. For example, when a result is obtained for a finite abelian group of rank 2, we are able to extend the argument to all rank, using this general property.

Theorem 2.5. *Let G and H be finite abelian groups. If there exist $X, Y \in ZS(G)$ such that $\lambda(X, Y) = \delta$ then there exist $X', Y' \in ZS(H \oplus G)$ such that $\lambda(X', Y') = \delta$. Moreover, if $X \in MZS(G)$ then we can find $X' \in MZS(H \oplus G)$ and if $Y \in MZS(G)$ then we can find $Y' \in MZS(H \oplus G)$.*

Proof. Let $X, Y \in ZS(G)$. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$ where $j \geq k$. Then $\lambda(X, Y) = \min \left\{ \frac{\beta_i}{\alpha_i} | 1 \leq i \leq k \right\}$. Let $h_i = (\mathbf{0}, g_i)$ for all i such that $1 \leq i \leq j$ where $\mathbf{0}$ is the zero element in H. Then, each $h_i \in H \oplus G$. Let $X' = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_k^{\alpha_k}$ and $Y' = h_1^{\beta_1} h_2^{\beta_2} \cdots h_j^{\beta_j}$. Then, $X', Y' \in ZS(H \oplus G)$, and if $X \in MZS(G)$ then by the way we constructed X', we have that $X' \in MZS(H \oplus G)$. Similarly, if $Y \in MZS(G)$, then, by the way we constructed Y', we have that $Y' \in MZS(H \oplus G)$.

Therefore, $\lambda(X', Y') = \min \left\{ \frac{\beta_i}{\alpha_i} | i \in \{1, 2, \ldots, k\} \right\} = \lambda(X, Y)$. ∎

From this theorem, we have that $\lambda(MZS(G), MZS(G)) \subseteq \lambda(MZS(H \oplus G), MZS(H \oplus G))$, $\lambda(MZS(G), ZS(G)) \subseteq \lambda(MZS(H \oplus G), ZS(H \oplus G))$, and $\lambda(ZS(G), MZS(G)) \subseteq \lambda(ZS(H \oplus G), MZS(H \oplus G))$.

2.2 Zero and One

This section explores four general properties that are more specific, as these address the properties of X and Y that yield $\lambda(X, Y) = 0$, $\lambda(X, Y) = 1$, $\lambda(X, Y) > 1$, or $0 < \lambda(X, Y) < 1$.

Theorem 2.6. *Let $X, Y \in ZS(G)$. $\lambda(X, Y) = 0$ if and only if $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$.*

Proof. Let $X, Y \in ZS(G)$. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$, where $j, k \in \mathbb{Z}^+$. Then, $\lambda(X, Y) = 0$ if and only if $\frac{\beta_i}{\alpha_i} = 0$ for some $g_i \in X$ if and only if $\beta_i = 0$ if and only if $(g_i)_Y = 0$ and $(g_i)_X = \alpha_i$ if and only if $g_i \in X$ such that $g_i \notin Y$ if and only if $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$. ∎

Theorem 2.7. *Let $X, Y \in ZS(G)$. $\lambda(X, Y) = 1$ if and only if $X \subseteq Y|_X$ and there exists at least one element $g_i \in X$ where $(g_i)_Y = (g_i)_X$.*

Proof. Let $X, Y \in ZS(G)$. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$, where $j, k \in \mathbb{Z}^+$ and $j \geq k$. Then, recall $Y|_X = g_1^{\beta_1} g_2^{\beta_2} \cdots g_k^{\beta_k}$. Then, $X \subseteq Y|_X$ such that for some $i \in \{1, 2, \ldots, k\}$, $(g_i)_{Y|_X} = (g_i)_Y = (g_i)_X$ if and only if $\frac{\beta_i}{\alpha_i} = 1$ and $\frac{\beta_l}{\alpha_l} \geq 1$ for all $l \neq i$ if and only if $1 = \min \left\{ \frac{\beta_m}{\alpha_m} | 1 \leq m \leq k \right\}$ if and only if $1 = \lambda(X, Y)$. ∎

Theorem 2.8. *Let $X, Y \in ZS(G)$. $\lambda(X, Y) > 1$ if and only if for all elements $g_i \in X, (g_i)_X < (g_i)_Y = (g_i)_{Y|_X}$.*

Proof. Let $X, Y \in ZS(G)$. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$, where j and k are positive integers and $j \geq k$. Then, recall $Y|_X = g_1^{\beta_1} g_2^{\beta_2} \cdots g_k^{\beta_k}$.

First, suppose that $\lambda(X, Y) > 1$. Then, by Theorem 2.1, $\min \left\{ \frac{\beta_i}{\alpha_i} | 1 \leq i \leq k \right\} > 1 \Rightarrow \frac{\beta_i}{\alpha_i} > 1$ for all i such that $1 \leq i \leq k$. Thus, $\beta_i > \alpha_i$ for all $i \Rightarrow$ for all elements $g_i \in X, (g_i)_X < (g_i)_Y = (g_i)_{Y|_X}$.

Second, suppose that for all elements $g_i \in X, (g_i)_X < (g_i)_Y = (g_i)_{Y|_X}$. Then, $\beta_i > \alpha_i$ for all i such that $1 \leq i \leq k$. Thus, $\frac{\beta_i}{\alpha_i} > 1$ for all $i \Rightarrow \min \left\{ \frac{\beta_i}{\alpha_i} | 1 \leq i \leq k \right\} > 1$. Then, according to Theorem 2.1, $\lambda(X, Y) > 1$. ∎

Theorem 2.9. *Let $X, Y \in ZS(G)$. $0 < \lambda(X, Y) < 1$ if and only if $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$ with at least one element $g_i \in X$ where $(g_i)_Y < (g_i)_X$.*

Proof. Let $X, Y \in ZS(G)$. Let $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ such that $\alpha > 0$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$, where $j, k \in \mathbb{Z}^+$ and $j \geq k$. (Since, trivially, if $j < k$, then $\lfloor X \rfloor \nsubseteq \lfloor Y \rfloor$, which implies that $\lambda(X, Y) = 0$.)

First, let $0 < \lambda(X, Y) < 1$. Since $0 < \lambda(X, Y)$, according to Theorem 2.6, $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$. Now, by Theorem 2.1 $\min \left\{ \frac{\beta_m}{\alpha_m} | 1 \leq i \leq k \right\} = \lambda(X, Y) < 1 \Rightarrow$ there exists at least one $g_i \in X$ such that $\frac{\beta_i}{\alpha_i} = \lambda(X, Y) < 1$. Thus, $\beta_i < \alpha_i \Rightarrow (g_i)_Y < (g_i)_X$.

Second, let $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$ with at least one element $g_i \in X$ where $(g_i)_Y < (g_i)_X$. Now, as shown in Theorem 2.6, since $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$, $\lambda(X, Y) > 0$. And, since for some $g_i \in X$, $(g_i)_Y < (g_i)_X \Rightarrow \beta_i < \alpha_i$. By algebra, we now have $\frac{\beta_i}{\alpha_i} < 1$. Thus, by Theorem 2.1, $\lambda(X, Y) \leq \frac{\beta_i}{\alpha_i} < 1$. ∎

2.3 X or Y Minimal Zero Sequence

As we continue to get more specific, when examining general properties of the extraction degree, we now place restrictions on X or Y, such that one of them is a minimal zero sequence. In doing so, we obtain the following nine results.

The following theorem makes it possible to only look at a special subset of the zero sequences of finite abelian groups, by creating an automorphism that is applicable as specified in the theorem. This theorem is useful in simplifying the proofs of later theorems.

Theorem 2.10. *Let G be a finite abelian group. Let $X \in MZS(G)$ such that $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $g_i = ({g_i}^{(1)}, {g_i}^{(2)}, \ldots, {g_i}^{(s)})$ for all i such that $1 \leq i \leq k$ where ${g_i}^{(j)} \in \mathbb{Z}_{n_j}$ for all $1 \leq j \leq s$. If there exists l where $1 \leq l \leq k$ such that $\alpha_l > \frac{n_s}{2}$ then there exists an automorphism $\phi : G \to G$ such that $\phi(g_l) = ({g_l}^{(1)}, {g_l}^{(2)}, \ldots, {g_l}^{(s-1)}, 1)$.*

Proof. Let $X \in MZS(G)$ such that $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $g_i = ({g_i}^{(1)}, {g_i}^{(2)}, \ldots, {g_i}^{(s)})$ for all i such that $1 \leq i \leq s$ where ${g_i}^{(j)} \in \mathbb{Z}_{n_j}$ for all j such that $1 \leq j \leq s$. Let $1 \leq l \leq k$ such that $\alpha_l > \frac{n_s}{2}$. Since $X \in MZS(G)$ and $\alpha_l > \frac{n}{2}$, $|{g_l}^{(s)}| = n_s$. Then there exists $r \in \mathbb{Z}_{n_s}$ such that $r \cdot {g_l}^{(s)} \equiv 0 \pmod{n_s}$.

Let $\phi : G \Rightarrow G$ be defined as $\phi((g^{(1)}, g^{(2)} \ldots, g^{(s)})) = (g^{(1)}, g^{(2)}, \ldots, g^{(s-1)}, 1)$. Since $\phi((1, 0, \ldots, 0)) = (1, 0, \ldots, 0)$ and $\phi((0, 0, \ldots, 0, 1)) = (0, 0, \ldots, 0, r)$, ϕ is an automorphism, and $\phi(g_l) = ({g_l}^{(1)}, {g_l}^{(2)}, \ldots, {g_l}^{(s-1)}, 1)$. ■

This next theorem and the resulting corollary construct minimal zero sequences in groups of rank two and higher. These specific constructions are used often in later proofs dealing with sets of extraction degrees.

Theorem 2.11. *Let $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ such that $n_1 | n_2$.
If $X = (0, 1)^{\alpha}(1, 1)^{n_2 - \alpha}(r, 0)^1$, where $\alpha \in \{1, 2, \ldots, n_2\}$, $r \equiv (n_1 - [n_2 - \alpha]) \pmod{n_1}$, and $(r, 0) \in X$ when $r > 0$, then $X \in MZS(G)$.*

Proof. Let $X = (0, 1)^{\alpha}(1, 1)^{n_2 - \alpha}(r, 0)^1$, where $1 \leq \alpha \leq n_2$, $r \equiv (n_1 - [n_2 - \alpha]) \pmod{n_1}$, and $(r, 0) \in X$ when $r > 0$.

First, suppose that $1 \leq \alpha \leq n_2$ such that $n_1 | (n_2 - \alpha)$. Thus, there exists some $m \in \mathbb{Z}^{\geq 0}$ such that $m\, n_1 = n_2 - \alpha$. Now, $r = n_1 - (n_2 - \alpha) = n_1 - (m\, n_1) = (1 - m)(n_1) \equiv 0 \pmod{n_1}$. Then, $X = (0, 1)^{\alpha}(1, 1)^{n_2 - \alpha}$. Thus, $X = \alpha(0, 1) + (n_2 - \alpha)(1, 1) = (n_2 - \alpha, \alpha + n_2 - \alpha) = (m\, n_1, n_2) = \mathbf{0}$. Since the second coordinate sums to exactly n_2, $X \in MZS(G)$.

Second, suppose that $1 \leq \alpha \leq n_2$ such that $n_1 \nmid (n_2 - \alpha)$. Since $n_1 \nmid (n_2 - \alpha)$, $n_1 \equiv (n_2 - \alpha) \pmod{n_1} \neq 0 \Rightarrow 1 \leq r \leq (n_1 - 1)$. Thus, $X =$

$(0,1)^{\alpha}(1,1)^{n_2-\alpha}(r,0)^1 = \alpha(0,1)+(n_2-\alpha)(1,1)+(r,0) = (n_2-\alpha+n_1-(n_2-\alpha),\alpha+n_2-\alpha) = (n_1,n_2) = \mathbf{0}$. Since the first and second coordinates sum to exactly n_1 and n_2 respectively, $X \in MZS(G)$. ∎

Corollary 2.12. *Let $G = \mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\cdots\oplus\mathbb{Z}_{n_s}$ where $s \geq 3$ and $n_i|n_j$ for all i,j such that $1 \leq i < j \leq k$. If $X = (0,\ldots,0,1)^{\alpha}(0,\ldots,0,1,1)^{n_k-\alpha}(0,\ldots,0,r,0)^1$, where $1 \leq \alpha \leq n_s$, $r = (n_{s-1}-[(n_s-\alpha)1]) \pmod{n_{s-1}}$, and $(0,\ldots,0,r,0) \in X$ when $r > 0$, then $X \in MZS(G)$.*

Please note that the structure and arguments from Theorem 2.11 are identical here, in Corollary 2.12. Slight notational changes are the following: The first and the second coordinates in Theorem 2.11 are now the $s-1$ and the s coordinates, respectively, and the i^{th} coordinates where $1 \leq i < s-1$ contain zeros for each element in X.

Continuing to examine what happens when either X or Y is a minimal zero sequence generates the following results.

Theorem 2.13. *If $X \in MZS(G)$ where $X = g_1^{\alpha_1}g_2^{\alpha_2}\cdots g_k^{\alpha_k}$, then $1 \leq \alpha_i \leq n_s$ for all i such that $1 \leq i \leq k$.*

Proof. Let $X \in MZS(G)$ where $X = g_1^{\alpha_1}g_2^{\alpha_2}\cdots g_k^{\alpha_k}$. Since X is a minimal zero sequence the maximum number of times one elements can appear in X is n_s. So, $\alpha_i \leq n_s$ for all i such that $1 \leq i \leq k$.

