Analyzing the General Linear Piecewise Lexicographic Programming Problem and an Extension of the Fundamental Theorem of Linear Programming

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Abstract

The General Linear Piecewise Lexicographic Programming (GLPwLgP) problem is a multiple objective optimization problem with piecewise linear objective functions. We generalize previous results by changing and eliminating certain assumptions. We also extend the fundamental theorem of Linear Programming (LP) to include certain nonlinear, noncontinuous functions. Then we formulate an LP that provides the same optimal set as a GLPwLgP, when there are two objectives. Moreover, we show that there exists a bijection that maps corner optimal solutions for the GLPwLgP problem to basic optimal solutions for the LP problem.

Key words: Linear Programming, Multiple Objective Programming, Piecewise Linear Functions, Lexicographic Optimization

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1 Introduction

Linear Programming is useful in many different fields. It has been applied to everything from minimizing business costs to maximizing the efficiency of cancer therapy. Multiple Objective Linear Programming (MOLP) is an extension of Linear Programming (LP), with the difference being that MOLPs consider many linear objective functions. The standard form of an LP is $\min\{c^T x : Ax = b, x \geq 0\}$, where $c \in \mathbb{R}^n, b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. The standard form of an MOLP is

$$\min\{Cx : Ax = b, x \geq 0\},$$

where $C \in \mathbb{R}^{p \times n}$, $A \in \mathbb{R}^{m \times n}$, and $m \leq n$. In both the LP and MOLP we assume that A has full row rank. The *feasible region* is a polyhedral set containing all points that satisfy the constraints of the problem. We denote the feasible set by $\mathcal{P} = \{x : Ax = b, x \geq 0\}$.

There are two standard ways to find a minimum in a multiple objective linear program. One technique uses weights to express a modeler's preferences. The idea is to give each objective a weight and the aggregate the weighted objectives to form one objective. So, if $w \in \mathbb{R}^p$ and $w > 0$, we transform the MOLP into the following LP,

$$\min\{w^T Cx : Ax = b, x \geq 0\}.$$

The solutions for every possible positive set of weights forms the *efficient frontier*, and the solutions are called *pareto optimal solutions*. In other words, x^1 is pareto optimal if $\nexists\, x^2 \ni Cx^2 \leq Cx^1$, with strict inequality in one component.

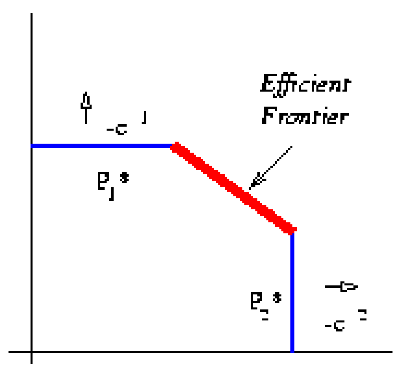

Figure 1: The efficient frontier for $\min \left\{ \left(c_1^T x, c_2^T x\right)^T : x \in \mathcal{P} \right\}$. The faces P_1^* and P_2^* are optimal for the single objectives $c_1^T x$ and $c_2^T x$, respectively.

Another way to find a minimum of a multiple objective linear program is to use a *lexicographic ordering*. The notation used to indicate a lexicographic ordering is the usual order symbols with a subscript L.

When using lexicographic ordering, the elements at the beginning of a vector are infinitely more important than the elements at the end of the vector. Thus, when two vectors are compared lexicographically, their first elements are compared first, their second elements are compared second, and so on. For example, because $1 > 0$, we have

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} >_L \begin{pmatrix} 0 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

It does not matter that the rest of the elements in the first vector are less than those in the second.

An example illustrating what minimization means when the objectives are lexicographically ordered is found in Figure [2]. In this figure, P_1^* is the set of solutions for the objective function $(c_1^T)x$, and hence, these are the only feasible elements consideredd when minimizing $c_2^T x$. The point that minimizes $c_2^T x$ is the one farthest to the right on P_1^* as indicated. In this paper we use lexicographic ordering to minimize linear piecewise objectives. We use the notation $\mathbb{R}_L^n$ to indicate a set of n lexicographically ordered vectors made up of real numbers.

