Where Has All the Value Gone?
Portfolio risk optimization using CVaR

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1 Introduction

Corporate securities are widely used as a means to boost the value of asset portfolios; however, the returns offered by securities do not come without risk. Earning a negative return on investments, or in the extreme case, losing all of the investment, is a very real scenario. Money managers worldwide are interested in the risk-return payoff of their investments. This concept is extended to the insurance industry. Insurance companies have significant investments in hard assets such as land, buildings, and vehicles. The upside of these investments is the premiums they receive for insuring the asset, while the downside risk is comprised of their obligation to pay for the reparation or loss of an asset. Thus, insurance companies are also interested in the risk-return payoff of their investments. One way to address this risk-return payoff is value-at-risk (VaR). VaR addresses the catastrophic events that may occur, such as the bankruptcy of a corporation, or a natural disaster that destroys numerous buildings. A return of -10% to -20% could be attributed to short term market trends, so the investor's portfolio would not be devastated by these fluctuations in the long run; however, the collapse of an asset would have a significant adverse affect on a portfolio. The investor should ask whether the return of his or her portfolio adequately compensates for a low-probability but large loss event.

Value-at-risk is a measure of default risk, where default is classified as the low-probability large-impact, event. Consider the following graph that depicts the probability of various portfolio returns.

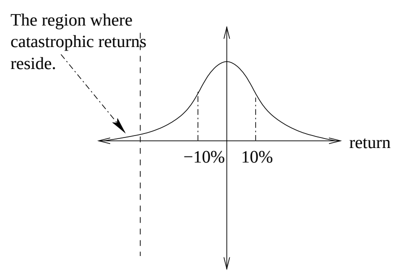

Figure 1: Frequency of portfolio returns

For the sake of illustration, we assume that the returns are normally distributed, which is not always the case. Notice that the probability of a return being between 0% and 10% is relatively high. As we move to the left, we enter the "tail" where the catastrophic returns reside. There is nothing efficient about an optimized portfolio obtained by ignoring the tails. In fact, by incorporating the tails into risk-return frontiers that previously ignored tail effects, efficient portfolios become inefficient [2]. VaR is one of many statistics that allow us to analyze the tail effects on portfolio risk. Conditional Value-at-Risk (CVaR), which builds on the concepts of VaR, is another measure of the monetary value that is at risk in the tails. While these two measures are closely related, CVaR is a more desirable statistic for optimization because it can be minimized using linear programming formulations. VaR is difficult to optimize when calculated using discrete scenarios because it is a non-convex, non-smooth, global optimization problem. For the remainder of this paper, we outline the development of VaR, and then apply the VaR concept to a portfolio of corporate securities. Following this example, we move into the drawbacks of VaR, and the benefits of using CVaR. Most importantly, we show that CVaR is efficiently calculated by solving a linear program.

2 Definition of VaR and CVaR

We initially assume that all random quantities take values from a finite set of scenarios. Let each scenario be indexed by a member of the set Ω, and suppose $x \in \mathbb{R}^n$ is a portfolio such that each x_i equals the number of shares held in security i. Define the loss function for each portfolio x with the initial portfolio value V_0 as:

$$L(x,y) \quad = \quad V_0 - V(x, P(y)), \tag{1}$$

where $y \in \Omega$, $P(y)$ is the pricing structure of scenario y, and $V(x, P(y))$ is the value of the portfolio x under prices $P(l)$. If the portfolio has earned a positive return, then the value of this loss function is negative, which indicates a gain. Let $p(y)$ be the probability associated with each scenario $P(y)$ so that the probability that the loss function does not exceed ζ is the cumulative probability:

$$\Psi(x,\zeta) \quad = \quad \sum_{\{y\in\Omega | L(x,P(y))\leq\zeta\}} p(y). \tag{2}$$

Using these functions, we define value-at-risk (VaR) and conditional value-at-risk (CVaR).

