A LOOK AT f-INVARIANT δ-SCRAMBLED SETS AND THEIR PLACEMENT IN SARKOVSKII'S STRATIFICATION OF THE REAL NUMBERS

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Abstract. We are interested in being able to pinpoint the existence of f-invariant δ-scrambled sets in the turbulent stratification presented by Block and Coppel[1] which builds on Sarkovskii's Ordering. In this paper, we give the properties which guarantee that map of odd period will admit an f-invariant δ-scrambled subset. We also demonstrate that any map f which admits such a set has the property that f^2 is turbulent. We follow with independent proofs that f-invariant δ-scrambled sets exist in all maps that are strictly turbulent, and that maps with f-invariant δ-scrambled sets have periodic points of period 2^k for every $k > 0$.

1. Introduction

Throughout this paper, let $f : I \to I$ be continuous where I is an interval in $\mathbb{R}$.

Definition 1.1. *The point x is a* fixed point *f if $f(x) = x$. The point x is a* periodic point *of period n if $f^n(x) = x$.*

As presented by Devaney [2], Sarkovskii was able to demonstrate when the existence of points with certain periods in a map imply the existance points with other periods for continuous maps in some interval of $\mathbb{R}$.

The following is *Sarkovskii's Ordering* of the natural number:

$$3 \rhd 5 \rhd 7 \rhd ... \rhd 2 * 3 \rhd 2 * 5 \rhd ... \rhd 2^2 * 3 \rhd 2^2 * 5 \rhd$$
$$... \rhd 2^3 * 5 \rhd ... \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

Theorem 1.2. [2] *Suppose $f : I \to I$ is continuous. Suppose f has a periodic point of period k. If $k \rhd \ell$ in the above ordering, then f also has a periodic point of period ℓ.*

Notice that maps with a periodic point of period 3, have periodic points of all other periods.

Definition 1.3. *A map $f : I \to I$ is* turbulent *if $\exists$ compact subintervals J and K with at most one common point where*

$$J \cup K \subseteq f(J) \cap f(K).$$

f is strictly turbulent *if J and K can be chosen disjoint.*

We can now define a chaotic map in the sense used by Block and Coppel [1].

Theorem 1.4. [1] *The following conditions are equivalent for f:*

(1) *f has a periodic point whose period is not a power of* 2
(2) *f^m is strictly turbulent for some positive integer m*
(3) *f^n is turbulent for some positive integer n*

Definition 1.5. [1] *A map f is* chaotic *if any of the above conditions hold for f.*

Let $\mathbb{S}_k$ denote the set of maps f for which f^k is strictly turbulent, $\mathbb{T}_k$ the set of maps f for which f^k is turbulent, $\mathbb{P}_k$ the set of maps with periodic points of period k, and $\mathbb{K}$ the set of all chaotic maps. With this notation, Block and Coppel [1] used Sarkovskii's Ordering to come up with the following *turbulent stratification.*

$$\mathbb{S}_1 \subset \mathbb{T}_1 \subset \mathbb{P}_3 \subset \mathbb{P}_5 \subset ... \subset \mathbb{S}_2 \subset \mathbb{T}_2 \subset \mathbb{P}_6 \subset \mathbb{P}_{10} \subset$$
$$... \subset \mathbb{S}_4 \subset \mathbb{T}_4 \subset \mathbb{P}_{12} \subset \mathbb{P}_{20} \subset$$
$$... \subset \mathbb{K} \subset ... \subset \mathbb{P}_8 \subset \mathbb{P}_4 \subset \mathbb{P}_2 \subset \mathbb{P}_1$$

We looked at the above stratification to investigate where specific sets might be in this stratification.

Definition 1.6. *A set $S \subset I$ is* δ-scrambled *if for $\delta > 0$, S is uncountable, and*
i. $\forall s_1, s_2 \in S$,

$$\limsup_{n\to\infty}(d[f^n(s_1), f^n(s_2)]) \geq \delta \tag{1}$$

$$\liminf_{n\to\infty}(d[f^n(s_1), f^n(s_2)]) = 0 \tag{2}$$

ii. $\forall s \in S$ *and for any periodic point z,*

$$\limsup_{n\to\infty}(d[f^n(s), f^n(z)]) \geq \delta \tag{3}$$

Definition 1.7. *A set $S \subset I$ is* f-invariant *if for any $s \in S$, $f(s) \in S$.*

Our goal is to say where in the turbulent stratification maps must admit δ-scrambled and f-invariant sets. It is well known that all chaotic maps admit a δ-scrambled set, but it was not previously known if they would admit one which is additionally f-invariant. We attacked this problem from two angles. First, tried to prove that certain positions in the turbulence stratification forced maps to admit f-invariant δ-scrambled sets, such as maps that are strictly turbulent or of odd period. Second, we tried to determine where in the stratification maps which admit δ-scrambled f-invariant sets were forced to be by that property.

2. Existence of f-Invariant δ-Scrambled Sets in Maps which Admit Points of Odd Period

It was shown by Harrison-Shermoen [4] that $\forall$ strictly turbulent maps $\exists$ an f-invariant δ-scrambled set , and by B-S Du [3] that $\forall$ turbulent maps there exists an f-invariant δ-scrambled set . We begin with an investigation into the properties under which we know that a map of odd period admits an f-invariant δ-scrambled subset instead of a f^2-invariant δ-scrambled set.