Furthermore, if G is cyclic there is a restriction on the number of times one element can appear in X (since $X \in MZS$); Let $G = \mathbb{Z}_{n_s}$. Then, since $\alpha_i \leq n_s$ for all i such that $1 \leq i \leq k$, suppose there exists m where $1 \leq m \leq k$ such that $\alpha_m = n_s - 1$; then $X = (g_m)^{n_s-1}g_p$. Then $(n_s-1)g_m + g_p = \mathbf{0}$ $\Rightarrow g_p = n_s - (n_s-1)g_m = g_m$. So, $X = (g_m)^{n_s-1}g_m = (g_m)^{n_s}$. So when G is cyclic, $\alpha_i \neq n_s - 1$ for all $1 \leq i \leq j$. Therefore, when G is cyclic, $\alpha_i \in \{1,2,\ldots,n_s-2,n_s\}$ for all $1 \leq i \leq k$.

When G is not cyclic, if $\alpha_m = n_s - 1$ for some $m \in \{1,2,\ldots,k\}$, then let $X = (0,\ldots,0,1)^{n_s-1}(0,\ldots,0,1,1)(0,\ldots,0,a,0)$ such that $a \equiv (n_{s-1}-1) \pmod{n}_{s-1}$. Then, since the last coordinate sums to exactly n_s and $X = \mathbf{0}$, $X \in MZS(G)$. Thus, when G is not cyclic $\alpha_i \in \{1,2,\ldots,n_s\}$ for all i such that $1 \leq i \leq k$.

∎

Corollary 2.14. *If $X \in MZS(G)$, $Y \in ZS(G)$ then*

$$\lambda(X,Y) = 0 \text{ or } \lambda(X,Y) \geq \frac{1}{n_s}.$$

Proof. Let $X \in MZS(G)$ and $Y \in ZS(G)$ such that $\lambda(X,Y) = \frac{\beta}{\alpha}$. Then from Theorem 2.13 $\alpha \leq n_s$. From the definition of $\lambda(X,Y)$, $\beta \in \mathbb{Z}^{\geq 0}$. If $\beta = 0$ then $\lambda(X,Y) = 0$. Now, let $\beta \in \mathbf{Z}^{+}$. Then $\beta \geq 1$ so $\frac{\beta}{\alpha} \geq \frac{1}{n_s}$. ∎

Theorem 2.15. *If $X \in ZS(G)$ and $Y \in MZS(G)$ then*

$$\lambda(X,Y) \leq 1.$$

Proof. Let $X \in ZS(G)$ and $Y \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$. If $j < k$ then $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$ and by Theorem 2.6, $\lambda(X,Y) = 0 < 1$. So, let $j \geq k$. Assume $\lambda(X,Y) > 1$. Let $\frac{\beta}{\alpha} = \lambda(X,Y) > 1$. Then $\alpha < \beta$. From Theorem 2.1, $\frac{\beta}{\alpha} = \min\left\{\frac{\beta_i}{\alpha_i} | 1 \leq i \leq k\right\}$, which implies that $\frac{\beta_i}{\alpha_i} \geq \frac{\beta}{\alpha} > 1$ so $\beta_i > \alpha_i$ for all i such that $1 \leq i \leq k$. Then $X \subsetneq Y$. However, $Y \in MZS(G)$, so $X \notin ZS(G)$ which is a contradiction so $\frac{\beta}{\alpha} \leq 1$. ∎

It is important to note that when Y is a minimal zero sequence of a cyclic group, there is another bound on the extraction degree. The following lemmas and theorem state this bound.

Lemma 2.16. *Let $\beta \leq n-2$ and $\alpha > \beta$. Then $\frac{\beta}{\alpha} > \frac{n-2}{n}$ if and only if $\beta = \alpha - 1$ and $\frac{n}{2} < \alpha < n$.*

Proof. First, let $\beta = \alpha - 1$ where $\frac{n}{2} < \alpha < n$. Since $\frac{n}{2} < \alpha \Rightarrow n < 2\alpha \Rightarrow n\alpha - n > n\alpha - 2\alpha \Rightarrow \frac{\alpha-1}{\alpha} > \frac{n-2}{n} \Rightarrow \frac{\beta}{\alpha} > \frac{n-2}{n}$.

Now, let $\frac{\beta}{\alpha} > \frac{n-2}{n}$. We want to show that $\beta = \alpha - 1$ and $\frac{n}{2} < \alpha < n$. Assume not, then $\beta \neq \alpha - 1$, $\alpha \leq \frac{n}{2}$, or $\alpha \geq n$.

First, let $\beta \neq \alpha - 1$. Since $\beta < \alpha$, we have that $\beta \leq \alpha - 2$. Now, $\alpha \leq n$ or $\alpha > n$. If $\alpha > n$ then $\frac{\beta}{\alpha} \leq \frac{n-2}{n}$, which is a contradiction. Now, let $\alpha \leq n$. Then $2\alpha \leq 2n \Rightarrow n\alpha - 2\alpha \geq n\alpha - 2n \Rightarrow \frac{n-2}{n} \geq \frac{\alpha-2}{\alpha} \geq \frac{\beta}{\alpha}$, which is a contradiction. Since we reached a contradiction for all α, we have that $\beta = \alpha - 1$.

Second, let $\alpha \leq \frac{n}{2}$. Then $2\alpha \leq n \Rightarrow n\alpha - 2\alpha \geq n\alpha - n \Rightarrow \frac{n-2}{n} \geq \frac{\alpha-1}{\alpha} = \frac{\beta}{\alpha}$, which is a contradiction, so $\alpha > \frac{n}{2}$.

Now, let $\alpha \geq n$. Since $\beta \leq n - 2$ we have that $\frac{\beta}{\alpha} \leq \frac{n-2}{n}$ which is a contradiction, so $\alpha < n$.

So, we have that $\beta = \alpha - 1$, and $\frac{n}{2} < \alpha < n$. ∎

The following term, is needed for this next lemma: If a zero sequence Y sums to $n \in \mathbb{Z}$, then we say that Y is *basic* [7].

Lemma 2.17. *Let $Y \in MZS(G)$. If $Y = 1^{\beta} g_2^{\beta_2} \cdots g_j^{\beta_j}$ where $\beta \geq \frac{n-1}{2}$ then Y is basic.*

Proof. If any combination of g_i's has a sum greater than $n - \beta$ in $\mathbb{Z}_n$, then because $(1)_Y = \beta$, we know we can find a proper subset of Y whose sum is zero in $\mathbb{Z}_n$. So, each element and each combination of elements, not including the combination with all the elements, has a sum less than $n - \beta$. So, if we look at the sequence $g_2, g_2^2, \ldots, g_2^{\beta_2}, g_2^{\beta_2} g_3, g_2^{\beta_2} g_3^2, \ldots, g_2^{\beta_2} g_3^{\beta_3} \cdots g_k^{\beta_k - 1}$, the sum of each term modulo n is less than $n - \beta$. So, in $\mathbb{Z}^+$ the sum of each term is less than $n - \beta$ or greater than n. Since $g_2 < n - \beta$, if there is a term in this sequence that is greater than n, we must have at some point jumped from being less than $n - \beta$ to being greater than n, which implies that we added on some g_i whose value is greater than β. Then $\beta < g_i < n - \beta \Rightarrow \beta < g_i \leq n - \beta - 1 \Rightarrow \beta < n - 1 - \beta$ $\Rightarrow 2\beta < n - 1 \Rightarrow \beta < \frac{n-1}{2}$, which is a contradiction, so every term is less than $n - \beta$ in $\mathbb{Z}^+$ so $\beta + \sum_{i=2}^{k} g_i \cdot \beta_i = n$ in $\mathbb{Z}$. Therefore, Y is basic. ■

Theorem 2.18. *Let $G = \mathbb{Z}_n$ where $n \geq 4$. If $Y \in MZS(G)$, $X \in ZS(G)$ then*

$$\lambda(X, Y) \leq \frac{n-2}{n} \text{ or } \lambda(X, Y) = 1.$$

Proof. Let $X \in ZS(G)$, $Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{\beta}{\alpha}$. From Theorem 2.13, $\beta \in \{1, 2, \ldots, n-2, n\}$ and from Theorem 2.15, $\frac{\beta}{\alpha} \leq 1$. So assume $\frac{n-2}{n} < \frac{\beta}{\alpha} < 1$.

Let $\frac{n-2}{n} < \frac{\beta}{\alpha} < 1$. Either $\beta = n$, or $\beta \leq n - 2$. If $\beta = n$, then the floor of Y consists of one element, call it g, where $|g| = n$. If $\lfloor X \rfloor \nsubseteq \lfloor Y \rfloor$ then by Theorem 2.6, $\lambda(X, Y) = 0 \leq \frac{n-2}{n}$ which is a contradiction, so let $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$. Then $\lfloor X \rfloor = \{g\}$ which implies that $X = g^{\alpha}$. In order for X to be a zero sequence, $\alpha = m \cdot n$ where $m \in \mathbb{Z}^+$. If $m = 1$ then $\frac{\beta}{\alpha} = 1$, which is a contradiction. So, $m \geq 2$. This implies that $\frac{\beta}{\alpha} \leq \frac{n}{2} \leq \frac{n-2}{n}$ since $n \geq 4$, which is also a contradiction, so $\beta \leq n - 2$.

Since $\frac{\beta}{\alpha} < 1$, $\beta < \alpha$. Then, since $\beta \leq n - 2$, we can apply Lemma 2.16. Since $\frac{n-2}{n} < \frac{\beta}{\alpha}$, we know that $\frac{\beta}{\alpha} = \frac{\alpha-1}{\alpha}$ and $\frac{n}{2} < \alpha < n$. Now, we work towards showing that this is not a valid extraction degree when $Y \in MZS(G)$. First, we will show that without loss of generality, we can assume that the critical element, call it g, is 1.

Since $\beta = \alpha - 1$ and $\alpha > \frac{n}{2}$, we have that $\beta > \frac{n-2}{2} \Rightarrow \beta \geq \frac{n-1}{2}$. If n is even then $\frac{n-1}{2} \notin \mathbb{Z}$ so $\beta \geq \frac{n}{2} \Rightarrow$. If $\beta = \frac{n}{2}$ then $|g| \geq \frac{n}{2}$. If $|g| = \frac{n}{2}$, then because $Y \in MZS(G)$, $Y = g^{\frac{n}{2}}$. Since $\lambda(X,Y) \neq 0$, $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$, so $X = g^{\frac{n}{2}+1}$, However, since $n \geq 4$ and $|g| = \frac{n}{2}$, $X \notin ZS(G)$ which is contradiction, so $|g| \neq \frac{n}{2} \Rightarrow |g| < \frac{n}{2}$,which is also a contradiction. Thus, $\beta \neq \frac{n}{2} \Rightarrow \beta > \frac{n}{2}$. So, from Theorem 2.10, we can assume $g = 1$. Now, if n is odd, then neither $\frac{n-1}{2}$ nor $\frac{n}{2}$ divides n, so $|g| > \frac{n}{2} \Rightarrow |g| = n$, so from Theorem 2.10, we can assume $g = 1$. So, we write $X = 1^{\alpha} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ and $Y = 1^{\beta} g_2^{\beta_2} \cdots g_j^{\beta_j}$ where $j \geq k$. Then from Lemma 2.17, Y is basic.

Now, we will show that $\alpha_i \leq \beta_i$ for all $i \in \{2, 3, \ldots, k\}$. Assume not, then there exists $m \in \{2, 3, \ldots, k\}$ such that $\alpha_m > \beta_m$. So, $\alpha_m \geq \beta_m + 1$. We will look at the cases when $\alpha_m > n$ and when $\alpha_m \leq n$.

If $\alpha_m > n$, then $\frac{\beta_m}{\alpha_m} < \frac{\frac{n}{2}}{n} = \frac{1}{2} \leq \frac{n-2}{n} < \frac{\beta}{\alpha}$ which implies that $\lambda(X,Y) \neq \frac{\beta}{\alpha}$, which is a contradiction so $\alpha_m \leq n$.

Now, let $\beta_m + 1 \leq \alpha_m \leq n$. Here we have two subcases: $\beta_m + 1 = \alpha_m$ and $\beta_m + 1 < \alpha_m$. In the first subcase, $\beta_m + 1 = \alpha_m$, which implies that $\beta_m = \alpha_m - 1$. Since the maximum number of elements in any minimal zero sequence with more than one distinct element is $n-1$ and $\beta \geq \frac{n-1}{2}$, we know that $\beta_i \leq \frac{n-1}{2}$ for all $i \in \{2, 3, \ldots, k\}$. Since $m \in \{2, 3, \ldots, k\}$, $\beta_m \leq \frac{n-1}{2}$. So, we have that $\alpha_m - 1 = \beta_m \leq \frac{n-1}{2} \leq \beta = \alpha - 1$. If $\beta_m = \frac{n-1}{2} = \beta$, then $Y = 1^{\frac{n-1}{2}} g_m^{\frac{n-1}{2}}$ and $X = 1^{\frac{n+1}{2}} g_m^{\frac{n+1}{2}}$ because $\alpha = \beta + 1$, $\alpha_m = \beta_m + 1$, and the maximum number of elements in Y is $n-1$. Then the sum of the elements in X is $n+1+g_m$. Since $g_m < n - \beta$ and $\beta \geq \frac{n-1}{2}$, $g_m < \frac{n+1}{2} < n-1$ since $n \geq 4$, so the sum of X does not equal zero. So, $\beta_m \neq \beta \Rightarrow \alpha_m - 1 < \alpha - 1$. Then $\frac{\beta_m}{\alpha_m} = \frac{\alpha_m - 1}{\alpha_m} < \frac{\alpha - 1}{\alpha} = \frac{\beta}{\alpha}$ which is a contradiction.

In the second subcase, $\beta_m + 1 < \alpha_m$. We then have that $\beta_m + 2 \leq \alpha_m$ so $\frac{\beta_m}{\alpha_m} \leq \frac{\alpha_m - 2}{\alpha_m} \leq \frac{n-2}{n} \leq \frac{\beta}{\alpha}$ which is a contradiction. Since we have reached a contradiction in all cases where $\alpha_m > \beta_m$ we know that $\alpha_i \leq \beta_i$ for all $i \in \{2, 3, \ldots, k\}$.