Definition: The value at risk (VaR) of a portfolio at the alpha probability level is the lowest possible value so that the probability of losses less than VaR exceeds $\alpha{\cdot}100\%$. It is given by:

$$\mathrm{VaR}(x,\alpha) = \min\{\zeta \in \mathbb{R} | \Psi(x,\zeta) > \alpha\}.$$

Definition: The conditional value at risk (CVaR) of the losses of a portfolio is the expected value of the losses, conditioned on the losses being in excess of VaR:

$$\begin{aligned} CVaR(x,\alpha) &= E[L(x,y)|L(x,y) > VaR(x,\alpha)] \\ &= \frac{\sum_{\{y\in\Omega|L(x,y)>VaR(x,\alpha)\}} p(y)L(x,y)}{\sum_{\{y\in\Omega|L(x,y)>VaR(x,\alpha)\}} p(y)} \end{aligned} \tag{3}$$

The above definitions are based on the assumption that all random values are discrete. Let $L(x,y)$ be the loss function for the continuous model, where $x \in \mathbb{R}^n$ represents a portfolio and $y \in \mathbb{R}$ represents the pricing scenario. Then $L(x,y)$ is a random variable with a p.d.f. induced by y. Similar to the discrete case, we have

$$\Psi(x,\zeta) \quad = \quad \int_{L(x,y)\leq\zeta} p(y)dy,$$

$$VaR(x,\alpha) \quad = \quad \min\{\zeta \in \mathbb{R} | \Psi(x,\zeta) \geq \alpha\},$$

$$CVaR(x,\alpha) \quad = \quad (1-\alpha)^{-1} \int_{L(x,y)\geq VaR(x,\alpha)} L(x,y)p(y)dy. \tag{4}$$

Equation (4) is the equation for CVaR, which is the conditional expected value of the loss $L(x,y)$ under the condition that it exceeds $\mathrm{VaR}(x,\alpha)$. In the continuous case, $\Psi(x,\zeta) = \alpha$; therefore, when extended to the continuous case, the denominator in equation (3) becomes $1-\alpha$. This leads to the coefficient of $(1-\alpha)^{-1}$ in equation (4). Notice that $\Psi(x,\zeta)$ is nondecreasing with respect to ζ and continuous from the right, but not necessarily from the left due to the possibility of jumps.

3 Numerical Example of VaR and CVaR

We now present a numerical example to highlight the topics of VaR and CVaR. Note that if we set α=90%, then 90% of the time we will not experience a loss greater than VaR. Consider the following example that illustrates the calculation of VaR for a portfolio of securities. The portfolio consists of any combination of the following stocks:

Company	Stock Ticker	Price per share
Chevron Texaco	CVX	\$61.00
Occidental Petroleum	OXY	\$70.00
PetroKazakhstan	PKZ	\$42.00
Exxon-Mobil	XOM	\$61.00

These four companies were chosen because they move in direct relation to each other. This enables us to define a price structure that corresponds to industry trends. Figure 2 is a graph of the standard business cycle that we use to define our pricing structures. The probability associated with each scenario is printed below each segment. In addition, the market conditions that characterize each scenario are listed.

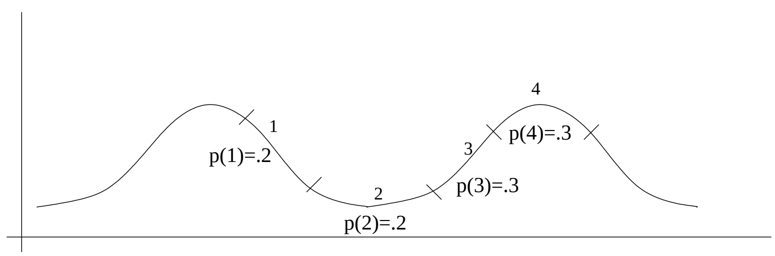

Figure 2: Pricing scenarios and their respective probabilities

Scenario 1: Declining demand for oil.
Scenario 2: Relatively low demand for oil.
Scenario 3: Increasing demand for oil.
Scenario 4: High demand for oil.