In the setup for this characterization, we will draw on a proof of Sarkovskii's Theorem which appears in Devaney [2]. For two closed intervals I_1 and I_2 we say

$I_1 \to I_2$ if and only if $I_2 \subset f(I_1)$. Let $x \in I$ be a periodic point of period n, where n is the minimum odd period of a point under f. $\forall i \in [1,n] \cup \mathbb{Z}^+$ let x_i be the points in the orbit of x, enumerated from left to right. Choose the largest i for which $f(x_i) > x_i$. Let I_1 be the interval $[x_i, x_{i+1}]$. Then we must have $I_1 \subset f(I_1)$, so $I_1 \to I_1$. Because n is odd, $n \neq 2$, so $f(I_1) \neq I_1$. Thus there must be some other such interval of the form $[x_j, x_{j+1}]$. Let Ω_2 be the union of intervals of this form that are covered by $f(I_2)$. Let some interval of this form be called $I_2 \subset \Omega_2$, and observe that $I_1 \to I_2$. Recursively let Ω_{k+1} be the union of intervals of the form $[x_j, x_{j+1}]$ covered by the image of some interval in Ω_k. Note that if I_{k+1} is any interval in Ω_{k+1}, there is a collection of intervals $I_2, ...I_k$ with $I_j \subset \Omega_j$ which satisfy $I_1 \to I_2 \to ... \to I_k \to I_{k+1}$. Because there are finitely many such x_js, there is some k for which $\Omega_k = \Omega_{k+1}$. Because x_j has period n for all $j \leq n$, we know that Ω_k contains all intervals of $[x_j, x_{j+1}]$. Because n is odd, there are more x_is on one side of I_1 than the other, so at least one must change sides under f, which means that there is some I_j for which $I_1 \subset f(I_j)$. Now consider a chain of intervals of the form $I_1 \to I_2 \to ... \to I_j \to I_1$, where all of the intervals are chosen distinct. By Devaney [2], these conditions mean that $j = n-1$. Let this chain determine the names of the $n-1$ base intervals we will use in this proof. We know that each point in I_i has a pre-image in $I_{i-1 \mod n}$ and that each point in I_1 has a pre-image in I_1.

Let $a_i \in [1, n-1] \cap \mathbb{Z}^+, \forall i \in \mathbb{Z}^+$. Given compact intervals $I_1, I_2, I_3, ...I_{n-1}$, let $I_{a_1 a_2}$ be the compact subinterval of minimum length for which $I_{a_1 a_2} \subset I_{a_1}, f(I_{a_1 a_2}) = I_{a_2}$. Recursively define $I_{a_1 a_2 ... a_n}$ as the compact subinterval of $I_{a_1 a_2 ... a_{n-1}}$ of minimum length for which $f(I_{a_1 a_2 ... a_n}) = I_{a_2 ... a_n}$.

Definition 2.1. *Let Σ^n be the space of infinite binary strings $(a_1 a_2 ...)$ with $a_i \in [1, n-1] \cap \mathbb{Z}^+$*

Definition 2.2. *Let σ be the shift operator on a binary string $\alpha = (a_1 a_2 a_3 ...) \in \Sigma^n$ such that $\sigma(\alpha) = (a_2 a_3 ...)$.*

Definition 2.3. *For $\alpha \in \Sigma^n$, define I_α as $\bigcap_{n=1}^{\infty} I_{a_1 a_2 ... a_n}$.*

Because I_α is the nested intersection of countably many compact intervals, I_α is either an interval or a point. Consider the case where I_α is a point. Observe $f(I_\alpha) = f(\bigcap_{n=1}^{\infty} I_{a_1 a_2 ... a_n}) \subset \bigcap_{n=0}^{\infty} f(I_{a_1 a_2 ... a_n}) = \bigcap_{n=1}^{\infty} I_{a_2 ... a_n} = I_{\sigma(\alpha)}$. But by uniform continuity, we can see that $I_{\sigma(\alpha)}$ is also a point, so in fact $f(\bigcap_{n=0}^{\infty} I_{a_1 a_2 ... a_n}) = \bigcap_{n=0}^{\infty} f(I_{a_1 a_2 ... a_n})$.

Observe that all we know about the intervals we are using is that $\forall i \in [1, n-1] \cap \mathbb{Z}^+, I_i$ has a pre-image in I_{i-1}, and I_1 has an additional pre-image in I_1. Therefore, the only strings of the above type of either finite or infinite length which we may be sure exist by construction of the above intervals have the property that if $a_i = h$, $a_{i+1} = (h - 1 \mod n-1)$ if $h \neq 1$, else a_{i+1} may equal 1 or $n-1$.

Proposition 2.4. *If f admits a periodic point of period n, and $\forall k \in [1, n-1] \cap \mathbb{Z}^+, I_k$ may be picked so that one or more of the following conditions hold:*
(1): $I_{1111...}$ is a point

(2): clo(int[$I_1 \backslash$int($I_{1111...}$)]) is connected

(3): WLOG assume I_2 is to the right of I_1, then $\exists x \in I_1 \backslash int I_{1111...}$ s.t. $f(x) > \max I_{1111...}, f(x) = \max I_2$

Then f admits an f-invariant δ-scrambled set.

Proposition 2.5. *If none of the above properties hold for f, then f admits an f^2-invariant δ-scrambled set.*

We will begin by proving the first proposition, and then sketch to the second.

Lemma 2.6. *If (1) does not hold but either (2) or (3) does, we may select $J \subset I_1$ for which $J \to J$ and $J \to I_2$, for which if we reassign $I_1 = J$, (1) will hold for the new I_1.*

Proof. Assume (2) holds. Then let $J = \text{clo}(\text{int}[I_1 \backslash \text{int}(I_{1111...})])$, and note that J is a compact interval. Because $I_1 \to I_1$, $I_1 \to I_2$, and $I_{1111...}$ is invariant under f, we know that the parts of I_1 whose image covers J and I_2 are in J. Thus $J \to J$ and $J \to I_2$.
Now assume that (3) holds and (2) does not. Let $y = \min\{x \in I_1 \backslash \text{int} I_{1111...} | x > \min I_{1111...}, f(x) = \max I_2\}$. Then let $z = \max\{x \in I_1 \backslash \text{int} I_{1111...} | x > \min I_{1111...}, x \leq y$. Then let $J = [y, z]$. By continuity, $J \to J, J \to I_2$. □

Redefine I_1 as J, changing the definition of I_α as necessary.