So, we have shown that when $\frac{\beta}{\alpha} > \frac{n-2}{n}$, Y is basic and $\alpha_i \leq \beta_i$ for all $i \in \{2, 3, \ldots, k\}$. Now, we will show that $\frac{\beta}{\alpha}$ is not a possible extraction degree when $Y \in MZS(G)$. We will first look at the case when $\alpha_i = \beta_i$ for all $i \in \{2, 3, \ldots, k\}$, then we will look at the case when there exists $m \in \{2, 3, \ldots, k\}$ such that $\alpha_m < \beta_m$.

Let $\alpha_i = \beta_i$ for all i. Then, since Y is basic we have $n = \sum_{i=2}^{k} g_i\beta_i + \alpha - 1 = \sum_{i=2}^{k} g_i\alpha_i + \alpha - 1 \equiv n - 1$. Since $n \geq 4$, we have a contradiction.

Let $\alpha_m < \beta_m$ for some $m \in \{2, 3, \ldots, k\}$. Then $\alpha_m \leq \beta_m - 1$. Since $g_m > 1$ we have, $g_m\alpha_m \leq g_m(\beta_m - 1) = g_m\beta_m - g_m < g_m\beta_m - 1$. Since Y is basic we have $n - 1 = \sum_{i=2}^{k} g_i\beta_i - 1 + \alpha - 1 > \sum_{i=2}^{k} g_i\alpha_i + \alpha - 1 = n - 1$, which is a contradiction. Since we have a contradiction in both cases, we know that when $\frac{bt}{\alpha} > \frac{n-2}{n}$, there is no $X \in ZS(G)$ and $Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{\beta}{\alpha}$. Therefore, $\frac{\beta}{\alpha} \leq \frac{n-2}{n}$. ■

3 Sets of Extraction Degrees

Moving beyond general theorems, we now examine the sets of possible extraction degrees for zero sequences, X and Y. This first theorem tells us what all the possible extraction degrees are when neither X nor Y have any restrictions. Then, the following theorems deal with specific subsets of $ZS(G)$. Furthermore, in this theorem and in many subsequent theorems $\lambda(ZS, ZS) = \lambda(ZS(G), ZS(G))$, so the G is implied.

Theorem 3.1. *Let $X, Y \in ZS(G)$ then*

$$\lambda(ZS, ZS) = \mathbb{Q}^+ \cup \{0\}.$$

Proof. It suffices to show that $\mathbb{Q}^+ \cup \{0\} \subseteq \lambda(ZS, ZS)$. Let $r \in \mathbb{Q}^+ \cup \{0\}$. Then there exists $\beta \in \mathbb{Z}^{\geq 0}$ and $\alpha \in \mathbb{Z}^+$ such that $r = \frac{\beta}{\alpha}$. Let X and Y be sequences such that $X = (0, 0, \ldots, 1)^\alpha (0, 0, \ldots, n_k - 1)^\alpha$ and $Y = (0, 0, \ldots, 1)^\beta$ $(0, 0, \ldots, n_k - 1)^\beta$. Then, $X, Y \in ZS(G)$. From Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha}$. Therefore, $\lambda(ZS, ZS) = \mathbb{Q}^+ \cup \{0\}$. ■

Although for Theorem 3.1, the set of possible extraction degrees is the same for all finite abelian groups, there are some situations where if X, Y, or both are restricted, then the set of possible extraction degrees is notably different for finite abelian cyclic groups. This is mainly because of the restriction for cyclic groups when working with minimal zero sequences as discussed in Theorem 2.13. Thus, when our set of extraction degrees is notably different for cyclic groups, it will also be mentioned.

3.1 X and Y Minimal

We will now restrict both X and Y to be minimal zero sequences. We will first look at certain results in rank two. This lemma explains about a hole that appears when $G = \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $n \in \mathbf{2}\mathbb{Z}^{\geq 4}$, which then leads to the theorem for the set of possible extraction degrees in this case. Also, in this lemma, since we are working in rank two, the maximum number of elements in $Y \in MZS(G)$ is $n+1$ [8].

Lemma 3.2. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $n \in \mathbf{2}\mathbb{Z}^{\geq 4}$. If $X, Y \in MZS(G)$ such that there exists an element, $g \in X$ where $(g)_X = n-1$ and $(g)_Y = n-2$, then $\lambda(X, Y) = 0$.*

Proof. Let $X, Y \in MZS(G)$ such that there exists an element, $g = (a, b) \in X$ where $(g)_X = n-1$ and $(g)_Y = n-2$. We need to show that we can use the automorphism defined in Theorem 2.10. First, suppose $g = (a, 0)$. Then, since $n \in \mathbf{2}\mathbb{Z}^{\geq 4}$, $n-1 > 2$. Thus, $(a,0)^2 = \mathbf{0}$ and $(a,0)^2 \subsetneq (a,0)^{n-1} \subseteq X$. This contradicts $X \in MZS(G)$. Thus, $b \neq 0$. Second, since $(g)_X = n-1$, we want $|b| > \frac{n}{2}$. Thus, suppose $|b| \leq \frac{n}{2} < n-1$. Then, since $(a,b)^{|b|} = 0$ and $(a,b)^{|b|} \subsetneq (a,b)^{n-1} \subset X$ this contradicts $X \in MZS(G)$. Thus, $|b| > \frac{n}{2}$. Now, according to Theorem 2.10, there exists an automorphism $\phi : \mathbb{Z}_2 \oplus \mathbb{Z}_n \to \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $\phi((a,b)) = (a,1)$. Since $a \in \mathbb{Z}_2$, $\phi((0,b)) = (0,1)$ and $\phi((1,b)) = (1,1)$.

Using this automorphism, we will construct the possible $X, Y \in MZS(G)$. For the first case, suppose that $g = (0,1)$. Then, $X = (0,1)^{n-1}(1,1)(1,0)$, which according to Theorem 2.11 is in $MZS(G)$. Now, according to Theorem 2.6, $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$ if and only if $\lambda(X,Y) = 0$. So, to prove by contradiction, assume $\lambda(X,Y) > 0$. Thus, $Y = (0,1)^{n-2}(1,1)^{\gamma}(1,0)^{\delta} g_m^{\epsilon}$, where $\gamma, \delta > 0$ and $g_m \in Y$ only when $\epsilon > 0$. As mentioned above, the maximum number of elements in $Y \in MZS(G)$ is $n+1$. Since $\gamma, \delta > 0$ we have two possibilities: First, $\gamma + \delta = 2$. Then, $\gamma = \delta = 1$. Then, $Y = (0,1)^{n-2}(1,1)(1,0) = (0, n-1) \neq \mathbf{0}$. Thus, we need $\epsilon = 1$ where $g_m = (0,1) = g$. Then, $Y = (0,1)^{n-1}(1,1)(1,0) = X$ and $(g)_Y \neq n-2$. Second, $\gamma + \delta = 3$. Then, either $\gamma = 1$ and $\delta = 2$ or $\gamma = 2$ and $\delta = 1$. If $\delta = 2$, then $(1,0)^2 \in MZS(G)$ and $(1,0)^2 \subsetneq Y$, this contradicts $Y \in MZS(G)$. Now, if $\gamma = 2$ and $\delta = 1$, then $\epsilon = 1$ where $g = (1,0)$. Thus, once again $(1,0)^2 \subsetneq Y \Rightarrow Y \notin MZS(G)$. Thus, if $g = (0,1)$, then $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor \Rightarrow \lambda(X,Y) = 0$.

For the second case, suppose that $g = (1,1)$. Then, $X = (1,1)^{n-1}(0,1)(1,0)$, which according to Theorem 2.11 is in $MZS(G)$. Similarly, assume $\lambda(X,Y) > 0$. Thus, $Y = (1,1)^{n-2}(0,1)^{\gamma}(1,0)^{\delta}g_m^{\epsilon}$, where $\gamma, \delta > 0$ and $g_m \in Y$ only when $\epsilon > 0$. Once again, the maximum number of elements in $Y \in MZS(G)$ is $n+1$. Since, $\gamma, \delta > 0$ we have the same two possibilities: First, $\gamma + \delta = 2 \Rightarrow \gamma = \delta = 1 \Rightarrow Y = (1,1)^{n-2}(0,1)(1,0)(1,1) = X \Rightarrow (g_1)_Y \neq n-2$. Second, $\gamma + \delta = 3$. If $\delta = 2 \Rightarrow (1,0)^2 \subsetneq Y \Rightarrow Y \notin MZS(G)$. Now, if $\gamma = 2$ and $\delta = 1$, then $\epsilon = 1$ where $g_m = (1,0) \Rightarrow (1,0)^2 \subsetneq Y \Rightarrow Y \notin MZS(G)$. Thus, if $g = (1,1)$, then $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor \Rightarrow \lambda(X,Y) = 0$. ■

Theorem 3.3. *If $G = \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $n \in \mathbf{2}\mathbb{Z}^{\geq 4}$, then*

$$\lambda(MZS, MZS) = \left\{ \frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \frac{\beta}{\alpha} \neq \frac{n-2}{n-1},\ 1 \leq \alpha \leq n \right\} \cup \{0\}.$$

Proof. First, let $\lambda(MZS, MZS) = \frac{\gamma}{\delta}$ (and show that $\frac{\gamma}{\delta} \subseteq \left\{ \frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \frac{\beta}{\alpha} \neq \frac{n-2}{n-1},\ 1 \leq \alpha \leq n \right\}$). Then, there exist $X \in MZS(G)$ and $Y \in MZS(G)$ such that $\frac{\gamma}{\delta} = \lambda(X,Y)$. From Theorem 2.13, since $X \in MZS(G)$ and $Y \in MZS(G)$, $\delta \leq n$ and $\gamma \leq n$. From Corollary 2.14, since $X \in MZS(G)$, $\frac{\gamma}{\delta} \geq \frac{1}{n}$ or $\frac{\gamma}{\delta} = 0$. From Lemma 3.2, since $X, Y \in MZS(G)$, $\frac{\gamma}{\delta} \neq \frac{n-2}{n-1}$. Last, suppose that $\frac{\gamma}{\delta} > 1$. Then, according to Theorem 2.8, for all $g \in X$, $(g)_X < (g)_Y$. This implies that $X \subsetneq Y$. Then, $Y \notin MZS(G)$, contradiction. Thus, $\frac{\gamma}{\delta} \leq 1$.

Now let $\frac{\gamma}{\delta} \in \left\{ \frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \frac{\beta}{\alpha} \neq \frac{n-2}{n-1},\ 1 \leq \alpha \leq n \right\} \cup \{0\}$ (and show that $\frac{\gamma}{\delta} \subseteq \lambda(MZS, MZS)$). First, we will show that $0 \in \lambda(MZS, MZS)$. Let $X = (0,1)^{n-2}(1,1)^2$ and let $Y = (0,1)^n$. Then $X, Y \in MZS(G)$ and $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$, thus $\lambda(X,Y) = 0$. Second, we will show that $1 \in \lambda(MZS, MZS)$. Let $X \in MZS(G)$. Let $Y = X$. Then $Y \in MZS(G)$, and $\lambda(X,Y) = 1$.

Third, we will show that $\frac{\gamma}{\delta} \in \left\{ \frac{\beta}{\alpha} | \frac{\beta}{\alpha} < 1,\ \frac{\beta}{\alpha} \neq \frac{n-2}{n-1},\ 1 \leq \alpha \leq n \right\}$. To show this, there are the following three cases, where δ and γ are positive integers: Case one, let $\delta = n$ then $0 < \gamma < \delta$. Let $X = (0,1)^n$ and $Y = (0,1)^{\gamma}(1,1)^{n-\gamma}(a,0)$ where $a = (2 - [n-\gamma]) \pmod 2$, and $(a,0) \in Y$ when $a > 0$. Then, according to Theorem 2.11, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} = \frac{\gamma}{n} < 1$. Case two, let $0 < \delta \leq n-1$ and δ be an odd positive integer then $0 < \gamma < \delta$ and $\gamma \neq n-2$. Let $X = (0,1)^{\delta}(1,1)^{n-\delta}(1,0)$ and when $\gamma \in \mathbf{2}\mathbb{Z}^+$, $Y = (0,1)^{n-2k_1}(1,1)^{2k_1-2}(1,0)(1,2)$ such that $k_1 \geq 2$ and when γ is an odd positive integer, $Y = (0,1)^{n-(2k_2-1)}(1,1)^{2k_2-1}(1,0)$ such that $k_2 \geq 1$. Then, $X, Y \in MZS(G)$ and for all such values of $\gamma, \frac{\gamma}{\delta} < 1$. Case three, let $0 < \delta \leq n-2$ and $\delta \in \mathbf{2}\mathbb{Z}^+$ then $0 < \gamma < \delta$. Let $X = (0,1)^{\delta}(1,1)^{n-\delta}$ and when $\gamma \in \mathbf{2}\mathbb{Z}^+$, $Y = (0,1)^{n-2k_3}(1,1)^{2k_3}$ such that $k_3 \geq 1$ and when γ is an odd positive integer, $Y = (0,1)^{n-(2k_4+1)}(1,1)^{2k_4+1}(1,0)$ such that $k_4 \geq 2$. Then, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} < 1$. ■

By excluding the case addressed in Theorem 3.3, we obtain this inclusive result.