The following chart lists the expected loss associated with each of the stocks depending on the pricing scenario. [1]

	$P(1)$	$P(2)$	$P(3)$	$P(4)$
CVX	3.72	0.00	-0.61	-0.31
OXY	8.05	0.28	-2.80	-0.84
PKZ	7.48	2.10	-16.40	-3.28
XOM	3.90	0.00	-0.61	-0.24

If we have a portfolio with only one share of PKZ, then in $P(3)$ we expect a loss of -\$16.40, which corresponds to a gain of \$16.40. Consider a portfolio with one share of each company and let \$10 be our threshold value. That is, let $x = (1, 1, 1, 1)^T$ and set ζ = \$10. We compute $\Psi((1, 1, 1, 1)^T, 10)$ by considering the expected losses and the probability associated with the pricing structures. Since our portfolio consists of one share of each security, we have the following:

[1] These losses were arbitrarily assigned based on the β of each security and the pricing scenario.

$$\begin{array}{ll} L(x,1) = 23.15 & p(1) = .2 \\ L(x,2) = 2.38 & p(2) = .2 \\ L(x,3) = -20.42 & p(3) = .3 \\ L(x,4) = -4.67 & p(4) = .3. \end{array}$$

Notice that $L(x,1) = 23.15 > 10 = \zeta$, so we exclude $p(1)$ from our summation. In scenarios 2-4, the loss is less than our cut off value, so

$$\begin{aligned} \Psi((1,1,1,1)^T, 10) &= \sum_{\{l \in \Omega | L(x,P(y)) \leq 10\}} p(y) && (5) \\ &= .2 + .3 + .3 && (6) \\ &= .8. && (7) \end{aligned}$$

Now, we compute VaR$(x, .79)$ by solving $\min\{\zeta \in \mathbb{R} | \Psi(x,\zeta) > .79\}$. Doing so, we have that

$$\begin{aligned} VaR((1,1,1,1)^T, .79) &= \min\{\zeta \in \mathbb{R} | \Psi((1,1,1,1)^T, \zeta) > .79\} \\ &= \$2.38. \end{aligned}$$

If we increase α by one basis point, then we have

$$\begin{aligned} VaR((1,1,1,1)^T, .80) &= \min\{\zeta \in \mathbb{R} | \Psi((1,1,1,1)^T, \zeta) > .80\} \\ &= \$23.15. \end{aligned}$$

In fact, this value for VaR holds for all $\alpha \geq .80$. This is precisely the reason that CVaR is often considered a better measure of risk than VaR. As we see in Figure 3, as α increases, VaR jumps from one value to another greater value. These jumps cause discontinuities, which make the function difficult to minimize. Notice that VaR is a continuous function from the right, but is discontinuous from the left.

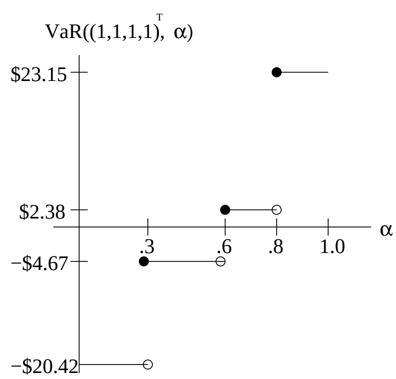

Figure 3: VaR for different α levels

To compute CVaR$(x, .79)$ using the same information, we first determine the scenarios $y \in \Omega$ for which $L(x, P(y)) > VaR(x, \alpha)$ with $\alpha = .79$. As we showed above, $VaR(x, .79) = \$2.38$. The only scenario in which $L((1,1,1,1)^T, P(y)) > \$2.38$ is scenario 1, so we have that

$$
\begin{aligned}
CVaR((1,1,1,1)^T, .79) &= \frac{p(1) \cdot L((1,1,1,1)^T, P(1))}{p(1)} \\
&= \frac{.2 \cdot \$23.15}{.2} \\
&= \$23.15.
\end{aligned}
$$

This result indicates that when conditions are outside of our confidence level of $\alpha = 79\%$, we expect to lose \$23.15. If we relied solely on VaR, we would only know that 79% of the time we would not experience a loss in excess of \$2.38. While this is useful information, CVaR provides a better indication of just how risky our portfolio is. In many cases, VaR provides an inaccurate premonition of how much the portfolio stands to lose in the worst case scenarios. Both VaR and CVaR are stress-tests of a portfolio to see what happens to the value in the worst $1-\alpha$ cases, but by using VaR we are considering only the smallest possible losses in those cases. In other words, we are ignoring whether or not the portfolio could lose \$1,000 or \$1,000,000 once we are outside of our α confidence level.