Let $v \in (0, 1)$. $\forall i \in \mathbb{Z}^+ \cup \{0\}$, let v_i be the ith digit after the decimal place in the decimal expansion of v, that is, $v_i = \lfloor (10^i * v \mod 10) \rfloor$. Let t_i be the string of ones of length $\lfloor 10^i * v \rfloor$. Let $r = (n-1)(n-2)(n-3)...(1)(1)$, that is, the decreasing string beginning with $n-1$ and ending with 11. Let w_i be a string of ones of length i. Then let $u_i = r w_i r w_i r \cdots r w_i r$, so that w_i is repeated i times. Consider the set:

$$S' = \{I_\alpha | \alpha = (t_1 u_1 t_2 u_2 t_3 u_3 t_4 u_4 ...), \mu(I_\alpha) = 0\}$$

Observe that α has no adjacent js, where $j \neq 1$, which means that $\alpha \in \Sigma^n$. Clearly, S' is uncountable, as its composition encodes uncountably many irrational numbers as long strings of ones, and there cannot be uncountably many intervals within a finite interval, so there must be uncountably many singletons. For each $\alpha \in S'$, let $v^\alpha, t_i^\alpha, u_i^\alpha$ be the numbers or sequences v, t_i, and u_i respectively for that particular α.

Let $S = \{f^n(s) | n \in \mathbb{Z}^+ \cup \{0\}, s \in S'\}$. By construction, S is uncountable and f-invariant.

Lemma 2.7. *Let $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$. Given distinct strings $\sigma^{k_1}(\alpha)$ and $\sigma^{k_2}(\beta)$ for which $I_\alpha, I_\beta \in S'$, $\exists$ some $n \in \mathbb{Z} \cup \{0\}$ for which $\sigma^{k_1+n}(\alpha)$ begins with $(n-1)(n-2)(n-3)...(1)(1)$ and $\sigma^{k_2+n}(\beta)$ is a string of ones of length $n+1$.*

Proof. If $\alpha \neq \beta$, $\exists j \in \mathbb{Z}^+$ for which $v_j^\alpha \neq v_j^\beta$. Without loss of generality say $v_j^\alpha \geq v_j^\beta$. Then $\forall i \geq j, |t_i|$ is greater in α than in β. Whatever the initial displacement of the strings at this point, from now on there will be longer strings of ones in the α string than in the β string. That is, there is some integer j_2 for which if $\sigma^{k_1+n_1}(\alpha)$ is the first character of the string $u_{j_2}^\alpha$ and $\sigma^{k_2+n_2}(\beta)$ is the first character of the string $u_{j_2}^\beta$, then $n_1 > n_2$. Redefine k_1 as $k_1 + n_2$, and redefine k_2 as $k_2 + n_2$. As the gaps between the sequences get arbitrarily large, we will be able to pick some i after this point for which the entire run $t_i^\beta u_i^\beta$ will line up entirely with ones from α.

Now pick an n to advance the beginnings of the sequences to the $n-1$ beginning u_i^β and the condition will be fulfilled.

If $\alpha = \beta$, let $k = |k_1 - k_2|$ pick some $j \in \mathbb{Z}^+$ for which $|t_j| \geq k + h$. Then if we pick an n which advances us to the beginning of this subsequence in the string that is further ahead, we still have at least h ones left at the beginning of each sequence, fulfilling the condition. □

Lemma 2.8. *Let $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$, and let $h \in \mathbb{Z}^+$. Then, given distinct binary strings $\alpha, \beta \in S^{'}$, $\forall h \in \mathbb{Z}^+$, $\exists$ an n for which the next h characters of both $\sigma^{n+k_1}(\alpha)$ and $\sigma^{n+k_2}(\beta)$ are all one.*

Proof. Let $j = |k_1 - k_2|$. Repeat the steps in the previous Lemma to redefine the sequence so that without loss of generality α's sequences of ones are longer. Then we will again be able to pick some i after this point for which the entire run $t_i^\beta u_i^\beta$ will line up entirely with ones from α. If $|t_i^\beta| \geq j + h$, pick an n which advances the sequence to the first character of t_i^β and the condition will be fulfilled. Else, because the sequence of lengths t_i^β is strictly increasing we will eventually be able to find another such t_i^β for which the property holds. □

Lemma 2.9. *Let $i, j, j_2, k \in \mathbb{Z}^+ \cup \{0\}, l \in [1, n-1] \cap \mathbb{Z}^+$, $\alpha \in S^{'}$, and $z \in I$ s.t. z is periodic of period k. Then let $z_i = f^{i-1}(z)$. Now we may pick some $n \in \mathbb{Z}^+$ for which $f^n(z) = z_k$ and $f^{n+j}(I_\alpha)$ begins with l.*

Proof. Pick some $i \in \mathbb{Z}^+ \cup \{0\}$ for which k divides $n + i$, $i > kn$. Let $l_2, i_3, i_4 \in [1, n-1] \cap \mathbb{Z}^+$. Then pick some n which advances α to the beginning of the string u_i. Then every $n + i$ times we iterate f, we will move all the z_ks forward by one relative to the string α. In particular, by the time we have iterated $fk(n+i)$ times, we will have lined up each element in the interval u_i with a z_k at least once. By construction, $k(n+i) < i(n-1+i)$, the size of the interval, so we do not leave the interval during this process. Then for some $n_2 \leq k(n+i)$ by iterating fn_2 times after the first n, we will line up an l with z_k. □

Now we are ready to prove the first proposition.

Theorem 2.10. *If f admits a periodic point of period n, and $\forall k \in [1, n-1] \cap \mathbb{Z}^+, I_k$ may be picked so that one or more of the following conditions hold:*
(1): $I_{1111...}$ is a point

(2): clo(int[$I_1 \backslash$int($I_{1111...}$)]) is connected

(3): WLOG assume I_2 is to the right of I_1, then $\exists x \in I_1 \backslash intI_{1111...}$ s.t. $f(x) >$ $\max I_{1111...}, f(x) = \max I_2$

Then f admits an f-invariant δ-scrambled set.

Lemma 2.11. *Given two points $f^k(I_\alpha), f^j(I_\beta) \in S$ such that $\alpha \neq \beta$ and $\epsilon \in \mathbb{R}^+$, $\exists n \in \mathbb{Z}^+$ such that $d(f^{j+n}(I_\alpha), f^{k+n}(I_\beta)) \leq \epsilon$.*

Proof. Without loss of generality assume $j < k$.