Theorem 3.4. *If $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ such that $n_1 \geq 3$ and $n_1 | n_2$, then*

$$\lambda(MZS, MZS) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ 1 \leq \alpha \leq n_2\right\} \cup \{0\}.$$

Proof. First, let $\lambda(MZS, MZS) = \frac{\gamma}{\delta}$ (and show that $\frac{\gamma}{\delta} \subseteq \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ 1 \leq \alpha \leq n_2\right\}$). Then, there exist $X \in MZS(G)$ and $Y \in MZS(G)$ such that $\frac{\gamma}{\delta} = \lambda(X, Y)$. From Theorem 2.13, since $X \in MZS(G)$ and $Y \in MZS(G)$, $\delta \leq n_2$ and $\gamma \leq n_2$. From Corollary 2.14, since $X \in MZS(G)$, $\frac{\gamma}{\delta} \geq \frac{1}{n_2}$ or $\frac{\gamma}{\delta} = 0$. Next, suppose that $\frac{\gamma}{\delta} > 1$. Then, according to Theorem 2.8, for all $g \in X$, $(g)_X < (g)_Y$. This implies that $X \subsetneq Y$. Then, $Y \notin MZS(G)$, contradiction. Thus, $\frac{\gamma}{\delta} \leq 1$.

Now, let $\frac{\gamma}{\delta} \in \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ 1 \leq \alpha \leq n\right\} \cup \{0\}$ (and show that $\frac{\gamma}{\delta} \subseteq \lambda(MZS, MZS)$). First, show that $0 \in \lambda(MZS, MZS)$. Let $X = (0,1)^{n_2-1}(1,1)(a_1,0)$ where $a_1 = (n_1 - 1)$ and let $Y = (0,1)^{n_2}$. Then, according to Theorem 2.11, $X, Y \in MZS(G)$. Then, $X, Y \in MZS(G)$ and $\lfloor X \rfloor \nsubseteq \lfloor Y \rfloor$, thus $\lambda(X, Y) = 0$. Now, show that $1 \in \lambda(MZS, MZS)$. Let $X \in MZS(G)$. Let $Y = X$. Then $Y \in MZS(G)$, and $\lambda(X, Y) = 1$.

Next, show that $\frac{\gamma}{\delta} \in \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} < 1,\ 1 \leq \alpha \leq n\right\}$. To show this, there are the following three cases, where δ and γ are positive integers: Case one, let $\delta = n_2$ then $\gamma < \delta$. Let $X = (0,1)^{n_2}$ and $Y = (0,1)^{\gamma}(1,1)^{n_2-\gamma}(a_2,0)$ where $a_2 = (n_1 - [n_2 - \gamma]) \pmod{n_1}$, and $(a_2, 0) \in Y$ when $a_2 > 0$. Then, according to Theorem 2.11, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} = \frac{\gamma}{n_2} < 1$. Case two, let $\delta < n_2 - 1$. First, let $\gamma < \delta - 1$. Then, let $X = (0,1)^{\delta}(0,b_1)$ where $b_1 = (n_2 - \delta) \pmod{n}_2$ and $Y = (0,1)^{\beta}(0,b_1)(0,b_2)$ where $b_2 = (\delta - \gamma) \pmod{n_2}$. Then, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} < 1$. Second, let $\gamma = \delta - 1$. Then, let $X = (1,1)^{\delta}(a_3,b_3)$ where $a_3 = (n_1 - \delta) \pmod{n}_1$ and $b_3 = (n_2 - \delta)$ and $Y = (1,1)^{\delta-1}(a_3,b_3)(1,0)(0,1)$. Then, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} < 1$. Case three, let $\delta = n_2 - 1$. First, let $\gamma = \delta - 1$. Then, let $X = (1,1)^{\delta}(1,0)(0,1)$ and $Y = (1,1)^{\delta-1}(1,0)^2(0,1)^2$. Then, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} < 1$. Second, let

$\gamma < \delta - 1$. Then, let $X = (1,1)^{\delta}(1,0)(0,1)$ and $Y = (1,1)^{\gamma}(1,0)(0,1)(a_4, b_4)$ where $a_4 = (n_1 - [\gamma+1]) \pmod{n_1}$ and $b_4 = n_2 - (\gamma+1)$. Then, $X, Y \in MZS(G)$ and for all such values of δ and γ, $\frac{\gamma}{\delta} < 1$. ■

Now we exclude rank 2, and discover the hole addressed in Lemma 3.2 is filled in for higher ranks in this following lemma.

Lemma 3.5. *If $G = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{n_s}$ such that $n_s \in \mathbf{2}\mathbb{Z}^{\geq 4}$ and $s \geq 3$, then there exists $X, Y \in MZS(G)$ such that $\lambda(X,Y) = \frac{n_s-2}{n_s-1}$.*

Proof. Denote $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_{s-1}} \oplus \mathbb{Z}_{n_s}$ where $2 = \{n_1, \ldots, n_{s-1}\}$ and $n_s \in \mathbf{2}\mathbb{Z}^{\geq 4}$ and suppose $Rank(G) = k \geq 3$. Now, let $G = H_1 \oplus H_2$ where $H_1 = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_{s-3}}$ and $H_2 = \mathbb{Z}_{n_{s-2}} \oplus \mathbb{Z}_{n_{s-1}} \oplus \mathbb{Z}_{n_s}$. Then, denote $\mathbf{0}_{H_1}$,$\mathbf{0}_{H_2}$, and $\mathbf{0}_G$ as the zero elements in H_1, H_2, and G respectively. (Note: if $k = 3$ then $G = H_2$ and there are no elements from an H_1 to be in X or Y.)

Now, let $X = (\mathbf{0}_{H_1}, 0, 0, 1)^{n_s-1}(\mathbf{0}_{H_1}, 0, 1, 1)(\mathbf{0}_{H_1}, 0, 1, 0)$.
According to Corollary 2.12, $X \in MZS(G)$.

Let $Y = (\mathbf{0}_{H_1}, 0, 0, 1)^{n_s-2}(\mathbf{0}_{H_1}, 0, 1, 1)(\mathbf{0}_{H_1}, 0, 1, 0)(\mathbf{0}_{H_1}, 1, 0, 1)(\mathbf{0}_{H_1}, 1, 0, 0)$. Since the sum of all the elements in Y equals $(\mathbf{0}_{H_1}, \mathbf{0}_{H_2}) = \mathbf{0}_G$ and the s-coordinate sums to exactly n_s, we know $Y \in MZS(G)$.

Now, for all $g \in X$, $\min\left\{\frac{\beta}{\alpha} | (g)_X = \alpha,\ (g)_Y = \beta\right\} = \frac{n_s-2}{n_s-1}$. Thus, according to Theorem 2.1, $\lambda(X,Y) = \frac{n_s-2}{n_s-1}$. ■

This next theorem utilizes Lemma 3.5 to generalize this set of extraction degrees to all finite abelian non-cyclic groups.

Theorem 3.6. *If $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, $n_i | n_{i+1}$ for all $1 \leq i < s \geq 3$, then*

$$\lambda(MZS(G), MZS(G)) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ 1 \leq \alpha \leq n_s\right\} \cup \{0\}.$$

Proof. Let $X, Y \in MZS(G)$. Since $X, Y \in MZS(G)$, from Theorem 2.13, for all $g \in X$, $(g)_X = \alpha \in \{1, \ldots, n_k\}$, and since all $g \in X$ may or may not be in Y, $(g)_Y = \beta \in \{0, 1, \ldots, n_k\}$. Also, suppose that $\frac{\beta}{\alpha} > 1$. Then, according to Theorem 2.8, for all $g \in X$, $(g)_X < (g)_Y$. This implies that $X \subsetneq Y$. Then, $Y \notin MZS(G)$, contradiction. Thus, $\frac{\beta}{\alpha} \leq 1$ for all $g \in X$. Therefore, $\lambda(MZS(G), MZS(G)) \subseteq \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_s\}\right\} \cup \{0\}$.

Now, we must consider the cases where $n_{s-1} > 2$ and where $n_{s-1} = 2$. First, let $n_{s-1} > 2$ and let $G = H_1 \oplus H_2$ where $H_1 = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_{s-2}}$ and

$H_2 = \mathbb{Z}_{n_{s-1}} \oplus \mathbb{Z}_{n_s}$. From Theorem 2.5, $\lambda(MZS(H_2), MZS(H_2)) \subseteq \lambda(MZS(H_1 \oplus H_2), MZS(H_1 \oplus H_2)) = \lambda(MZS(G), MZS(G))$. Then, since $n_{s-1} \neq 2$, from Theorem 3.4, $\lambda(MZS(H_2), MZS(H_2)) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_s\}\right\} \cup \{0\}$. Thus, $\left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_k\}\right\} \cup \{0\} = \lambda(MZS(H_2), MZS(H_2)) \subseteq \lambda(MZS(G), MZS(G))$.
Therefore, $\lambda(MZS(G), MZS(G)) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_s\}\right\} \cup \{0\}$.

Second, let $n_{s-1} = 2$. Again, let $G = H_1 \oplus H_2$ where $H_1 = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_{s-2}}$ and $H_2 = \mathbb{Z}_{n_{s-1}} \oplus \mathbb{Z}_{n_s}$. And, as shown above and from Theorem 2.5, $\lambda(MZS(H_2), MZS(H_2)) \subseteq \lambda(MZS(G), MZS(G)) \subseteq \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_s\}\right\} \cup \{0\}$. Then, since $n_{s-1} = 2$, from Theorem 3.3, $\lambda(MZS(H_2), MZS(H_2)) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \frac{\beta}{\alpha} \neq \frac{n_s-2}{n_s-1},\ \alpha \in \{1, 2, \ldots, n_s\}\right\} \cup \{0\}$. Now, from Theorem 3.5, since $k \geq 3$ there exists $X, Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{n_s-2}{n_s-1}$. Thus, $\lambda(MZS(G), MZS(G)) = \left\{\frac{\beta}{\alpha} | \frac{\beta}{\alpha} \leq 1,\ \alpha \in \{1, \ldots, n_s\}\right\} \cup \{0\}$. ■

Now, in the cyclic group case, we obtain a similar but noteworthy result.

Theorem 3.7. *Let $G = \mathbb{Z}_n$ where $n \geq 4$. Then*

$$\lambda(MZS, MZS) = \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \alpha \in \{1, 2, \ldots, n-2, n\}\right\} \cup \{0, 1\}.$$

Proof. $\lambda(MZS, MZS) \subseteq \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \alpha \in \{1, 2, \ldots, n-2, n\}\right\} \cup \{0, 1\}$ follows directly from Theorem 2.13, Theorem 2.18, and the definition of the extraction degree.

Now, let $\delta \in \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \alpha \in \{1, 2, \ldots, n-2, n\}\right\} \cup \{0, 1\}$. Then we can represent δ as $\frac{\beta}{\alpha}$ where $\frac{\beta}{\alpha} \leq \frac{n-2}{n}$ and $\alpha \in \{1, 2, \ldots, n-2, n\}$. We want to show that there exist $X \in MZS(G)$ and $Y \in ZS(G)$ such that $\lambda(X, Y) = \frac{\beta}{\alpha}$. Now, $\lambda(1^{n-2}2, 1^n) = 0$ and $\lambda(1^n, 1^n) = 1$, so let $0 < \frac{\beta}{\alpha} \leq \frac{n-2}{n}$. Then we have four cases: $\alpha = n$, $\alpha = \frac{n}{2}$, $\alpha < \frac{n}{2}$, and $\frac{n}{2} < \alpha \leq n-2$.

First, let $\alpha = n$. Then let $X = 1^\alpha$ and let $Y = 1^\beta(n-1)^\beta$. Then, from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha}$.

Next, let $\alpha = \frac{n}{2}$. Then let $X = 2^\alpha$ and let $Y = 2^\beta 1^{n-2\beta}$. Then, from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha}$.

Now, let $\alpha < \frac{n}{2}$. Since $\frac{\beta}{\alpha} \leq \frac{n-2}{n}$ we know that $\frac{\beta}{\alpha} < 1$, so $\beta < \alpha \Rightarrow 2\beta < 2\alpha < n \Rightarrow n - 2\beta > n - 2\alpha$ so $\frac{n-2\beta}{n-2\alpha} > 1$. Let $X = 2^\alpha 1^{n-2\alpha}$ and let $Y = 2^\beta 1^{n-2\beta}$. Then $\frac{\beta}{\alpha} < 1 < \frac{n-2\beta}{n-2\alpha}$. then from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha}$.

Finally, let $\frac{n}{2} < \alpha \leq n-2$. Then, from Lemma 2.16, $\beta \neq \alpha - 1$, so $\beta < \alpha - 1$, so $\alpha - \beta > 1$. Also, since $\alpha \leq n-2$ we know that $n - \alpha > 1$. Let $X = 1^{\alpha}(n-\alpha)$ and let $Y = 1^{\beta}(n-\alpha)(\alpha-\beta)$. From Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$. ■

3.2 X Minimal

In this section we only place the restriction that X has to be a minimal zero sequence. Again, there is a different result for the cyclic case that is addressed in the second theorem, as we would expect from Theorem 2.13.

Theorem 3.8. *Let $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ where $n_j | n_{j+1}$ for all $1 \leq j < s$ and $s \geq 2$. Then*

$$\lambda(MZS, ZS) = \left\{ \frac{\beta}{\alpha} | 1 \leq \alpha \leq n_s,\ \beta \in \mathbb{Z}^{\geq 0} \right\}.$$

Proof. $\lambda(MZS, ZS) \subseteq \left\{ \frac{\beta}{\alpha} | 1 \leq \alpha \leq n_s,\ \beta \in \mathbb{Z}^{\geq 0} \right\}$ follows directly from Theorem 2.13 and the definition of the extraction degree.