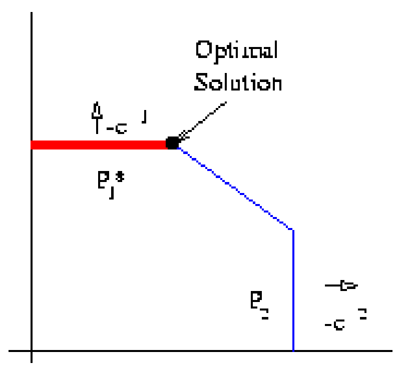

Figure 2: The bold line labeled P_1^* represents the optimal region for the objective function $c_1^T x$ and hence, is the feasible region for minimizing P_2.

We generalize the results of Ukovich, Pastore, and Premoli [1], as well as provide new results that extend their problem statement. We begin by defining a *three-piece linear function* as follows

$$d(x) = d(x|h^-, h^+, c^-, c^+) = \begin{cases} c^-(h^- - x) & \text{if } x < h^- \\ 0 & \text{if } h^- \leq x \leq h^+ \\ c^+(x - h^+) & \text{if } h^+ < x, \end{cases}$$

where c^- and c^+ are positive real numbers, and h^- and h^+ are such that $h^- \leq h^+$, with at least one being finite. The three-piece linear function is non-negative, convex, and it's domain is $\mathbb{R}$. Three-piece linear functions are used by modelers to incorporate penalties. An example of this could be a grocery store that loses money if it does not keep h^- cans on the shelf and is charged a storage fee if it has more than h^+ cans. Having an amount between h^- and h^+ is desirable because no penalty is accrued throughout the interval $[h^-, h^+]$.

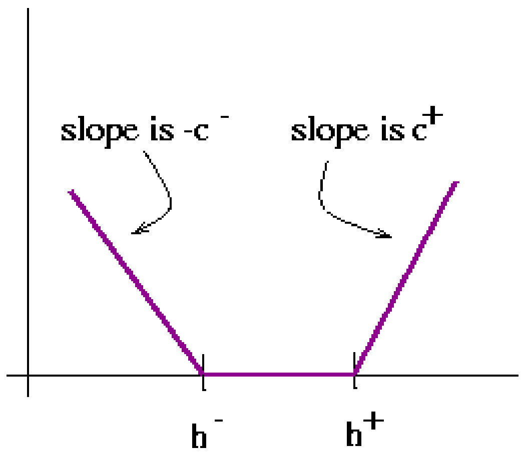

Figure 3: Three-piece linear function

There is a corresponding collection of h^-, h^+, c^-, and c^+ for each variable and each objective function. The penalty for each separate variable is added together to calculate the total penalty of x for the corresponding objective, forming a *separable piecewise function.* Thus, the i^{th} objective function is:

$$\begin{aligned} D_i(x) &= d_{(i,1)}(x_1|h^-_{(i,1)}, h^+_{(i,1)}, c^-_{(i,1)}, c^+_{(i,1)}) + d_{(i,2)}(x_2|h^-_{(i,2)}, h^+_{(i,2)}, c^-_{(i,2)}, c^+_{(i,2)}) + \\ &\quad \ldots + d_{(i,n)}(x_n|h^-_{(i,n)}, h^+_{(i,n)}, c^-_{(i,n)}, c^+_{(i,n)}) \\ &= \sum_{j=1}^{n} d_{(i,j)}(x_j|h^-_{(i,j)}, h^+_{(i,j)}, c^-_{(i,j)}, c^+_{(i,j)}). \end{aligned}$$