Because $I_{111...}$ is a point, the sequence of intervals $I_{a_1 a_2 ... a_n}$ where $a_i = 1 \, \forall i \in \mathbb{Z}^+$ has length converging to zero as n approaches infinity. Then given any $\epsilon \in \mathbb{Z}^+$, we may find an $i \in \mathbb{Z}^+$ for which the length of $I_{a_1 a_2 ... a_n}$ is less than $\epsilon \forall n \geq i$

By lemma, pick some $n_1 \in \mathbb{Z}^+$ for which the first i terms of $\sigma^{n+k_1}(\alpha)$ are 1 and overlap with the first i terms of $\sigma^{n+k_2}(\beta)$. Then $d(f^{j+n}(I_\alpha), f^{k+n}(I_\beta)) \leq \epsilon$, where $n = \sum_{h=1}^{\infty} n_h$. Because ϵ and the starting points were arbitrary, this proves that $\liminf_{n\to\infty}(d[f^n(s_1), f^n(s_2)]) = 0$. □

Now we will divide into cases based on the period of n:

Case I: $n = 3$.

We will begin by restricting what we mean by I_1 and I_2.

Case I A: $\min(I_1^{'}) < \min(I_2^{'})$. Let $a = \max\{x \in I_1 | f(x) = \max(I_2)\}$. Then let $b = \min\{x \in I_1 | f(x) = a, x > a\}$. Redefine $I_1 = [a, b]$, and observe that we still have $I_1 \to I_1$ and $I_1 \to I_2$.

Case I B: $\min(I_1^{'}) > \min(I_2^{'})$. Let $b = \min\{x \in I_1 | f(x) = \min(I_2)\}$. Then let $a = \max\{x \in I_1 | f(x) = a, x < b\}$. Redefine $I_1 = [a, b]$, and observe that we still have $I_1 \to I_1$ and $I_1 \to I_2$.

If $d(I_1, I_2) = 0$, let $\delta = \min\{d(I_{211}, I_1), d(I_{12}, I_2), \mu(I_1), \mu(I_2)\}$. If $d(I_1, I_2) > 0$, let $\delta = \min\{d(I_{12}, I_2), d(I_{211}, I_1), \mu(I_1), \mu(I_2), \frac{d(I_0,I_1)}{3}\}$.
Then $\delta > 0$ iff $\min\{d(I_{12}, I_2), d(I_{211}, I_1)\} > 0$.

Lemma 2.12. $d(I_{211}, I_1) > 0$

As above, $I_1 = [a, b]$, let $I_2 = [c, d]$.
Case I a: $a < \min(I_2)$.

Proof. We know $f(b) = a$, and that $\forall x \in I_1 \backslash b, f(x) \neq a$, so $b \in I_{11}$. Because I_{11} is a subinterval of $I_1, \min(I_{11}) > a$.We know that $f(c) \leq a$, so $c \notin I_{211}$. Because f is uniformly continuous, and $d(\min(I_{11}), a) > 0$, we know that $d(I_{211}, c) > 0$. Then because c is the closest point in I_2 to I_1, we have that $d(I_{211}, I_1) > 0$, which completes the proof. □

Case I b: $a < \min(I_2)$.

Proof. We know $f(a) = b$, and that $\forall x \in I_1 \backslash a, f(x) \neq b$, so $a \in I_{11}$. Because I_{11} is a subinterval of $I_1, \max(I_{11}) < b$.We know that $f(d) \geq b$, so $d \notin I_{211}$. Because f is uniformly continuous, and $d(\max(I_{11}), b) > 0$, we know that $d(I_{211}, d) > 0$. Then because d is the closest point in I_2 to I_1, we have that $d(I_{211}, I_1) > 0$, which completes the proof. □

The proof that $d(I_{12}, I_2) > 0$ is similar.

Lemma 2.13. *Given two points $f^{k_1}(I_\alpha), f^{k_2}(I_\beta) \in S$ such that $\alpha \neq \beta$, $\exists n \in \mathbb{Z}^+$ such that $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$.*

Proof. By Lemma 1, pick some n for which $\sigma^{k_1+n}(\alpha)$ begins with 211 and $\sigma^{k_2+n}(\beta)$ begins with 1. Then by definition and the nested property of our intervals, $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$. □

Lemma 2.14. *Given some periodic point z and some $f^j(I_\alpha) \in S$, $\limsup_{n\to\infty}(d[f^n(z), f^{n+j}(I_\alpha)]) \geq \delta$.*

Proof. Case I a: The orbit of z at some point leaves the series of intervals we are considering.
WLOG, let this point be the $(k-1)$st iterate of z, where z has period k. Then by construction of $\delta, \exists$ some $l \in \{1, 2\}$ for which $d(z_k, I_l) > \delta$. By Lemma 2.9, we know we may find an n for which $f^{n+j}(I_\alpha) \subset I_l$ and $f^n(z) = z_k$.

Case I b: The orbit of z is constantly within the series of intervals we are considering.
Then we may encode z's trajectory through intervals in a sequence of 2 variables. Let $\gamma \in \Sigma^2$, for which $\gamma = (c_1c_2c_3...)$ such that $f^i(z) \in I_{c_i}$. Because the trajectory of z is periodic, γ must also be periodic. Assume γ contains a 1. WLOG, say $z_k \in I_1$. Then we may use Lemma 2.9 to pick an n which forces $f^{n+j}(I_\alpha) \subset I_2$ (which means it will be a subset of I_{211}, by construction) and $f^n(z) = z_k$, which will force our Lemma to be true. Otherwise, γ must contain a 2, we may assign $z_k \in I_2$, and we may again apply Lemma 2.9 to force z_k to line up with an element of I_12, proving our Lemma.

Case I c: The point z is an endpoint of more than one interval.
Then z will be the periodic point x on which these intervals are based, and we may easily line up one of the points in its orbit that does not bound I_1 with an arbitrarily long string of 1s, giving a distance greater than δ.

□

Case II: $n \neq 3$

Because there are 4 or more intervals we are considering, and each has at most 1 point in common, we know that there are some pairs of intervals which are completely disjoint. Let $\delta = \inf\{d(I_j, I_k)|k, j \in [1, n-1] \cap \mathbb{Z}^+, d(I_j, I_k) > 0\}$.