Now, let $\delta \in \left\{ \frac{\beta}{\alpha} | 1 \leq \alpha \leq n_s,\ \beta \in \mathbb{Z}^{\geq 0} \right\}$. Then δ is of the form $\frac{\beta}{\alpha}$ where $1 \leq \alpha \leq n_s$ and $\beta \in \mathbb{Z}^{\geq 0}$. If $\alpha = n_s$ then we can let $X = (\mathbf{0}, 0, 1)^{n_s}$ and $Y = (\mathbf{0}, 0, 1)^{\beta}(\mathbf{0}, n_{s-1} - 1, n_s - 1)^{\beta}(\mathbf{0}, 1, 0)^{\beta}$. Then, from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Let $\alpha < n_s$. Then, let $X = (\mathbf{0}, 0, 1)^{\alpha}(\mathbf{0}, 1, n_s - \alpha)(\mathbf{0}, n_{s-1}, 0)$ and let $Y = (\mathbf{0}, 0, 1)^{\beta}(\mathbf{0}, 1, n_s - \alpha)^{1+m_1 \cdot n_s}(\mathbf{0}, n_{s-1}, 0)^{2+m_2 \cdot n_s}(\mathbf{0}, 1, d)$ where $d = \alpha - \beta \pmod{n}$, and $m_1, m_2 \in \mathbb{Z}^{\geq 0}$ such that $1 + m_1 n_s > \frac{\beta}{\alpha}$ and $2 + m_2 n_s > \frac{\beta}{\alpha}$. Then, from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$. So, $\left\{ \frac{\beta}{\alpha} | \alpha \in \{1, 2, \ldots, n_s\},\ \beta \in \mathbb{Z}^{\geq 0} \right\} \subseteq \lambda(MZS, ZS)$. ■

Theorem 3.9. *Let $G = \mathbb{Z}_n$ where $n \geq 3$. Then,*

$$\lambda(MZS, ZS) = \left\{ \frac{\beta}{\alpha} |\ \alpha \in \{1, 2, \ldots, n-2, n\},\ \beta \in \mathbb{Z}^{\geq 0} \right\}.$$

Proof. $\lambda(MZS, ZS) \subseteq \left\{ \frac{\beta}{\alpha} |\ \alpha \in \{1, 2, \ldots, n-2, n\},\ \beta \in \mathbb{Z}^{\geq 0} \right\}$ follows directly from Theorem 2.13, and the definition of extraction degree.

Let $\frac{\beta}{\alpha} \in \left\{ \frac{\beta}{\alpha} |\ \alpha \in \{1, 2, \ldots, n-2, n\},\ \beta \in \mathbb{Z}^{\geq 0} \right\}$. Now, $\alpha = n$ or $\alpha \leq n-2$. Let $\alpha = n$. Then let $X = 1^n$ and let $Y = 1^{\beta}(n-1)^{\beta}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$. Now, let $\alpha \leq n-2$. Let $X = 1^{\alpha}(n-\alpha)$ and let $Y = 1^{\beta}(n-\alpha)^{\beta_2}(n-1)^{\beta_3}$ where $\beta_2 = 1 + mn$ with $m \in \mathbb{Z}^+$ such that $\beta_2 \geq \frac{\beta}{\alpha}$, and $\beta_3 \in \mathbb{Z}^+$ such that $\beta_3(n-1) \equiv \alpha - \beta \pmod{n}$. Then, from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$. ■

3.3 Y Minimal

Now, we place the restriction that Y must be a minimal zero sequence. Referring to Theorem 2.6, $\lfloor X \rfloor \nsubseteq \lfloor Y \rfloor$ yields $\lambda(X,Y)=0$. In order to obtain non trivial results when restricting Y, this imposes restrictions on X. This makes it difficult to obtain a complete answer. Here, we work through the cyclic case first. It is important to note that in the following theorem we use $\bar{\alpha}$ to denote $\alpha \pmod{n}$.

Theorem 3.10. *Let $G=\mathbb{Z}_n$ where $n \geq 4$. Then*

$$\left\{\frac{\beta}{\alpha} | \beta < \frac{n}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS, MZS)\right\} = \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\} \cup \{1\}.$$

Proof. $\left\{\frac{\beta}{\alpha} | \beta < \frac{n}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS, MZS)\right\} \subseteq \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\}$ follows directly from the definition of the extraction degree and Theorem 2.18.

We want to show that $\left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\} \cup \{1\} \subseteq \left\{\frac{\beta}{\alpha} | \beta < \frac{n}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS, MZS)\right\}$. First, let $\beta < \frac{n}{2}$, $X = 2^{\beta}1^{n-2\beta}$ and $Y = 2^{\beta}1^{n-2\beta}$ Then $\lambda(X,Y)=1$. Now, let $\frac{\beta}{\alpha} \in \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\}$. Then $\frac{\beta}{\alpha} < 1$, $\beta < \frac{n}{2}$, and $\alpha \in \mathbb{Z}^+$. Since $\frac{\beta}{\alpha} < 1$, $\beta < \alpha$. Then all the possible cases for α are: $\beta = \bar{\alpha}$, $\bar{\alpha} = 0$, $\beta < \bar{\alpha} < \frac{n}{2}$, $\bar{\alpha} = \frac{n}{2}$, $\bar{\alpha} = n-1$, $\frac{n}{2} < \bar{\alpha} \leq n-2$, or $\bar{\alpha} < \beta$.

First, let $\beta = \bar{\alpha}$. Since $\beta < \frac{n}{2}$, $2\beta < n$. So, let $X = 2^{\alpha}1^{n-2\beta}$ and let $Y = 2^{\beta}1^{n-2\beta}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\bar{\alpha} = 0$ or $\bar{\alpha} = \frac{n}{2}$. Since $\beta < \frac{n}{2}$, $2\beta < n$. So, let $X = 2^{\alpha}$ and let $Y = 2^{\beta}1^{n-2\beta}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\beta < \bar{\alpha} < \frac{n}{2}$. Then $2\beta < 2\bar{\alpha} < n$ so $n - 2\beta > n - 2\bar{\alpha} > 0$. So, let $X = 2^{\alpha}1^{n-2\beta}$ and let $Y = 2^{\beta}1^{n-2\bar{\alpha}}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\bar{\alpha} = n-1$. Since $\beta < \frac{n}{2}$ we know that $\beta \leq \frac{n-1}{2}$ and $n - 2\beta \geq 1$. So, let $X = 2^{\alpha}1^{2}$ and let $Y = 2^{\beta}1^{n-2\beta}$. Since $n-2\beta \geq 1$, $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$, and since $\beta \leq \frac{n-1}{2}$, we have that $\frac{\beta}{\alpha} \leq \frac{\frac{n-1}{2}}{n-1}$. From Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\frac{n}{2} < \bar{\alpha} \leq n-2$. Here, we have two subcases: $\beta = \bar{\alpha} - 1$ and $\beta \leq \bar{\alpha} - 2$. First, let $\beta = \bar{\alpha} - 1$. Since $\bar{\alpha} > \frac{n}{2}$ and $\beta = \bar{\alpha} - 1$ we know from Lemma 2.16 that $\bar{\alpha} \neq \alpha$ because if it did $\frac{\beta}{\alpha} > \frac{n-2}{n}$, so $\alpha > n$. Now, we know that $\bar{\alpha} - 1 = \beta < \frac{n}{2} < \bar{\alpha}$ so $\beta = \frac{n-1}{2}$ and $\bar{\alpha} = \frac{n+1}{2}$. So, let $X = 2^{\alpha}1^{n-1}$ and let $Y = 2^{\beta}1$. We need to show that $\frac{\beta}{\alpha} \leq \frac{1}{n-1}$. Since $\alpha > n$ we know that $\alpha \geq n + \frac{n+1}{2} = \frac{3n+1}{2}$. So, $\frac{\beta}{\alpha} \leq \frac{\frac{n-1}{2}}{\frac{3n+1}{2}} = \frac{n-1}{3n+1} < \frac{n}{3n} = \frac{1}{3} \leq \frac{1}{n-1}$ since $n \geq 4$. Then by Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Second, let $\beta \leq \bar{\alpha} - 2$. Then $\bar{\alpha} - \beta \geq 2$ so we can write $\bar{\alpha} - \beta = 2^{\beta_2}3^{\beta_3}$ where

$\beta_2, \beta_3 \in \mathbb{Z}^{\geq 0}$. Since $\bar{\alpha} \leq n-2$ we have $n-\bar{\alpha} \geq 2$ so we can write $n-\bar{\alpha} = 2^{\alpha_2}3^{\alpha_3}$ where $\alpha_2, \alpha_3 \in \mathbb{Z}^{\geq 0}$. Then let $X = 1^{\alpha}2^{\alpha_2}3^{\alpha_3}$ and let $Y = 1^{\beta}2^{\alpha_2+\beta_2}3^{\alpha_3+\beta_3}$. Then we have that $\beta + 2(\alpha_2+\beta_2) + 3(\alpha_3+\beta_3) = \beta + n - \bar{\alpha} + \bar{\alpha} - \beta = n$ so $Y \in MZS(G)$. Also, if $\alpha_2 > 0$ then $\frac{\alpha_2+\beta_2}{\alpha_2} \geq 1$ and if $\alpha_3 > 0$ then $\frac{\alpha_3+\beta_3}{\alpha_3} \geq 1$. Since $\frac{\beta}{\alpha} < 1$, by Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$

Finally, let $0 < \bar{\alpha} < \beta$. Then $\alpha > n$. These next few paragraphs construct $X \in ZS(G)$, $Y \in MZS(G)$ such that $\lambda(X,Y) = \frac{\beta}{\alpha}$ for all $n \geq 12$.

Assume $n-\beta$ and $n-\alpha$ are odd. Then let $X = 1^{\alpha}2^{a}3$ where $2^{a}3 = n-\alpha$ and let $Y = 1^{\beta}2^{b}3$ where $2^{b}3 = n-\beta$. Then $X \in ZS(G)$ and $Y \in MZS(G)$. Then $\frac{b}{a} = \frac{\frac{n-\beta-3}{2}}{\frac{n-\alpha-3}{2}} = \frac{n-\beta-3}{n-\alpha-3}$ If n is odd $n-\beta \geq \frac{n+1}{2}$ and $n-\alpha \leq n-2$. So, $\frac{n-\beta-3}{n-\alpha-3} \geq \frac{\frac{n+1}{2}-3}{n-2-3} = \frac{n-5}{2(n-5)} = \frac{1}{2}$. If n is even $n-\beta \geq \frac{n+2}{2}$ and $n-\alpha \leq n-1$. So, $\frac{n-\beta-3}{n-\alpha-3} \geq \frac{\frac{n+1}{2}-3}{n-1-3} = \frac{n-4}{2(n-4)} = \frac{1}{2}$.

Assume $n-\beta$ is odd and $n-\alpha$ is even. Then let $X = 1^{\alpha}2^{a}3^{2}$ where $2^{a}3^{2} = n-\alpha$ and let $Y = 1^{\beta}2^{b}3$ where $2^{b}3 = n-\beta$. Then $X \in ZS(G)$ and $Y \in MZS(G)$. Then $\frac{b}{a} = \frac{\frac{n-\beta-3}{2}}{\frac{n-\alpha-6}{2}} = \frac{n-\beta-3}{n-\alpha-6}$ If n is odd $n-\beta \geq \frac{n+1}{2}$ and $n-\alpha \leq n-1$. So, $\frac{n-\beta-3}{n-\alpha-6} \geq \frac{\frac{n+1}{2}-3}{n-1-6} = \frac{n-5}{2(n-7)} \geq \frac{1}{2}$. If n is even $n-\beta \geq \frac{n+2}{2}$ and $n-\alpha \leq n-2 < n-1$. So, $\frac{n-\beta-3}{n-\alpha-6} \geq \frac{\frac{n+1}{2}-3}{n-1-3} = \frac{n-4}{2(n-7)} \geq \frac{1}{2}$.

Assume $n-\beta$ and $n-\alpha$ are even. Then let $X = 1^{\alpha}2^{a}$ where $2^{a} = n-\alpha$ and let $Y = 1^{\beta}2^{b}$ where $2^{b} = n-\beta$. Then $X \in ZS(G)$ and $Y \in MZS(G)$. Then $\frac{b}{a} = \frac{\frac{n-\beta}{2}}{\frac{n-\alpha}{2}} = \frac{n-\beta}{n-\alpha}$ Then whether n is odd or even $n-\beta \geq \frac{n+1}{2}$ and $n-\alpha < n-1$. So, $\frac{n-\beta}{n-\alpha} > \frac{\frac{n+1}{2}}{n} > \frac{1}{2}$.

Assume $n-\beta$ is even and $n-\alpha$ is odd. Then let $X = 1^{\alpha}2^{a}3^{3}$ where $2^{a}3^{3} = n-\alpha$ and let $Y = 1^{\beta}2^{b}3^{2}$ where $2^{b}3^{2} = n-\beta$. Then $X \in ZS(G)$ and $Y \in MZS(G)$. Then $\frac{b}{a} = \frac{\frac{n-\beta-6}{2}}{\frac{n-\alpha-9}{2}} = \frac{n-\beta-6}{n-\alpha-9}$ If n is odd $n-\beta \geq \frac{n+1}{2}$ and $n-\alpha \leq n-2$. So, $\frac{n-\beta-6}{n-\alpha-9} \geq \frac{\frac{n+1}{2}-6}{n-2-9} = \frac{n-11}{2(n-11)} = \frac{1}{2}$. If n is even $n-\beta \geq \frac{n+2}{2}$ and $n-\alpha \leq n-1$. So, $\frac{n-\beta-6}{n-\alpha-9} \geq \frac{\frac{n+2}{2}-6}{n-1-9} = \frac{n-10}{2(n-10)} = \frac{1}{2}$.

So $\frac{b}{a} \geq \frac{1}{2}$ and $\frac{\beta}{\alpha} \leq \frac{1}{2}$. So $\frac{\beta}{\alpha} \leq \frac{b}{a}$ in all cases so $\frac{\beta}{\alpha} = \lambda(X,Y)$.

Since this works for $n \geq 12$ we need to address $4 \leq n \leq 11$. Because $n \leq 11$ and $\beta < \frac{n}{2}$ we know that $\beta \leq 5$. Then, since $\bar{\alpha} < \beta$, $\beta - \bar{\alpha} \leq 4$.