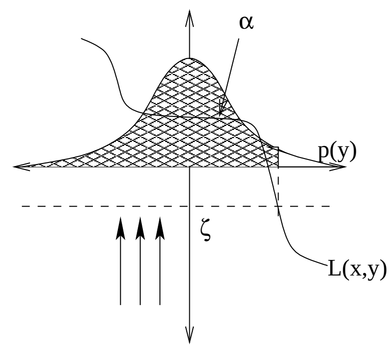

Figure 4: Graphical calculation of VaR with $\alpha = 95\%$

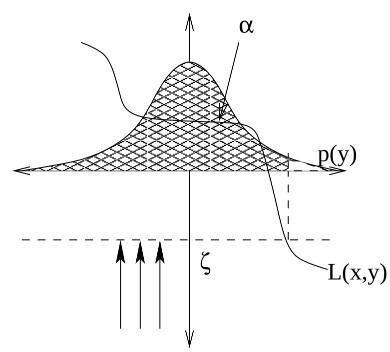

Figure 5: Graphical calculation of VaR with $\alpha = 99\%$

Figures 4 and 5 illustrate how VaR and CVaR are computed in the continuous case. Figure 4 illustrates the computation of VaR for $\alpha = .95$, and Figure 5 shows the graph for $\alpha = .99$. Notice that as we increase α, the VaR (ζ) grows. Given α, we move ζ up or down to the point that we reach the desired area under the curve, which in this case is α. This point is precisely VaR.

Again, while VaR provides a measure of how likely it is that a portfolio will experience losses less than some threshold ζ, it fails to provide an analysis of the losses that occur in excess of ζ. In the above graphs, the CVaR corresponds to the portion of the curve $L(x, y)$ that falls below the line determined by ζ. While these losses occur relatively infrequently, as indicated by the unshaded portion under $p(y)$, they could be so devastating that the portfolio would cease to exist. Thus, we must account for these potential losses which reside in the tails. CVaR provides us with a model that incorporates this tail effect.

4 Minimization of CVaR

In section 3, we discussed how to calculate VaR and CVaR for a given portfolio. In this section we outline an approach to minimize CVaR. This optimization problem yields a portfolio x that minimizes our exposure to losses. We define our portfolio in such a way that the value for each x_i is equal to the fraction of the total portfolio value that security i comprises. Thus, our portfolio is in $X = \{x : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, 2, ..., n\}$. We begin by defining

$$\begin{aligned} G(\zeta) &= \int_{y \in \mathbb{R}} [L(x, y) - \zeta]^+ p(y) dy \text{ and} \\ F(x, \zeta, \alpha) &= \zeta + (1 - \alpha)^{-1} \int_{y \in \mathbb{R}} [L(x, y) - \zeta]^+ p(y) dy \\ &= \zeta + (1 - \alpha)^{-1} G(\zeta), \end{aligned}$$

where

$$[t]^+ = \max\{0, t\}.$$

Theorem 4.5 and its supporting lemma establish the process by which we minimize CVaR. To establish Theorem 4.5, we make the assumption that $\Psi(x, \zeta)$ is continuous everywhere with respect to ζ. In addition, the following definitions are useful for understanding the proofs.

Convex Function: The function $f : X \mapsto \mathbb{R}$ is convex if

$$f((1 - \alpha)x_1 + \alpha x_2) \leq (1 - \alpha) f(x_1) + \alpha f(x_2),$$

for all x_1 and x_2 in X and α in [0,1].

Smooth: A function f is n-times smooth if the n^{th} derivative of f exists and is continuous. The set of n-times smooth functions is denoted by $\mathcal{C}^n$, where $\mathcal{C}^0$ is the set of continuous functions.