Lemma 2.15. *Given two points $f^{k_1}(I_\alpha), f^{k_2}(I_\beta) \in S$ such that $\alpha \neq \beta$, $\exists n \in \mathbb{Z}^+$ such that $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$.*

Proof. Assume β and α are distinct, then by Lemma 2.8, we know that we may pick some n for which without loss Of generality $\sigma^{k_1+n}(\alpha)$ begins with $(n-1)(n-2)...(1)(1)$ and $\sigma^{k_2+n}(\beta)$ is constantly 1. Because $1 < j \leq n-1$, we know we may pick some slightly greater n for which $\sigma^{k_1+n}(\alpha)$ begins with j and $\sigma^{k_2+n}(\beta)$ begins with 0. Then by definition of δ and the nested property of our intervals, $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$.

Now assume β and α are identical.
Case I a: The displacement $|k_1 - k_2| = 1$.
We know by Devaney [2] that there is some $j \in [2, n-1] \cap \mathbb{Z}^+$ for which I_j is on a different side of I_1 from I_{j+1}. When the strings are displaced by 1, we will have this j lined up with $j+1$ every time they occur, so if we advance the sequence by n to that place, we will find $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\alpha)) \geq 0$, and thus by definition of δ, $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\alpha)) \geq \delta$.

Case I b: The displacement $|k_1 - k_2| > 1$.
Before and after the strings u_i, we will have displacements of 2 or more on each side. This means that we will have a minimum of 4 intervals lined up with ones. Because the intervals are distinct, there are at most 2 which are adjacent to I_1, so by advancing the sequence by n to an overlap of some j for which $d(I_j, I_1) > 0$ with 1, we will have $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$.

Lemma 2.16. *Given some periodic point z and some $f^j(I_\alpha) \in S$,* $\limsup_{n\to\infty}(d[f^n(z), f^{n+j}(I_\alpha)]) \geq \delta$.

Proof. Case I a: The orbit of z at some point leaves the series of intervals we are considering. WLOG, let this point be the $k-1$st iterate of z, where z has period k. Then by construction of $\delta, \exists$ some $l \in [1, n-1] \cap \mathbb{Z}^+$ for which $d(z_k, I_l) > \delta$. By Lemma 2.9, we know we may find an n for which $f^{n+j}(I_\alpha) \subset I_l$ and $f^n(z) = z_k$.

Case I b: The orbit of z is constantly within the series of intervals we are considering. Then we may encode z's trajectory through intervals in a sequence of $n-1$ variables. Let $\gamma \in \Sigma^n$, for which $\gamma = (c_1c_2c_3...)$ such that $f^i(z) \in I_{c_i}$. Because the trajectory of z is periodic, γ must also be periodic. Since we have 4 or more intervals, we may pick any $h \in [1, n-1] \cap \mathbb{Z}^+$ for which $z_k \in I_h$, and we know that there will be an $l \in [1, n-1] \cap \mathbb{Z}^+$ for which $d(I_h, I_l) > \delta$. Then we may apply Lemma 2.9 to find the appropriate n for which $f^n(z) = z_k$ and $f^{n+j}(I_\alpha) \in I_h$ to complete the proof.

Case I c: The point z is an endpoint of more than one interval.
Then z will be the periodic point x on which these intervals are based, and we may easily line up one of the points in its orbit that does not bound I_1 with an arbitrarily long string of 1s, giving a distance greater than δ. □

□

Now consider the case where none of the three properties of the first proposition hold. Then let $I_{1'} = [\max I_{1111...}, \max I_1]$ and redefine $I_1 = [\min I_1, \min I_{1111...}]$. With this notation, observe that $I_{11'11'11'...}$ will be a point on the boundary of the removed invariant interval and $I_{1'11'11'1...}$ will be its image. We may define a set S as before, except now instead of strings of ones we will use strings of double length of $11'$s. Then if we allow only f^2 iterates of points in the set to be used, we may line up arbitrarily long sequences of $11'$s and reproduce the above lemmas for lim inf. Because $I_{n-1} \to I_1$ and $I_{n-1} \to I_{11'}$, we may follow either ones or $1'$s by $n-1$s, so we may alternate between sequences of even and odd length. This will insure that we may line up an I_1 or $I_{1'}$ with some I_j and get a distance greater than δ in a method parallel to the above construction.

3. Maps with f-invariant δ-scrambled Sets are in $\mathbb{T}_2$

Here we use a theorem proved in Block and Coppel [1] to argue by contradiction that any map f which admits an f-invariant δ-scrambled set is in $\mathbb{T}_2$. Here are some relivant definitions.

Definition 3.1. [1] *A trajectory $\{f^n(c)\}$ will be said to be* alternating *if either $f^k(c) < f^j(c)$ for all even k and all odd j, or $f^k(c) > f^j(c)$ for all even k and all odd j.*

Definition 3.2. [1] *We define the* limit set *of a point $x \in I$ to be the set*

$$\omega(x) = \omega(x, f) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} f^n(x)}$$

.

Equivalently, $y \in \omega(x)$ if and only if y is a limit point of the trajectory $\gamma(x)$, i.e. $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$.

The following is a useful Lemma from Block and Coppel:

Lemma 3.3 (Block and Coppel IV.4)**.** *A limit sets $\omega(x)$ contains only finitely many points $\iff$ x is asymptotically periodic.*

If $\omega(x)$ contains infinitely many periodic points, then no isolated point of $\omega(x)$ is periodic.

Theorem 3.4. [1] *Suppose f^2 is not turbulent. Then $\forall c \in I$ exactly one of the following alternatives holds:*

(i): the trajectory $\{f^k(c)\}$ is bimonotinic and converges to a fixed point of f

(ii): the trajectory $\{f^k(c)\}$ is alternating from some point on and the limit set $\omega(c, f)$ is an f-orbit of period 2.

(iii): the trajectory $\{f^k(c)\}$ is alternating from some point on and if we set

$$\alpha = \min \omega(c, f^2), \beta = \max \omega(c, f^2), \tag{4}$$

$$\gamma = \min \omega(f(c), f^2), \delta = \max \omega(f(c), f^2), \tag{5}$$

then $[\alpha, \beta]$ and $[\gamma, \delta]$ are disjoint non-degenerate intervals which contain no fixed point of f.