First, let $\beta - \bar{\alpha} = 1$. Since $\beta < \frac{n}{2}$ we have $2\beta \leq n-1 \Rightarrow 3\beta \leq n+\beta-1 \Rightarrow \frac{\beta}{n+\beta-1} \leq \frac{1}{3}$. Since $2\beta \leq n-1 \Rightarrow 2\beta+1 \leq n \Rightarrow 4\beta+2 \leq 2n \Rightarrow 4\beta+2+n \leq 3n \Rightarrow n-2\beta+2 \leq 3n-6\beta \Rightarrow \frac{1}{3} \leq \frac{n-2\beta}{n-2\beta+2}$. Let $Y = 2^{\beta}1^{n-2\beta}$ and $X = 2^{\alpha}1^{n-2\bar{\alpha}}$. Since $\bar{\alpha} = \beta - 1$, we know $\alpha > n+\beta-1$, and $n-2\bar{\alpha} = n-2\beta+2$. So we have that $\frac{\beta}{\alpha} \leq \frac{\beta}{n+\beta-1} \leq \frac{1}{3} \leq \frac{n-2\beta}{n-2\beta+2} = \frac{n-2\beta}{n-2\bar{\alpha}}$. Then from Theorem 2.1,

$\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\beta - \bar{\alpha} = 2$. Then $\beta \geq 3$. Since $\beta < \frac{n}{2}$, we know $n - \beta \geq 4$ so we can write $n-\beta = 2^{\beta_2}3^{\beta_3}$ where $\beta_2 \geq 1$ and $\beta_3 \geq 0$. We also have that $n-\bar{\alpha} = n-\beta+2$ so we can write $n - \bar{\alpha} = 2^{\beta_2+1}3^{\beta_3}$. Let $X = 1^{\alpha}2^{\beta_2+1}3^{\beta_3}$ and $Y = 1^{\beta}2^{\beta_2}3^{\beta_3}$. Since $\alpha > n$, $\alpha \geq n + \beta - 2$. So, $\frac{\beta}{\alpha} \leq \frac{\beta}{n+\beta-2} \leq \frac{\beta}{3\beta-2} \leq \frac{\beta}{2\beta} = \frac{1}{2} \leq \frac{\beta_2}{\beta_2+1}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, let $\beta - \bar{\alpha} = 3$. Then $\beta \geq 4$, and $n - \bar{\alpha} = n - \beta + 3$. Then $\alpha \geq n + \beta - 3 \geq 3\beta - 3 \geq 3\beta - bt = 2\beta$. Since $\beta \geq 4$ and $\beta < \frac{n}{2}$, we know $n - \beta \geq 5$. Then we can write $n - \beta = 2^{\beta_2}3^{\beta_3}$ where $\beta_3 \geq 1$ and $\beta_2 > 0$. We can also write $n - \bar{\alpha} = 2^{\beta_2}3^{\beta_3+1}$. Let $X = 1^{\alpha}2^{\beta_2}3^{\beta_3+1}$ and $Y = 1^{\beta}2^{\beta_2}3^{\beta_3}$. Then $\frac{\beta}{\alpha} \leq \frac{\beta}{n+\beta-3} \leq \frac{\beta}{2\beta} = \frac{1}{2} \leq \frac{\beta_3}{\beta_3+1}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Finally, let $\beta - \bar{\alpha} = 4$. Then $\beta \geq 5$ and $n - \bar{\alpha} = n - \beta + 4$. So, $\alpha \geq n + \beta - 4 \geq 3\beta - 4 \geq 3\beta - \beta = 2\beta$. Since $\beta \geq 5$ and $\beta > \frac{n}{2}$ we know that $n - \beta \geq 6$. Since $n \leq 11$ we know that $n - \beta \leq 6$ so we have that $6 \leq n - \beta \leq 6 \Rightarrow n - \beta = 6$. Then $n - \bar{\alpha} = 10$. Let $X = 1^{\alpha}2^{5}$ and $Y = 1^{\beta}2^{3}$. Then $\frac{\beta}{\alpha} \leq \frac{\beta}{2\beta} = \frac{1}{2} < \frac{3}{5}$. Then from Theorem 2.1, $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, that we have covered all cases, we have shown that there exists $X \in ZS(G)$ and $Y \in MZS(G)$ such that $\lambda(X,Y) = \frac{\beta}{\alpha}$ for all $\frac{\beta}{\alpha} \in \left\{\frac{\beta}{\alpha} \leq \frac{n-2}{n} | \beta < \frac{n}{2},\, \alpha \in \mathbb{Z}^+\right\}$. ■

Furthermore, we were able to construct $X \in ZS(G)$ where $\lfloor X \rfloor \subseteq \{1,2,3\}$ and $Y \in MZS(G)$ where $\lfloor Y \rfloor \subseteq \{1,2,3\}$. This leads directly to the next theorem.

Theorem 3.11. *Let $X \in ZS(G)$ and $Y \in MZS(G)$ such that $\lambda(X,Y) = \frac{\beta}{\alpha}$ where $\beta \geq \frac{n}{2}$ then there exists $X' \in ZS(G)$ and $Y' \in MZS(G)$ where $\lfloor X' \rfloor \subseteq \{1,2,3\}$ and $\lfloor Y' \rfloor \subseteq \{1,2,3\}$ such that $\lambda(X',Y') = \frac{\beta}{\alpha}$.*

Proof. Let $X \in ZS(G)$, $Y \in MZS(G)$ such that $\lambda(X,Y) = \frac{\beta}{\alpha}$ where $\beta \geq \frac{n}{2}$. Since $\beta \geq \frac{n}{2}$ we can let $X = 1^{\alpha}g_1^{\alpha_1}\cdots g_k^{\alpha_k}$ and $Y = 1^{\beta}g_1^{\beta_1}\cdots g_j^{\beta_j}$ where $j \geq k$. Since $\frac{\beta}{\alpha} = \lambda(X,Y)$, $\frac{\beta}{\alpha} \leq \frac{\beta_i}{\alpha_i}$ $\forall i \in \{2,3,\ldots,k\}$. Now, for all $i \in \{2,3,\ldots,k\}$ we know that each $g_i = 2^{a_i}3^{b_i}$ so $g_i^{\alpha_i} = (2^{a_i}3^{b_i})^{\alpha_i} = 2^{\alpha_i a_i}3^{\alpha_i b_i}$. Now, let $X' = 1^{\alpha}(2^{a_1}3^{b_1})^{\alpha_1}\cdots(2^{a_k}3^{b_k})^{\alpha_k}$ and $Y' = 1^{\beta}(2^{a_1}3^{b_1})^{\beta_1}\cdots(2^{a_j}3^{b_j})^{\beta_j}$. . So, $\alpha' = (2)_{X'} = \sum_{i=1}^{k}\alpha_i a_i$, $\alpha'' = (3)_{X'} = \sum_{i=1}^{k}\alpha_i b_i$, $\beta' = (2)_{Y'} \geq \sum_{i=1}^{k}\beta_i a_i$, and $\beta'' = (3)_{Y'} \geq \sum_{i=1}^{k}\beta_i b_i$. Then $\frac{\beta'}{\alpha'} = \frac{\sum_{i=1}^{k}\beta_i a_i}{\sum_{i=1}^{k}\alpha_i a_i} \geq \frac{\sum_{i=1}^{k}\frac{\beta}{\alpha}\alpha_i a_i}{\sum_{i=1}^{k}\alpha_i a_i} = \frac{\beta}{\alpha}$ and $\frac{\beta''}{\alpha''} = \frac{\sum_{i=1}^{k}\beta_i b_i}{\sum_{i=1}^{k}\alpha_i b_i} \geq$

$\frac{\sum_{i=1}^{k} \frac{\beta}{\alpha}\alpha_i b_i}{\sum_{i=1}^{k} \alpha_i b_i} = \frac{\beta}{\alpha}$. Therefore, $\frac{\beta}{\alpha} = \lambda(X', Y')$. ■

Since when $\beta \geq \frac{n}{2}$, we can form every possible extraction degree only using $\{1, 2, 3\}$ for the floor of X and Y, we now define a function that compares every possible combination of 2's and 3's for $n - \beta$ and $n - \alpha$ when given β and α. This function allows us to answer the question of which extraction degrees are possible when $\beta \geq \frac{n}{2}$.

Definition 3.12. Given $\frac{\beta}{\alpha}$ where $\beta \geq \frac{n}{2}$ and $\alpha > n$ then $n-\beta = 2^{\beta'}3^{\beta''}(2^{-3}3^2)^a$ and $n - \alpha = 2^{\alpha'}3^{\alpha''}(2^{-3}3^2)^b$ where $\beta''\ \alpha'' \in \{0, 1\}$, $0 \leq a \leq \frac{\beta'}{3}$, and $0 \leq b \leq \frac{\alpha'}{3}$. Then $\gamma_{a,b}$ is defined to be $\min\left\{\frac{\beta'-3a}{\alpha'-3b}, \frac{\beta''+2a}{\alpha''+2b}\right\}$, and $\rho(\beta, \alpha)$ is defined to be $\max\left\{\gamma_{a,b} \middle|\ 0 \leq a \leq \frac{\beta'}{3},\ 0 \leq b \leq \frac{\alpha'}{3}\right\}$.

Theorem 3.13. *Let $G = \mathbb{Z}_n$. Let $\beta \geq \frac{n}{2}$, and $\alpha > n$. Then there exist $X \in ZS(G)$, $Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{\beta}{\alpha}$ if and only if $\frac{\beta}{\alpha} \leq \rho(\beta, \alpha)$.*

Proof. Assume there exists $X \in ZS(G)$, $Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{\beta}{\alpha}$. Then from Theorem 3.11, then there exists $X' \in ZS(G)$ and $Y' \in MZS(G)$ where $\lfloor X' \rfloor \subseteq \{1, 2, 3\}$ and $\lfloor Y' \rfloor \subseteq \{1, 2, 3\}$ such that $\lambda(X', Y') = \frac{\beta}{\alpha}$. Then we can write $X' = 1^{\alpha}2^{\alpha_2}3^{\alpha_3}$ and $Y' = 1^{\beta}2^{\beta_2}3^{\beta_3}$. Since $\frac{\beta}{\alpha} = \lambda(X', Y')$, from Theorem 2.1, $\frac{\beta}{\alpha} \leq \frac{\beta_2}{\alpha_2}$ and $\frac{\beta}{\alpha} \leq \frac{\beta_3}{\alpha_3}$. Then $\frac{\beta}{\alpha} \leq \min\left\{\frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right\}$. Note that we can find $\alpha' \in \mathbb{Z}^{\geq 0}$, $\alpha'' \in \{0, 1\}$, $b \geq 0$ such that $2^{\alpha_2}3^{\alpha_3} = 2^{\alpha'}3^{\alpha''}(2^{-3}3^2)^b$. We can also find $\beta' \in \mathbb{Z}^{\geq 0}$, $\beta'' \in \{0, 1\}$, $a \geq 0$ such that $2^{\beta_2}3^{\beta_3} = 2^{\beta'}3^{\beta''}(2^{-3}3^2)^a$. Then $\frac{\beta}{\alpha} \leq \min\left\{\frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right\} = \min\left\{\frac{\beta'-3a}{\alpha'-3b}, \frac{\beta''+2a}{\alpha''+2b}\right\} = \gamma_{a,b} \leq \rho(\beta, \alpha)$. This last inequality follows directly from the definition of $\rho(\beta, \alpha)$.

Now, let $\frac{\beta}{\alpha} \leq \rho(\beta, \alpha)$. We can write $n - \bar{\alpha} = 2^{\alpha'}3^{\alpha''}$ and $n - \beta = 2^{\beta'}3^{\beta''}$ where $\alpha'', \beta'' \in \{0, 1\}$ and $\alpha', \beta' \in \mathbb{Z}^{\geq 0}$. Then from the definition of $\rho(\beta, \alpha)$, there exist a where $0 \leq a \leq \frac{\beta'}{3}$ and b where $0 \leq b \leq \frac{\alpha'}{3}$ such that $\rho(\beta, \alpha) = \gamma_{a,b} = \min\left\{\frac{\beta'-3a}{\alpha'-3b}, \frac{\beta''+2a}{\alpha''+2b}\right\}$. So, $\frac{\beta}{\alpha} \leq \min\left\{\frac{\beta'-3a}{\alpha'-3b}, \frac{\beta''+2a}{\alpha''+2b}\right\}$. Let $X = 1^{\alpha}2^{\alpha'-3b}3^{\alpha''+2b}$ and $Y = 1^{\beta}2^{\beta'-3a}3^{\beta''+2a}$. Because $\min\left\{\frac{\beta'-3a}{\alpha'-3b}, \frac{\beta''+2a}{\alpha''+2b}\right\} = \min\left\{\frac{(2)_Y}{(2)_X}, \frac{(3)_Y}{(3)_X}\right\}$, and Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha}$. ■

Although this function completely answers the question of which extraction degrees are possible for cyclic groups when Y is a minimal zero sequence, it is not nice. We were not able to determine a simple formula for calculating $\rho(\beta, \alpha)$

or even its minimum. Now, we look to higher ranks, and use some of our results from the cyclic case.

Theorem 3.14. *Let $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ where $n_j | n_{j+1}$ for all $1 \leq j < s$ and $s \geq 2$, $n_s \geq 4$. Then*

$$\left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS, MZS)\right\} = \left\{\frac{\beta}{\alpha} \leq 1 | \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+\right\}.$$

Proof. $\left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS, MZS)\right\} \subseteq \left\{\frac{\beta}{\alpha} \leq 1 | \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+\right\}$ follows directly from the definition of the extraction degree and Theorem 2.15.

Now, $\left\{\frac{\beta}{\alpha} \leq 1 | \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+\right\} \setminus \left(\left\{\frac{\beta}{\alpha} \leq \frac{n_s-2}{n_s} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\} \cup \{1\}\right) = \left\{\frac{\beta}{\alpha} | \frac{n_s-2}{n_s} < \frac{\beta}{\alpha} < 1,\ \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+\right\}$. From Lemma 2.16, since $\frac{\beta}{\alpha} > \frac{n_s-2}{n_s}$ we know that $n_s > \alpha > \frac{n_s}{2}$ and $\beta = \alpha - 1$. Then we have that $\alpha - 1 = \beta < \frac{n_s}{2} < \alpha$. This can only happen if n_s is odd, $\alpha = \frac{n_s+1}{2}$ and $\beta = \frac{n_s-1}{2}$.