The function we optimize is $D : \mathbb{R}^n \mapsto \mathbb{R}^p_L$, where

$$D(x) = \begin{pmatrix} D_1(x) \\ \\ D_2(x) \\ \\ \vdots \\ \\ D_p(x) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n d_{(1,j)}(x_j | h^-_{(1,j)}, h^+_{(1,j)}, c^-_{(1,j)}, c^+_{(1,j)}) \\ \\ \sum_{j=1}^n d_{(2,j)}(x_j | h^-_{(2,j)}, h^+_{(2,j)}, c^-_{(2,j)}, c^+_{(2,j)}) \\ \\ \vdots \\ \\ \sum_{j=1}^n d_{(p,j)}(x_j | h^-_{(p,j)}, h^+_{(p,j)}, c^-_{(p,j)}, c^+_{(p,j)}) \end{pmatrix}.$$

$D(x)$ takes an n-vector and returns a p-vector in a lexicographically ordered space. The order in which the individual objectives are arranged is determined by the modeler.

The Linear Piecewise Lexicographic Programming (LPwLgP) problem statement in [1] is

$$\min\{D(x) : Ax = 0\},$$

where the assumption is made that $[h^-_{(i+1,j)}, h^+_{(i+1,j)}] \subseteq [h^-_{(i,j)}, h^+_{(i,j)}]$. We generalize this problem statement in two ways. First, we remove the restriction that $[h^-_{(i+1,j)}, h^+_{(i+1,j)}] \subseteq [h^-_{(i,j)}, h^+_{(i,j)}]$, and second we allow the right-hand side of the constraints to be non-zero. The General Linear Piecewise Lexicographic Programming (GLPwLgP) problem is

$$\min\{D(x) : Ax = b\},$$

where $A \in \mathbb{R}^{m \times n}$, $m \leq n$, and A has full row rank.

If we want to include the constraint $x \geq 0$, we do this by adding $D_0(x)$, such that:

$$D(x) = \begin{pmatrix} D_0(x) \\ \\ D_1(x) \\ \\ \vdots \\ \\ D_p(x) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n d_{(0,j)}(x_j | 0, \infty, 1, 1) \\ \\ \sum_{j=1}^n d_{(1,j)}(x_j | h^-_{(1,j)}, h^+_{(1,j)}, c^-_{(1,j)}, c^+_{(1,j)}) \\ \\ \vdots \\ \\ \sum_{j=1}^n d_{(p,j)}(x_j | h^-_{(p,j)}, h^+_{(p,j)}, c^-_{(p,j)}, c^+_{(p,j)}) \end{pmatrix}.$$

Since $D(x)$ is lexicographically ordered, and $D_0(x)$ is its first element, $D_0(x)$ is the most important objective to minimize. We have arranged the function so that if $\exists\, x \ni Ax = b$ and $x \geq 0$, $\min\{D_0(x) : Ax = b\} = 0$. On the other hand, if $\nexists\, x \ni Ax = b$ and $x \geq 0$, $\min\{D_0(x) : Ax = b\} > 0$. Thus, if the problem statement without D_0,

$$\min\{D(x) : Ax = b, x \geq 0\},$$

produced no solutions, then $D_0(x) \neq 0$. $D_0(x)$ may or may not be used, depending on whether we want to guarantee non-negativity.

Throughout the following sections we refer to matrices and vectors with set subscripts. A vector with a set subscript is a subvector of the original vector containing the components whose indices are in the set. A matrix with a set subscript is a submatrix containing the columns of the original matrix whose indices are in the set. For example,

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}_{\{1,2,4\}} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}_{\{1,3\}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

2 Corners and an Analog to the Fundamental Theorem of LP

In this section we develop an analog to the fundamental theorem of linear programming using functions that have much weaker conditions. We begin by introducing the ideas behind the fundamental theorem of LP. The following theorem helps to develop the fundamental theorem.

Theorem 1 (See [2] for proof) *If a linear program has an optimal solution, there exists an extreme point optimal solution.*

Thereom 1 is important because there is a convenient algebraic description of an extreme point. Let

$$N(x) = \{j : x_j = 0\}, \text{ and } B(x) = \{j : x_j \neq 0\}.$$

These sets allow the following definition of a basic solution, which are equivalent to extreme points as indicated in Lemma 1.