Proposition 3.5. *If f admits an f-invariant δ-scrambled set, then f^2 is turbulent.*

Proof. Let $S \subset I$ be an f-invariant δ-scrambled set. Let $c \in S$. TAssume (i) or (ii) holds. But then the limit set is finite, so c is asymptotically periodic by the above Lemma from B and C, so we have a contradiction against the property of δ-scrambled that $\limsup_{n\to\infty}(d[f^n(s), f^n(z)]) \geq \delta$ for any $s \in S$ and periodic $z \in I$.

If the trajectory $\{f^k(c)\}$ is not alternating from some point on then (iii) fails. Assume that the trajectory $\{f^k(c)\}$ is alternating from some point on. Then we know that if k_1 is odd and k_2 is even, then without loss of generality $f^{k_1}(c) < f^{k_2}(c)$.

Let $w_1 = \sup\{f^{2k+1}(c) | k \in \mathbb{N}\}$ and $w_2 = \inf\{f^{2k}(c) | k \in \mathbb{N}\}$. Because $\liminf_{n\to\infty}(d[f^n(c), f^{n+1}(c)]) = 0$, we must have $w_1 = w_2 = w$. But then by continuity, $w \in \omega(c, f^2)$ and $w \in \omega(f(c), f^2)$, so (iii) still fails. Thus f^2 must be turbulent. □

4. All Strictly Turbulent Maps Admit f-invariant δ-scrambled subsets

As mentioned previously, [4] showed that all strictly turbulent maps admit f-invariant δ-scrambled sets. We produced an independent proof of this result.

Let f be strictly turbulent. Then $\exists$ compact intervals $I_0^{'} = [a^{'}, b^{'}]$, $I_1^{'} = [c^{'}, d^{'}]$, s.t. $I_0^{'} \cup I_1^{'} \subset f(I_0^{'}) \cap f(I_1^{'})$ Thus $I_0^{'} \subset f(I_0^{'})$, so there is at least one fixed point in $I_0^{'}$. Call the set of such fixed points F, and observe that the set is non-empty and closed. Because $I_1^{'} \subset f(I_0^{'})$, the set $K = \{x \in I_0^{'} | f(x) = d^{'}\}$ is also non-empty and closed by continuity. By these two conditions, $\exists$ two points $a^{''} \in F, b^{''} \in K$ for which $d(a^{''}, b^{''}) = d(F, K)$, where $d(F, K)$ has the usual definition $\inf_{x \in F, y \in K} d(x, y)$. Let $a = \min\{a^{''}, b^{''}\}$, $b = \max\{a^{''}, b^{''}\}$. Let $I_0 = [a, b]$. Similarly select $c, d \in I_1^{'}$ and let $I_1 = [c, d]$. Now by construction, $I_0 \cup I_1 \subset f(I_0) \cap f(I_1)$

Let $a_i \in \{0, 1\}$, $\forall i \in \mathbb{Z}^+ \cup \{0\}$. By continuity of f, we know $\exists$ intervals in I_{a_0} which are pre-images of I_0 and I_1 under f, for either value of a_0. Now let $I_{a_0 a_1}$ be the subinterval of I_{a_0} of minimum width for which $f(I_{a_0 a_1}) = I_{a_1}$. Without Loss Of Generality force $I_{a_0 a_1}$ to include its endpoints, then $I_{a_0 a_1}$ is compact.

Definition 4.1. *Define $I_{a_0 a_1 a_2 \ldots a_n}$ as the subinterval of $I_{a_0 a_1 a_2 \ldots a_{n-1}}$ of minimum length for which $f(I_{a_0 a_1 a_2 \ldots a_n}) = I_{a_1 a_2 \ldots a_n}$.*

All of these intervals will be constructed to be compact by repetition of the above argument. In addition, it follows from the definition that the intervals are nested, that is, $I_{a_0 a_1 a_2 \ldots a_{n-1} a_n} \subset I_{a_0 a_1 a_2 \ldots a_{n-1}} \forall n \in \mathbb{Z}^+$.

Definition 4.2. *Let $\Sigma = \{\alpha | \alpha = (a_0 a_1 a_2 \ldots)$, for $a_i \in \{0, 1\}\}$ be the set of all infinite binary strings.*

Definition 4.3. *For some $\alpha \in \Sigma$, define I_α as $\bigcap_{n=0}^{\infty} I_{a_0 a_1 a_2 \ldots a_n}$.*

Because I_α is the nested intersection of countably many compact intervals, I_α is either an interval or a point. For consider the case where I_α is a point. Observe $f(I_\alpha) = f(\bigcap_{n=0}^{\infty} I_{a_0 a_1 a_2 \ldots a_n}) \subset \bigcap_{n=0}^{\infty} f(I_{a_0 a_1 a_2 \ldots a_n}) = \bigcap_{n=0}^{\infty} f(I_{a_1 a_2 \ldots a_n}) = I_{\sigma(\alpha)}$. But by uniform continuity, we can see that $I_{\sigma(\alpha)}$ is also a point, so in fact in this case $f(\bigcap_{n=0}^{\infty} I_{a_0 a_1 a_2 \ldots a_n}) = \bigcap_{n=0}^{\infty} f(I_{a_0 a_1 a_2 \ldots a_n})$.

Definition 4.4. *Let σ be the operation on a binary string $\alpha = (a_0 a_1 a_2 a_3 \ldots) \in \Sigma$ such that $\sigma(\alpha) = (a_1 a_2 a_3 \ldots)$. Observe that $f(I_\alpha) = f(I_{a_0 a_1 a_2 a_3 \ldots}) = I_{a_1 a_2 a_3 \ldots} = I_{\sigma(\alpha)}$.*

$\forall i \in \mathbb{Z}^+ \cup \{0\}$, let t_i be the string of zeros of length 2^i, u_i be the string of ones of length 2^i. Consider the set:

$$S' = \{I_\alpha \mid \alpha = (*_0 t_0 *_1 *_0 u_1 *_2 *_1 *_0 t_2 *_3 *_2 *_1 *_0 u_3 \ldots), \text{ for } *_i \in \{0, 1\}\}$$

This set is uncountable, so it must contain uncountably many α's for which I_α is a singleton, i.e. $I_\alpha = x_\alpha$. Let $S = \{f^n(I_\alpha) \mid I_\alpha \in S', \mu(I_\alpha) = 0, n \in \mathbb{Z}^+ \cup \{0\}\}$. Observe that then if $f^n(I_\alpha) \in S$, $f^n(I_\alpha) = \{x_{\sigma^n(\alpha)}\}$, a singleton.