Now, we will show that $\frac{\beta}{\alpha} \in \lambda(ZS(G), MZS(G))$. Since $n_s \geq 4$ and n_s is odd, we know that $n_s \geq 5$. Then $\frac{n_s-1}{2} \geq 2$ so we can write $n_s - \alpha = 2^{\alpha_2}3^{\alpha_3}$ where $\alpha_2, \alpha_3 \in \mathbb{Z}^{\geq 0}$. So, let $X = (0,\ldots,0,1)^{\alpha}(0,\ldots,0,2)^{\alpha_2}(0,\ldots,0,3)^{\alpha_3}$. If $n_{s-1} = 2$, let $Y = (0,\ldots,0,1)^{\beta}(0,\ldots,0,2)^{\alpha_2}(0,\ldots,0,3)^{\alpha_3}(0,\ldots,0,1,1)$ $(0,\ldots,0,1,0)$. If $n_{s-1} \geq 3$, then $n_{s-1} - 1 = 2^{\beta_2}3^{\beta_3}$ where $\beta_2, \beta_3 \in \mathbb{Z}^{\geq 0}$, so let $Y = (0,\ldots,0,1)^{\alpha}(0,\ldots,0,2)^{\alpha_2}(0,\ldots,0,3)^{\alpha_3}(0,\ldots,0,1,1)(0,\ldots,0,2,0)^{\beta_2}$ $(0,\ldots,0,3,0)^{\beta_3}$.
Then either way, $Y \in MZS(G)$ and $\lambda(X,Y) = \frac{\beta}{\alpha}$.

Now, we will show that
$\left(\left\{\frac{\beta}{\alpha} \leq \frac{n_s-2}{n_s} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\} \cup \{1\}\right) \subseteq \left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(G), MZS(G))\right\}$.
If we write $G = H_1 \oplus H_2$ where $H_1 = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_{s-1}}$ and $H_2 = \mathbb{Z}_{n_s}$, then from Theorem 2.5,

$$\left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(H_2), MZS(H_2))\right\}$$
$$\subseteq \left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(G), MZS(G))\right\}.$$

Now, from Theorem 3.10,

$$\left(\left\{\frac{\beta}{\alpha} \leq \frac{n_s - 2}{n_s} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+\right\} \cup \{1\}\right)$$
$$\subseteq \left\{\frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(H_2), MZS(H_2))\right\}$$

$$\subseteq \left\{ \frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(G), MZS(G)) \right\}.$$

Thus,

$$\left\{ \frac{\beta}{\alpha} | \frac{n_s - 2}{n_s} < \frac{\beta}{\alpha} < 1,\ \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+ \right\}$$

$$= \left\{ \frac{\beta}{\alpha} \leq 1 | \beta < \frac{n_s}{2},\ \alpha \in \mathbb{Z}^+ \right\} \setminus \left(\left\{ \frac{\beta}{\alpha} \leq \frac{n_s - 2}{n_s} | \beta < \frac{n}{2},\ \alpha \in \mathbb{Z}^+ \right\} \cup \{1\} \right)$$

$$\subseteq \left\{ \frac{\beta}{\alpha} | \beta < \frac{n_s}{2},\ \frac{\beta}{\alpha} \in \lambda(ZS(G), MZS(G)) \right\}, \text{as needed.}$$

■

Extending Theorem 3.10 to all finite abelian groups is straightforward. However, finding a function similar to $\rho(\beta, \alpha)$ for all finite groups is a possible area of further study. This could include simplifying the problem as Theorem 3.11 does for cyclic groups.

4 Fixing A Zero Sequence

In this section we fix either X or Y to be a specific zero sequence and allow the other to vary within a subset of $ZS(G)$. We then analyze the set of possible extraction degrees.

4.1 Fixing X

To begin with, allowing Y to be any zero sequence, and fixing X as a zero sequence produces the following theorem.

Theorem 4.1. *If $X \in ZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ then*

$$\lambda(X, ZS) = \left\{ \frac{\beta}{\alpha_i} |\ 1 \leq i \leq k,\ \beta \in \mathbb{Z}^{\geq 0} \right\}.$$

Proof. $\lambda(X, ZS) \subseteq \left\{ \frac{\beta}{\alpha_i} |\ 1 \leq i \leq k,\ \beta \in \mathbb{Z}^{\geq 0} \right\}$ follows directly from Theorem 2.1.

Now, we will show that $\left\{ \frac{\beta}{\alpha_i} |\ 1 \leq i \leq k,\ \beta \in \mathbb{Z}^{\geq 0} \right\} \subseteq \lambda(X, ZS)$. We need to construct a $Y \in ZS(G)$ such that some element $g_i \in X$, $\frac{(g_i)_Y}{(g_i)_X} \leq \frac{(g_j)_Y}{(g_j)_X}$ for all j such that $1 \leq j \leq k$. This construction is different for cyclic and non-cyclic finite abelian groups.

If $g_j \in G$, then we write $g_j = (g_j^{(1)}, g_j^{(2)}, \ldots, g_j^{(s)})$. Now, let $Y = g_1^{\beta_1} \cdots g_i^{\beta} \cdots g_k^{\beta_k} h$, where for all j such that $1 \leq j \leq k$, $\beta_j = \alpha_j + d_j \cdot n_s$, and d_j is chosen such that $\frac{\beta_j}{\alpha_j} \geq \frac{\beta}{\alpha_i}$. For $h = (h^{(1)}, h^{(2)}, \ldots, h^{(s)})$ we define each $h^{(m)}$ as $h^{(m)} = \sum_{j=1}^{i-1} \beta_j \cdot g_j^{(m)} + \sum_{j=i+1}^{k} \beta_j \cdot g_j^{(m)} - \beta \cdot g_i^{(m)} \pmod{n_m}$ for all $1 \leq m \leq s$. Thus, $Y \in ZS(G)$. Now, if $h \neq g_i$, then from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha_i}$. So, let $h = g_i$, then either g_i has at least two non-zero coordinates or g_i has exactly one non-zero coordinate.

First, suppose that g_i contain at least two non-zero coordinates. Then we can write $g_i = (0, \ldots, 0, g_i^{(u)}, 0, \ldots, 0, g_i^{(v)}, 0, \ldots, 0)$ such that $u \neq v$ and $u, v \in \{1, 2, \ldots, s\}$. Then, since $Y \in ZS(G)$, we can replace $h \in Y$ with h_1 and h_2 where $h_1 = (0, \ldots, 0, g_i^{(u)}, 0, \ldots, 0)$ and $h_2 = (0, \ldots, 0, g_i^{(v)}, 0, \ldots, 0)$. Then, from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha_i}$.

Second, suppose that g_i has exactly one non-zero coordinate. Here we have two subcases: when G is not cyclic and when G is cyclic. To start, we will assume that G is not cyclic. Then we can write $g_i = (0, \ldots, 0, g_i^{(u)}, 0, \ldots, 0)$ such that $1 \leq u \leq s$. Then, since $Y \in ZS(G)$, we can replace $h \in Y$ with h_1 and h_2 where $h_1 = (0, \ldots, 0, g_i^{(u)}, 0, \ldots, 0, 1^{(w)}, 0, \ldots, 0)$ and $h_2 = (0, \ldots, 0, (n-1)^{(w)}, 0, \ldots, 0)$ such that $u \neq w$ and $1 \leq w \leq s$. Then from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha_i}$. Now, we will assume that G is cyclic. Then we can write $G = \mathbb{Z}_n$. Since $Y \in ZS(G)$ we can replace h with f^r where $f \neq g_i$, $|f| = n$ and $f \cdot r \equiv h$. Then from Theorem 2.1, $\lambda(X, Y) = \frac{\beta}{\alpha_i}$. ∎

Corollary 4.2. *If $X \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$, then*

$$\lambda(X, MZS) = \left\{ \frac{\beta}{\alpha_i} |\ 1 \leq i \leq k,\ \beta \in \mathbb{Z}^{\geq 0} \right\}.$$

Proof. Theorem 4.1 applies here, because $X \in MZS(G) \subseteq ZS(G)$. ∎

Theorem 4.3. *Let $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ such that $n_j | n_{j+1}$ for all $1 \leq j < s$. If $X \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ then*

$$\lambda(X, MZS) \subseteq \left\{ \frac{\beta_i}{\alpha_i} |\ 1 \leq i \leq k,\ \beta_i \leq \alpha_i \right\} \cup \{0\}.$$

Proof. Let $\frac{\beta_i}{\alpha_i} \in \lambda(X, MZS)$. Then there exists a $Y \in MZS(G)$ such that $\lambda(X, Y) = \frac{\beta_i}{\alpha_i}$. According to Theorem 2.1, this implies that there exists $g_i \in X$ such that $(g_i)_X = \alpha_i$ and $(g_i)_Y = \beta_i$. Also, since $Y \in MZS(G)$, as shown in Theorem 2.15, $\frac{\beta_i}{\alpha_i} \leq 1$. Since $\alpha_i > 0$, $\frac{\beta_i}{\alpha_i} \leq 1$ implies that $\beta_i \leq \alpha_i$. Thus, $\frac{\beta_i}{\alpha_i} \in \left\{ \frac{\beta_i}{\alpha_i} \leq 1 | \; 1 \leq i \leq k, \; \beta_i \leq \alpha_i \right\}$. ■

Note that Theorem 4.3 is not a complete answer for the case of X, a fixed minimal zero sequence, and $Y \in MZS(G)$. Based on the general properties and the results from Theorem 3.6, we believe that this containment is actually an equality, as shown in the Conjectures below. Furthermore, the second conjecture is needed because of Lemma 3.2.

Conjecture 4.4. *Let $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ such that $n_j | n_{j+1}$ for all $1 \leq j < s$, excluding the case where $s = 2$ with $n_1 = 2$ and $n_2 \in \mathbf{2}\mathbb{Z}^{\geq 4}$. If $X \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ then*

$$\lambda(X, MZS) = \left\{ \frac{\beta_i}{\alpha_i} | \; 1 \leq i \leq k, \; \beta_i \leq \alpha_i \right\} \cup \{0\}.$$

Conjecture 4.5. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $n \in \mathbf{2}\mathbb{Z}^{\geq 4}$. If $X \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$ then*

$$\lambda(X, MZS) = \left\{ \frac{\beta_i}{\alpha_i} | \; 1 \leq i \leq k, \; \beta_i \leq \alpha_i, \text{ and } \frac{\beta_i}{\alpha_i} \neq \frac{n-2}{n-1} \right\} \cup \{0\}.$$

Although we do not have the complete answer for all finite abelian groups, we do have an answer for cyclic groups.

Theorem 4.6. *$X \in MZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k}$, $\alpha_1 > \alpha_2 \cdots \geq \alpha_k$. If $\exists Y \in MZS$ such that $X g_1^{-1} \subseteq Y$, then*

$$\lambda(X, MZS) = \{0, 1\} \cup \left\{ \frac{\beta}{\alpha_i} | \; 1 \leq i \leq k, 1 \leq \beta \leq \alpha_i - 1 \right\}.$$

Otherwise,

$$\lambda(X, MZS) = \{0, 1\} \cup \left\{ \frac{\beta}{\alpha_1} | \; \beta \leq \alpha_1 - 2 \right\} \cup \left\{ \frac{\beta}{\alpha_i} | \; 2 \leq i \leq k, 1 \leq \beta \leq \alpha_i - 1 \right\}.$$

Proof. According to Theorem 2.18 and Theorem 2.13, $\lambda(X, MZS) \subseteq \{0, 1\} \cup \left\{ \frac{\beta}{\alpha_i} | \; 1 \leq i \leq k, \; 0 \leq \beta \leq \alpha_i \right\}$.

$\{0, 1\} \subseteq \lambda(X, MZS)$ follows directly from Theorem 2.6 and Theorem 2.7. To show that $\left\{ \frac{\beta}{\alpha_i} | \; 1 \leq i \leq k, \; \beta \leq \alpha_i - 1 \right\} \setminus \left\{ \frac{\alpha_1 - 1}{\alpha_1} \right\} \subseteq \lambda(X, MZS)$, we will first construct a Y that satisfies $\frac{\beta}{\alpha_1} = \lambda(X, Y)$, for any $\beta \leq \alpha_1 - 2$. Let $Y = g_1^{\beta} g_2^{\alpha_2} \cdots g_k^{\alpha_k} h$, $h = (\alpha_1 - \beta) g_1 \pmod{n}$, and $\beta \leq \alpha_1 - 2$. It is important that we show that $h \neq g_1$. Assume that $h = g_1$, then $Y = g_1^{\beta} g_2^{\alpha_2} \cdots g_k^{\alpha_k} h =$

$g_1^{\beta+1}g_2^{\alpha_2}\cdots g_k^{\alpha_k} \subseteq X$. This contradicts the fact that $X \in MZS(G)$, and hence $h \neq g_1$. Clearly, whether $h = g_i \neq g_1$ or not, $\frac{\beta}{\alpha_1}$ is the minimum of $\left\{\frac{\beta_i}{\alpha_i}\right\}$, where $\beta_i = (g_i)_Y$. Therefore, $\frac{\beta}{\alpha_1} \in \lambda(X, MZS)$. Now, we will construct a Y' that satisfies $\frac{\beta}{\alpha_i} = \lambda(X, Y')$, for any $\beta \leq \alpha_i - 1$ and $i \neq 1$. Let $Y' = g_1^{\alpha_1-1}g_2^{\alpha_2}\cdots g_i^{\beta}\cdots g_k^{\alpha_k}h$, $h = (\alpha_i - \beta)g_i + g_1 \pmod{n}$, and $\beta \leq \alpha_i - 1$. We will show that $\frac{\alpha_i-1}{\alpha_i} \leq \frac{\alpha_1-1}{\alpha_1}$. $\frac{\alpha_1-1}{\alpha_1} = 1 - \frac{1}{\alpha_1}$, $\frac{\alpha_i-1}{\alpha_i} = 1 - \frac{1}{\alpha_i}$. Since $\frac{1}{\alpha_1} \leq \frac{1}{\alpha_i}$, we get $\frac{\alpha_i-1}{\alpha_i} \leq \frac{\alpha_1-1}{\alpha_1}$. Using the same reasoning as above, it follows that $\frac{\beta}{\alpha_i}$ is the minimum of $\left\{\frac{\alpha_1-1}{\alpha_1}, \frac{\beta_j}{\alpha_j}\right\}$, where $\beta_i = (g_i)_Y$. Therefore, $\frac{\beta}{\alpha_i} \in \lambda(X, MZS)$.