Lemma 4.1 *If $f(x)$ and $g(x)$ are continuous functions, then the function $h(x) = \max\{f(x), g(x)\}$ is continuous.*

Proof: Let $f(x)$ and $g(x)$ be continuous functions, and pick $\epsilon > 0$. Then there exist a $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \text{ implies that } |f(x) - f(x_0)| < \epsilon \tag{8}$$

and

$$|x - x_0| < \delta_2 \text{ implies that } |g(x) - g(x_0)| < \epsilon. \tag{9}$$

Now,

$$\begin{aligned} h(x) - h(x_0) &= \max\{f(x), g(x)\} - \max\{f(x_0), g(x_0)\} \\ &\leq \max\{f(x) - f(x_0), f(x) - g(x_0), g(x) - f(x_0), g(x) - g(x_0)\}. \end{aligned}$$

Let $x \in (x_0 - \min\{\delta_1, \delta_2\}, x_0 + \min\{\delta_1, \delta_2\})$.

Case 1: Suppose there does not exist an $\hat{x} \in (x_0 - \min\{\delta_1, \delta_2\}, x_0 + \min\{\delta_1, \delta_2\})$ such that $f(\hat{x}) = g(\hat{x})$. Then $h(x) - h(x_0)$ is $f(x) - f(x_0)$ or $g(x) - g(x_0)$. From inequalities (8) and (9), we have that $h(x) - h(x_0) < \epsilon$ in either case, so $h(x)$ is continuous.

Case 2: Suppose there does exist an $\hat{x} \in (x_0 - \min\{\delta_1, \delta_2\}, x_0 + \min\{\delta_1, \delta_2\})$ such that $f(\hat{x}) = g(\hat{x})$. If $\max\{f(x) - f(x_0), f(x) - g(x_0), g(x) - f(x_0), g(x) - g(x_0)\} = f(x) - f(x_0)$ or $g(x) - g(x_0)$, then we are done. Suppose this max is $f(x) - g(x_0)$. Then,

$$\begin{aligned} |f(x) - g(x_0)| &= |f(x) - f(\hat{x}) + g(\hat{x}) - g(x_0)| \\ &\leq |f(x) - f(\hat{x})| + |g(\hat{x}) - g(x_0)| \\ &< \epsilon + \epsilon = 2\epsilon, \qquad (10) \end{aligned}$$

where (10) follows from the fact that $|x - \hat{x}| < \min\{\delta_1, \delta_2\}$. This implies that $h(x)$ is continuous when the max is $f(x) - g(x_0)$. Similarly, when the max is $g(x) - f(x_0)$, we conclude that $h(x)$ is continuous. From the results in Case 1 and Case 2, we conclude that $h(x)$ is continuous for all x. ■

Corollary 4.2 *$g(\zeta) = [L(x, y) - \zeta]^+$ is continuous.*

The proof of this corollary follows directly from Lemma 4.1. To see this, rewrite $g(\zeta) = [L(x, y) - \zeta]^+$ as $\max\{0, L(x, y) - \zeta\}$. Since $L(x, y) - \zeta$ is continuous, we apply Lemma 4.1 to conclude that $[L(x, y) - \zeta]^+$ is continuous. While we have shown that $g(\zeta)$ is continuous, Lemma 4.3 that appears in [3] provides us with the fact that G is continuously differentiable. The proof of this, along with the proof of convexity requires topics outside the scope of this project. Such topics include a complete understanding of subdifferentials and conjugate functions. Computing $G'(\zeta)$ would be rather trivial by *Theorem 9.42* in [4] if we assumed that $[L(x, y) - \zeta]^+ p(y) \in \mathcal{C}^1$ with respect to ζ. However, we know that $[L(x, y) - \zeta]^+ p(y)$ is not in $\mathcal{C}^1$ since the derivative does not exist at the point when $L(x, y)$ falls below ζ. At this point, the function shifts from having a positive value to having a value of zero. See [5] for a proof of Lemma 4.3.