Lemma 4.5. *Given our construction of I_0, I_{t_∞} is a point and not an interval.*

Proof. Recall that $I_{t_\infty} = \bigcap_{n=0}^{\infty} I_{a_0 a_1 a_2 \ldots a_n}$, where $a_i = 0$, $\forall i \in \mathbb{Z}^+$. Because f is continuous, and $I_0 \cup I_1 \subset f(I_0)$, some of the range of I_0 under f is not in I_0, so the length of I_{00} is smaller than the length of I_0. By a similar argument, it can be seen the that length of $I_{a_0a_1a_2\ldots a_{n-1}}$ is smaller than the length of $I_{a_0a_1a_2\ldots a_n}$, $\forall n \in \mathbb{Z}^+$ so we have that the sequence of lengths of these intervals is strictly decreasing.

Assume I_{t_∞} is an interval, i.e. that there is a positive lower limit to the decreasing of the length of the interval. Then I_{t_∞} is preserved under f, that is $f(I_{t_\infty}) = I_{\sigma(t_\infty)} = I_{t_\infty}$. We know a is a fixed point by construction, and because f does not cross the line $y = x$ again in I_0, we know it cannot in I_{t_∞}. But this is a contradiction as it is graphically impossible to have a fixed interval under a continuous mapping which has a fixed endpoint and no other fixed points, so I_{t_∞} must be a point. □

Lemma 4.6. *Let $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$. Given distinct binary strings $\sigma^{k_1}(\alpha)$ and $\sigma^{k_2}(\beta)$, $\exists$ an $n \in \mathbb{Z} \cup \{0\}$ for which the first digit of $\sigma^{n+k_1}(\alpha)$ is not equal to the first digit of $\sigma^{n+k_2}(\beta)$.*

Proof. Let $j = |k_1 - k_2|$.

Case 1: $j > 0$: The strings have the same initial structure, and are displaced relative to each other by j applications of σ, so if we look far enough ahead in the string, we will see a place in which the string of constant zeroes t_i overlaps exactly the last j of the string of $*$s. If $*_1 = 1$, apply n to both strings until the first character of the binary string is $*_1$, lined up with a 0 from t_i. Else, apply n until $*_1$ is lined up with a 1 from u_{i+1}.

Case 2: $j = 0$: Because the strings are lined up, the indices on the strings of *s will be lined up. Because the strings are not identical, $\exists n \in \mathbb{Z}^+ \cup \{0\}$ such that $*_n$ in α is different from $*_n$ in β. When σ is applied enough times to bring the matched $*_n$s to the front of their binary strings, the Lemma's condition will be fulfilled. □

Lemma 4.7. *Let $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$, Without loss of generality let $k_1 > k_2$. Then, given binary strings $\sigma^{k_1}(\alpha)$ and $\sigma^{k_2}(\beta)$, $\forall j \in \mathbb{Z}^+$, $\exists$ an n for which the next i characters of $\sigma^{n+k_1}(\alpha)$ and $\sigma^{n+k_2}(\beta)$ are both zero.*

Proof. Again, let $j = |k_1 - k_2|$. Find some n_1 in the even integers for which $2^{n_1} > j + i$. Then if we apply σ enough times to $\sigma^{k_1}(\alpha)$ to advance i positions into t_{n_1} or some longer sequence of zeroes, there will still be at least i zeros which the strings will have in common. □

Theorem 4.8. *If f is strictly turbulent, then $\exists$ a δ-scrambled f-invariant subset of S under f.*

Proof. Let $\delta = \frac{d(I_0,I_1)}{3}$, then $\delta > 0$ because f is strictly turbulent. Then, if two points I_α and I_β are in different intervals, $d(I_\alpha, I_\beta) \geq \delta$. Recall that the I_α is in I_0 if the first entry in the string is 0, else I_α is in I_1

Lemma 4.9. *Given two points $f^{k_1}(I_\alpha), f^{k_2}(I_\beta) \in S$ such that $\alpha \neq \beta$, $\exists n \in \mathbb{Z}^+$ such that $d(f^{k_1+n}(I_\alpha), f^{k_2+n}(I_\beta)) \geq \delta$.*

Proof. By Lemma, there is an n for which the first digit of $\sigma^{n+k_1}(\alpha)$ is not equal to the first digit of $\sigma^{n+k_2}(\beta)$. We know $\sigma^{n+k_1}(\alpha) = f^{n+k_1}(I_\alpha)$ and $\sigma^{n+k_2}(\beta) = f^{n+k_2}(I_\beta)$, so we may apply fn times to $f^{k_1}(I_\alpha)$ and $f^{k_2}(I_\beta)$ and find that
$d(f^{j+n}(I_\alpha), f^{k+n}(I_\beta)) \geq \delta$. □

By applying this lemma to arbitrary points and iterates of them, we may see that $\limsup_{n\to\infty}(d[f^{k+n}(I_\alpha), f^{j+n}(I_\beta)]) \geq \delta$.

Lemma 4.10. *Given two points $f^k(I_\alpha), f^j(I_\beta) \in S$ such that $\alpha \neq \beta$ and $\epsilon \in \mathbb{R}^+$, $\exists n \in \mathbb{Z}^+$ such that $d(f^{j+n}(I_\alpha), f^{k+n}(I_\beta)) \leq \epsilon$.*

Proof. Without loss of generality assume $j < k$.