Now consider the case where $\frac{\alpha_1-1}{\alpha_1} \in \lambda(X, MZS)$. We will first show that $\frac{\alpha_1-1}{\alpha_1}$ is not in the set of $\lambda(X, MZS)$ if $\alpha_1 > \frac{n}{2}$. Assume that $\frac{\alpha_1-1}{\alpha_1} \in \lambda(X, MZS)$. $\frac{\alpha_1-1}{\alpha_1} = 1 - \frac{1}{\alpha_1} > 1 - \frac{1}{\frac{n}{2}} = 1 - \frac{2}{n} = \frac{n-2}{n}$ is a contradiction to Theorem 2.18. If $\alpha_1 \leqslant \frac{n}{2}$ and $\frac{\alpha_1-1}{\alpha_1} \in \lambda(X, MZS)$, then there must exist a $Y \in MZS(G)$ such that $\frac{\alpha_1-1}{\alpha_1}$ is the minimum of $\left\{\frac{\beta_i}{\alpha_i}\right\}$, $\beta_i = (g_i)_Y$. We will show that Y must contain a subsequence Xg_1^{-1}. By Theorem 2.1, if $\lambda(X, Y) = \frac{\alpha_1-1}{\alpha_1}$, then $(g_1)_X = \alpha_1$ and $(g_1)_Y = \alpha_1 - 1$. Also, all remaining elements $h \in X$ and Y must have $\frac{(h)_Y}{(h)_X} \geq \frac{\alpha_1-1}{\alpha_1}$. Hence, $(h)_Y \geq (h)_X$. Therefore, $\frac{\alpha_1-1}{\alpha_1} \in \lambda(X, MZS)$ only if $\alpha_1 \leq \frac{n}{2}$ and $\exists Y \in MZS(G)$ such that $Xg_1^{-1} \subseteq Y$. ∎

The final case is when X is a fixed zero sequence and $Y \in MZS(G)$. Since we do not have a complete answer for $\lambda(ZS, MZS)$, generating one when X is fixed is an area for further study. However, we were able to find a bound for the cyclic case.

Theorem 4.7. *$X \in ZS(G)$ where $X = g_1^{\alpha_1}g_2^{\alpha_2}\cdots g_k^{\alpha_k}$. If $k = 1$, then $\lambda(X, MZS) = \left\{0, \frac{(g_1)_Y}{\alpha_1}\right\}$. Otherwise,*

$$\lambda(X, MZS) \subseteq \left\{\frac{\beta}{\alpha} \middle| \frac{\beta}{\alpha} \leq \min\left\{\frac{|g_i| - 1}{\alpha_i}\right\}\right\}.$$

Proof. If $k = 1$, $\lambda(X, MZS) = \left\{0, \frac{(g_1)_Y}{\alpha_1}\right\}$ follows directly from the characterization of $\lambda(X, Y)$.

Let Y be any minimal zero sequence. Assume that $\lambda(X, Y) > \min\left\{\frac{|g_i|-1}{\alpha_i}\right\}$. If g_j is the critical element and $\frac{|g_j|-1}{\alpha_j} = \min\left\{\frac{|g_i|-1}{\alpha_i}\right\}$, then $\frac{\beta}{\alpha_j} > \min\left\{\frac{|g_i|-1}{\alpha_i}\right\}$ implies that there are at least $|g_j|$ copies of g_j in Y. Since $g_j^{|g_j|} \in MZS(G)$ and $g_j^{|g_j|} \in Y$, this contradicts the fact that $Y \in MZS(G)$. If g_j is the critical element and $\frac{|g_j|-1}{\alpha_j} \neq \min\left\{\frac{|g_i|-1}{\alpha_i}\right\}$, then by definition of $\lambda(X, Y)$, $\frac{\beta}{\alpha_j}$ is the minimum of $\left\{\frac{(g_i)_Y}{\alpha_i}\right\}$. However, Y must have β_i copies of g_i, $\beta_i < |g_i|$. Since $\min\left\{\frac{\beta_i}{\alpha_i}\right\} \leq \min\left\{\frac{|g_i|-1}{\alpha_i}\right\} < \frac{\beta}{\alpha_j}$, it contradicts the assumption that $\frac{\beta}{\alpha_j}$ is the minimum of $\left\{\frac{\beta_i}{\alpha_i}\right\}$. Therefore $\lambda(X, Y) \leq \min\left\{\frac{|g_i|-1}{\alpha_i}\right\}$. ■

By restricting X to have only two distinct elements, we were able to prove a theorem for the special case where $X = g_1^{\alpha_1} g_2^{\alpha_2}$ and there doesn't exist a $Y \in MZS$ such that $\lfloor Y \rfloor = \{g_1, g_2\}$. The following example illustrates the set of extraction degrees for such an X in $\mathbb{Z}(12)$. Let $X = 3^4 4^3$, which is obviously a zero sequence. The possible minimal zero sequences Y in $\mathbb{Z}(12)$ that give $\lambda(X, Y) \neq 0$ or 1, are: $1^1 3^3 4^1 10^1$; $3^3 4^1 11^1$; $3^3 4^2 7^1$; $3^2 4^1 7^2$; $3^1 4^1 5^1$; $3^2 4^2 10^1$. The resulting $\lambda(X, Y) \in \left\{\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}\right\}$.

Theorem 4.8. *$X \in ZS(G)$ where $X = g_1^{\alpha_1} g_2^{\alpha_2}$. If there does not exist a $Y \in MZS$ such that $\lfloor Y \rfloor = \{g_1, g_2\}$, then*

$$\lambda(X, MZS) = \{1, 0\} \cup \left\{\frac{\beta}{\alpha_i} | \ \frac{\beta}{\alpha_i} \leq min\left\{\frac{|g_1| - 1}{\alpha_1}, \frac{|g_2| - 1}{\alpha_2}\right\}, i \in \{1, 2\}\right\}.$$

Proof. $\{0, 1\} \subseteq \lambda(X, MZS)$ follows directly from Theorem 2.6 and Theorem 2.7.

Now, we will show that $\lambda(X, MZS) \subseteq \left\{\frac{\beta}{\alpha_i} | \ \frac{\beta}{\alpha_i} \leq \min\left\{\frac{|g_1|-1}{\alpha_1}, \frac{|g_2|-1}{\alpha_2}\right\} .i \in \{1, 0\}\right\}$. Let $\frac{\beta}{\alpha} \in \lambda(X, MZS)$, and $Y = g_1^{\beta_1} g_2^{\beta_2}$ with $\lambda(X, Y) = \frac{\beta}{\alpha}$. According to Theorem 2.1, there must exist an element $g_i \in X$, such that $\beta = (g_i)_Y$ and $\alpha = (g_i)_X = \alpha_i$. Since $Y \in MZS(G)$, β is less than or equal to $|g_i| - 1$. Otherwise, $g_i^{|g_i|} \in MZS(G)$ is a subsequence of Y, which contradicts the fact that $Y \in MZS(G)$ and that $\lfloor Y \rfloor = \{g_1, g_2\}$. Therefore, $\frac{\beta}{\alpha} \leq \frac{|g_i|-1}{\alpha_i}$. It is important to show that $\frac{\beta}{\alpha} \leq \frac{|g_j|-1}{\alpha_j}, j \neq i$. Using the same reasoning as above, we have $\beta_j \leq |g_j| - 1$, where $\beta_j = (g_j)_Y$ and $j \neq i$. By definition of $\lambda(X, Y)$, $\frac{\beta}{\alpha_i} \leq \frac{\beta_j}{\alpha_j} \leq \frac{|g_j|-1}{\alpha_j}$. Therefore, $\frac{\beta}{\alpha} = \frac{\beta}{\alpha_i} \leq \min\left\{\frac{|g_1|-1}{\alpha_1}, \frac{|g_2|-1}{\alpha_2}\right\}$.

To show that $\left\{\frac{\beta}{\alpha_i} | \ \frac{\beta}{\alpha_i} \leq min\left\{\frac{|g_1|-1}{\alpha_1}, \frac{|g_2|-1}{\alpha_2}\right\}\right\} \subseteq \lambda(X, MZS)$, it suffices to show that $\frac{\beta}{\alpha_1} \in \lambda(X, Y)$. Construct $Y = g_1^{\beta} g_2^{|g_2|-1} h$, where $\beta < |g_1|$, $h = n - \{\beta g_1 + (|g_2| - 1) g_2 \pmod{n}\}$. Obviously, Y is a zero sequence. We will now show that $Y \in MZS(G)$. Assume that $Y \notin MZS(G)$, then there must exist a $Y' = g_1^{\beta_1} g_2^{\beta_2} h$ such that $Y' \in MZS(G)$, and $\beta_1 = \beta, \beta_2 < |g_2| - 1$, or $\beta_1 < \beta, \beta_2 = |g_2| - 1$, or $\beta_1 < \beta, \beta_2 < |g_2| - 1$. All three cases imply that $g_1^{\beta-\beta_1} g_2^{(|g_2|-1)-\beta_2} \in MZS(G)$. If both $\beta - \beta_1$ and $(|g_2| - 1) - \beta_2$ are non-zero, then $g_1^{\beta-\beta_1} g_2^{(|g_2|-1)-\beta_2} \in MZS(G)$ contradicts the assumption that there does not exist a Y such that $\lfloor Y \rfloor = \{g_1, g_2\}$. We will next show that $\beta - \beta_1$ or $(|g_2| - 1) - \beta_2$ being zero is not possible. Both β, β_1 are less than $|g_1|$, and hence $\beta - \beta_1 < |g_1|$. Similarly, $(|g_2| - 1) - \beta_2 < |g_2|$. Trivially, either $g_1^{\beta-\beta_1}$ or $g_2^{\beta_2-(|g_2|-1)}$ cannot be zero sequence. Therefore, $Y \in MZS(G)$. Then $h \neq g_1$ or $h \neq g_2$ follows directly from the hypothesis, because Y cannot contain only the elements g_1 and g_2. Therefore, $\frac{\beta}{\alpha_1} \leq min \left\{ \frac{|g_1|-1}{\alpha_1}, \frac{|g_2|-1}{\alpha_2} \right\} \in \lambda(X, Y)$.

■

4.2 Y Fixed

The final sets that we investigated are those in which Y is a fixed zero sequence and X is contained in either $MZS(G)$ or $ZS(G)$. As discussed previously, by placing restrictions on Y, we impose restrictions on X. Using some of our previous results we were able to state some bounds for these sets.

Proposition 4.9. *If $Y \in ZS(G)$ where $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$ and X is any minimal zero sequence, then*

$$\lambda(X, Y) = 0, \text{ or } \min \left\{ \frac{\beta_i}{|g_i|} | 1 \leq i \leq j \right\} \leq \lambda(X, Y) \leq \max \{\beta_i | 1 \leq i \leq j\}.$$

Proof. Let $Y = g_1^{\beta_1} g_2^{\beta_2} \cdots g_j^{\beta_j}$ and let $X \in MZS(G)$. If $\lfloor X \rfloor \not\subseteq \lfloor Y \rfloor$, then from Theorem 2.6, $\lambda(X, Y) = 0$. So, assume $\lfloor X \rfloor \subseteq \lfloor Y \rfloor$. Then there exist m where $1 \leq m \leq j$ such that $\lambda(X, Y) = \frac{(g_m)_Y}{(g_m)_X} = \frac{\beta_m}{(g_m)_X}$. Since $X \in MZS(G)$, $1 \leq (g_m)_X \leq |g_m|$. Then $\frac{\beta_m}{|g_m|} \leq \frac{\beta_m}{(g_m)_X} \leq \frac{\beta_m}{1}$. Then we have that $\min \left\{ \frac{\beta_i}{|g_i|} | \ 1 \leq i \leq j \right\} \leq \frac{\beta_m}{|g_m|} \leq \lambda(X, Y) \leq \frac{\beta_m}{1} \leq \max \{\beta_i | \ 1 \leq i \leq j\}$. ■

For the case when Y is a fixed minimal zero sequence, note that if $Y \in MZS(G)$, $\lambda(X, Y) \leq 1$ from Theorem 2.15. Specifically, if $G = \mathbb{Z}_n$, $\lambda(X, Y) \leq \frac{n-2}{n}$ from Theorem 2.18 .

Now, for the last two cases where X is any zero sequence and Y is either a fixed minimal zero sequence or zero sequence, we can combine the previous upper bounds with the fact that if X is any zero sequence, $\lambda(X, Y)$ can be arbitrarily close to zero. For example, let $X = mY$ where $m \in \mathbb{Z}^+$. Then $\lambda(X, Y) = \frac{1}{m}$.

5 Conclusion

We set out to find the sets of possible extraction degrees for all combinations of two zero sequences. In the end, we were able to determine complete results for many of these combinations. However, difficulties arose when placing restrictions on Y, as addressed in Section 3.3 and Section 4.2. These problems are open to further study.

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