Lemma 4.3 (Rockafellar, R.T., and S. Uryasev [3]) *With x fixed, G is a convex continuously differentiable function with derivative*

$$G'(\zeta) = \Psi(x, \zeta) - 1.$$

Lemma 4.4 *Assume $f \in \mathcal{C}^0$. Then $\{x : f(x) = 0\}$ is closed.*

Proof: Pick $\hat{x}$ so that $f(\hat{x}) \neq 0$. Without loss of generality, assume that $f(\hat{x}) > 0$. Now, let $\epsilon = \frac{1}{2} f(\hat{x})$. Since f is continuous, there exists a δ so that

$$|x - \hat{x}| < \delta \text{ implies } |f(x) - f(\hat{x})| < \epsilon.$$

This is true for all $x \in (\hat{x} - \delta, \hat{x} + \delta)$ such that $f(x) > 0$. Therefore, $\{x : f(x) \neq 0\}$ is open, which implies that $\{x : f(x) = 0\}$ is closed. ■

Theorem 4.5 (Rockafellar, R.T., and S. Uryasev [3]) *As a function of* ζ, $F(x,\zeta,\alpha)$ *is convex and continuously differentiable. The* α*-CVaR of the loss associated with any* $x \in X$ *can be determined from the formula*

$$CVaR(x,\alpha) = \min_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha). \tag{11}$$

In this formula, the set consisting of the values of ζ *for which the minimum is attained, namely*

$$A(x,\alpha) = \operatorname*{argmin}_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha), \tag{12}$$

is a nonempty closed bounded interval (perhaps reducing to a single point), and the α*-VaR of the loss is given by*

$$VaR(x,\alpha) = glb\ A(x,\alpha). \tag{13}$$

In particular, it is always true that

$$VaR(x,\alpha) \in \operatorname*{argmin}_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha) \quad and \quad CVaR(x,\alpha) = F(x, VaR(x,\alpha),\alpha).$$

Proof: Let f_1 and f_2 be convex functions on the set X. Then for any x_1 and x_2 in X and $\alpha \in [0,1]$,

$$\begin{aligned}
(f_1+f_2)((1-\alpha)x_1+\alpha x_2) &= f_1((1-\alpha)x_1+\alpha x_2) + f_2((1-\alpha)x_1+\alpha x_2)\\
&\leq (1-\alpha)f_1(x_1)+\alpha f_1(x_2)+(1-\alpha)f_2(x_1)+\alpha f_2(x_2)\\
&= (1-\alpha)(f_1(x_1)+f_2(x_1))+\alpha(f_1(x_2)+\alpha f_2(x_2))\\
&= (1-\alpha)(f_1+f_2)(x_1)+\alpha(f_1+f_2)(x_2).
\end{aligned}$$

This result shows that the sum of f_1 and f_2 is convex. Given the defining formula for $F(x,\zeta,\alpha)$, it is immediate from this result and Lemma 4.3 that $F(x,\zeta,\alpha)$ is convex and continuously differentiable with derivative

$$\frac{\partial}{\partial\zeta}F(x,\zeta,\alpha) = 1+(1-\alpha)^{-1}[\Psi(x,\zeta)-1] = (1-\alpha)^{-1}[\Psi(x,\zeta)-\alpha]. \tag{14}$$

Thus, $F(x,\zeta,\alpha) : \mathbb{R}^n \to \mathbb{R}$ is smooth in ζ. Theorem 1.15 in [1] implies that for the values ζ^* that minimize $F(x,\zeta,\alpha)$, we have that $\frac{\partial}{\partial\zeta} F(x,\zeta*,\alpha) = 0$. From (14) we have that $(1-\alpha)^{-1}[\Psi(x,\zeta)-\alpha] = 0$, and since $(1-\alpha)^{-1} > 0$, $\Psi(x,\zeta^*)-\alpha = 0$. The values ζ^* that furnish the minimum of $F(x,\zeta,\alpha)$ are precisely the values in the set $A(x,\alpha)$ defined in (12). By Lemma 4.4 they form a closed interval since $\Psi(x,\zeta)$ is continuous and nondecreasing in ζ with limit 1 as $\zeta \to \infty$ and limit 0 as $\zeta \to -\infty$. The value ζ^* is precisely VaR, which follows from (13). Now, we have