Because $I_{000...}$ is a point, the sequence of intervals $I_{a_0a_1a_2...a_n}$ where $a_i = 0 \forall i \in \mathbb{Z}^+$ has length converging to zero as n approaches infinity. Then given any $\epsilon \in \mathbb{Z}^+$, we may find an $i \in \mathbb{Z}^+$ for which the length of $I_{a_0a_1a_2...a_n}$ is less than $\epsilon \forall n \geq i$ By lemma, pick some $n_1 \in \mathbb{Z}^+$ for which the first i terms of $\sigma^{n+k_1}(\alpha)$ are 0 and overlap with the first i terms of $\sigma^{n+k_2}(\beta)$. Then $d(f^{j+n}(I_\alpha), f^(k+n)(I_\beta)) \leq \epsilon$, where $n = \sum_{h=1}^{\infty} n_h$. Because ϵ and the starting points were arbitrary, this proves that $\liminf_{n\to\infty}(d[f^n(s_1), f^n(s_2)]) = 0$. □

Lemma 4.11. *$\exists \delta \geq 0$, $n \in \mathbb{Z}^+$ $\forall s \in S, z \in I$ s.t. z is periodic, $\limsup_{n\to\infty}(d[f^n(s), f^n(z)]) \geq \delta$*

Proof. Let δ be defined as above. Then pick some periodic point z and some $s = x_\alpha \in S$.

Case 1: The trajectory of z under f never leaves $I_0 \cup I_1$. Then we may encode z's trajectory between these intervals in a binary sequence. Let $\gamma \in \Sigma$, for which $\gamma = (c_0c_1c_2c_3...)$ such that $f^i(s) \in I_{c_i}$. Because the trajectory of z is periodic, γ must also be periodic. We know that α is not periodic or eventually periodic, so there must some place in the string in which γ and α differ. If we use the σ map to bring these to the front of the strings, we will have the corresponding points in different intervals under the f map. Thus $\exists$ an n for which $d[f^n(s), f^n(z)] > \delta$.

Case 2: The trajectory of z under f leaves $I_0 \cup I_1$ at some point w. Let k be the period of z. By construction of S, there exists an n for which $f^n(s)$ has an arbitrary length string of zeros or ones. If $d(w, I_0) > d(w, I_1)$, then $d(w, I_0) > \delta$, so if we select a high enough n to get an overlapping string of zeros with length greater than k, we can find some n_2 for which $\sigma^{n_2}(\alpha)$ begins with a 0 and $f^{n_2}(z) = w$. Then $d(f^{n_2}(s), f^{n_2}(z)) > \delta$. If $d(w, I_1) > d(w, I_0)$, we make a similar construction with a string of ones.

Because both these points are arbitrary, this proof may be applied countably many times to the iterates of the points under this method, so
$\limsup_{n\to\infty}(d[f^n(s), f^n(z)]) \geq \delta$. □

The set is uncountable and f-invariant by construction. □

5. Periodic Points in Maps with f-invariant δ-scrambled Sets

Finally, we show that maps with f-invariant δ-scrambled sets must have periodic points of period 2^k for every $k > 0$.

Lemma 5.1. *If $f : I \to I$ admits a δ-scrambled f-invariant set $S \subset I$ for some $\delta > 0$, then f^t has a periodic point of period 2 for every $t \geq 0$.*

Proof. Let $f : I \to I$ admit a δ-scrambled f-invariant set $S \subset I$ for some $\delta > 0$.

Now, let $t \geq 0$ be arbitrary and consider the map f^t. Consider the theorem below that comes from Block and Coppel.

Theorem 5.2. [1] *If f has no periodic point of period 2 then, for every $c \in I$, the trajectory $\{f^k(c)\}$ is bimonotonic and converges to a fixed point of f.*

If f^t does not have a periodic point of period 2, then according to the theorem above from [1], for every $c \in I$ the trajectory $\{f^{kt}(c)\}$ converges to a fixed point of f^t. Now take a point $s \in S$. Since s is obviously in I we can say,

$$f^{kt}(s) \to z$$

where z is a fixed point of f^t. By continuity,

$$\begin{gathered} f(f^{kt}(s)) \to f(z) \\ f^2(f^{kt}(s)) \to f^2(z) \\ \vdots \\ f^n(f^{kt}(s)) \to f^n(z) \end{gathered}$$

Now $f^{kt} = s' \in S$ because S is f-invariant, and notice that z must be a periodic point of f since it is a fixed point of f^t.

$$\begin{gathered} f^n(s') \to f^n(z) \\ \lim_{n\to\infty} f^n(s') = \lim_{n\to\infty} f^n(z) \\ \lim_{n\to\infty} (d[f^n(s'), f^n(z)]) = 0 \\ \limsup_{n\to\infty}(d[f^n(s'), f^n(z)]) = 0 \not\geq \delta \end{gathered}$$

since $\delta > 0$

This contradicts the final part of the definition of a δ-scrambled set. Therefore, f^t must admit a periodic point of period 2.

□

Corollary 5.3. *If f admits an f-invariant δ-scrambled set for some $\delta > 0$, then f has a periodic point of period 2.*

Proposition 5.4. *If $f : I \to I$ admits a δ-scrambled f-invariant set $S \subset I$ for some $\delta > 0$, then it has periodic points of periods 2^k for every $k > 0$.*

Proof. From Lemma 5.1 above we know that f^t must admit a periodic point of period 2 for every $t > 0$. Let $t = 2^j$ where $j > 0$.

Now consider the below theorem from Block and Coppel.

Theorem 5.5. [1] *If c is a periodic point of f^h with period m, then c is a periodic point of f with period mh/d, where $d|h$ and $gcd(d, m) = 1$.*

Now using the theorem above, because f^{2^k} has a periodic point of period 2, then f will have a periodic point with period $\frac{2(2^j)}{d}$ where $d|(2^j)$ and $gcd(d, 2) = 1$. Now because $gcd(d, 2) = 1$ we know that d must be odd.

We will restrict our search to when $d > 0$ since if $d < 0$ we could simply take $|d|$ as our new d and the above conditions would still hold true.

Now, there is only one odd integer, 1, that divides an integer in the form 2^j. So, $d = 1$ and f must have periodic points of periods $2(2^j)$. Because this works for every $j > 0$, we can say that f must have period points of periods $2(2^j) = 2^{j+1}$ for every $j > 0$. Let $j + 1 = k$. Thus, f has periodic points of periods 2^k for every $k > 1$.

So combining this with Corollary 5.3 from above, f has periodic points of periods 2^k for every $k > 0$. □

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