$\min_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha)$
$= F(x, VaR(x,\alpha),\alpha)$
$= VaR(x,\alpha) + (1-\alpha)^{-1}\int_{y\in\mathbb{R}}[L(x,y)-VaR(x,\alpha)]^+p(y)dy$
$= VaR(x,\alpha) + (1-\alpha)^{-1}\int_{L(x,y)\geq VaR(x,\alpha)}[L(x,y)-VaR(x,\alpha)]^+p(y)dy$

$= VaR(x,\alpha) + (1-\alpha)^{-1}\left(\int_{L(x,y)\geq VaR(x,\alpha)} L(x,y)p(y)dy - VaR(x,\alpha)\int_{L(x,y)\geq VaR(x,\alpha)} p(y)dy\right).$

By definition, we know that

$$\begin{aligned}\int_{L(x,y)\geq VaR(x,\alpha)} L(x,y)p(y)dy &= (1-\alpha)CVaR(x,\alpha), \text{ and}\\ \int_{L(x,y)\geq VaR(x,\alpha)} p(y)dy &= 1-\Psi(x,VaR(x,\alpha)).\end{aligned}$$

In addition, $\Psi(x, VaR(x,\alpha)) = \alpha$. Thus,

$$\begin{aligned}\min_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha) &= VaR(x,\alpha) + (1-\alpha)^{-1}[(1-\alpha)CVaR(x,\alpha) - VaR(x,\alpha)(1-\alpha)]\\ &= CVaR(x,\alpha).\end{aligned}$$

Thus, we have confirmed the formula for $CVaR(x,\alpha)$ in (11), and have proven Theorem 4.5. ■

Using the results of this theorem, we have a method to find the optimal $CVaR$. The following outlines the method:

$$\begin{aligned}\min_{x\in X,\zeta\in\mathbb{R}} F(x,\zeta,\alpha) &= \min_{x\in X}\min_{\zeta\in\mathbb{R}} F(x,\zeta,\alpha)\\ &= \min_{x\in X} F(x, VaR(x,\alpha),\alpha)\\ &= \min_{x\in X} CVaR(x).\end{aligned}$$

Thus, by minimizing $F(x,\zeta,\alpha)$ with respect to x and ζ, we are able to find the minimum $CVaR$. Notice also that after we compute the first minimum with respect to ζ, we have found the VaR.

We now construct a linear program (LP) to compute the minimal CVaR and corresponding VaR. To begin, we model

$$G(\zeta) = \sum_{y\in\Omega}[L(x,y)-\zeta]^{+}p(y)dy \tag{15}$$

linearly, where (15) follows from the fact that Ω is discrete. We model the + operator as

$$\begin{aligned}L(x,y)-\zeta &\leq s_y, &(16)\\ s_y &\geq 0, &(17)\\ \sum_{y\in\Omega} s_y p(y) &= q, &(18)\end{aligned}$$

from which we have the following optimization problem:

$$\begin{aligned}\min_{x\in X,\zeta\in\mathbb{R}} F(x,\zeta,\alpha) &= \min\{\zeta + (1-\alpha)^{-1}G(\zeta) : \sum x_i = 1, x_i \geq 0, \zeta\in\mathbb{R}\} \qquad (19)\\ &= \min\{\zeta + (1-\alpha)^{-1}q : \sum x_i = 1, x_i\geq 0,\\ &\qquad L(x,y)-\zeta\leq s_y, s_y\geq 0, \forall y\in\Omega,\\ &\qquad \sum_{y\in\Omega} s_y p(y) = q\}.\end{aligned}$$

This LP generates a value for VaR, and also calculates a portfolio x that minimizes $CVaR$. The fact that this problem is an LP enables us to efficiently calculate both VaR and an optimal portfolio.

5 Conclusion

As shown in section 3, the calculation of VaR can be fairly complicated. In addition, the minimization of VaR is difficult because the function is non-convex and non-smooth. The optimization problem in (19) is very useful because it provides a way to efficiently calculate CVaR. Furthermore, the value of ζ that results from minimizing $F(x, \zeta, \alpha)$ with respect to ζ is precisely VaR. Thus, by solving (19) we simultaneously find VaR and the portfolio x that minimizes $CVaR$. These results are very useful to all investors trying to shield their portfolio from extraordinary losses.

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