Definition 1 (Basic Solution) *A vector $x^0 \in \mathcal{P}$ is* basic *if $A_{B(x^0)}$ has linearly independent columns.*

If $A_{B(x^0)}$ is invertible, the columns of $A_{B(x^0)}$ form a unique basis, and x^0 is *non-degenerate.* If x^0 is basic and $A_{B(x^0)}$ is not invertible, the columns of $A_{B(x^0)}$ do not form a unique basis. In this case, x^0 is called *degenerate.*

Lemma 1 (See [2] for proof) *A feasible element is basic if, and only if, it is an extreme point of the feasible region.*

From Lemma 1, we conclude that a basic solution is the same as an extreme point solution. Since every linear program with an optimal solution has an extreme point optimal solution, we use Lemma 1 to conclude the following theorem.

Theorem 2 (See [2] for proof) *If a linear program has an optimal solution, then there exists a basic optimal solution.*

Theorem 2 is known as the fundamental theorem of linear programming, and we now show that this can be extended to accomodate lexicographic minimization.

In this section we consider the multiple objective program (MOP) $\min\{F(x) : Ax = b\}$, where $F : R^n \to R^p_L$. The goal of this section is to develop properties for F that permit our extension of Theorem **??**. The properties fall into two catagories. First, we need a way to define an object similar to a basic solution. The difficulty here is that the minimization for (MOP) is over an affine space, and hence, there are no inequality constraints. This means that all the variables are unbounded and that the index sets $B(x)$ and $N(x)$ do not make sense. These index sets rely only on the feasible element x and the coefficient matrix A, and are defined independent of the objective function. Our approach is different in that the index sets used in this section depend on the objective function. These new index sets allow us to define a **corner**, which is an extension of the idea of a basic solution. Second, we require that the objective function in (MOP) has a monotonicity property over line segments within the optimal set. This condition is precisely defined in Definition 3.

We assume that F has the following form,

$$\begin{aligned} F(x) &= (F_1(x), F_2(x), \ldots, F_p(x))^T, \text{ where} \\ F_i(x) &= \sum_{j=1}^{n} f_{(i,j)}(x_j), \text{ for } i \in \{1, 2, \ldots, p\} \text{ and } x \in R^n. \end{aligned}$$

$F_i(x)$ is **seperable**, meaning that $f_{(i,j)}$ has only x_j as its argument.

We describe the behavior of a function "near" a point by saying that a function exhibits a behavior **locally at a point** if there exists a neighborhood of that point over which that behavior holds. For example, a function f is locally monotonic at x if there exists a neighborhood of x over which f is monotonic. Similarly, f is locally constant at x if there exists a neighborhood of x over which f is constant. We also state that a function is **strongly monotonic** if it is either strictly increasing, strictly decreasing, or constant.

Unless $n = p = 1$, the concept of monotonicity does not make sense for F. However, we use the property that F is seperable to capture, in some sense, where F changes monotonicity over each axis. We now want to examine those points at which F "changes monotonicity." Let

$$\begin{aligned} H_j = \{x_j : & f_{(i,j)} \text{ is not locally strongly monotonic at } x_j \\ & \text{for some } i \in \{1, 2, \ldots, p\}\}. \end{aligned}$$

For example, suppose that $F : R^2 \to R^3_L$ is defined by

$$F(x) = (F_1(x), F_2(x), F_3(x))^T = \left(\sum_{j=1}^{2} f_{(1,j)}(x_j), \sum_{j=1}^{2} f_{(2,j)}, \sum_{j=1}^{2} f_{(3,j)} \right)^T, \text{ where}$$

$$\begin{aligned} f_{(1,1)}(x_1) &= x_1 + \sin x_1, \\ f_{(1,2)}(x_2) &= (x_2)^3, \\ f_{(2,1)}(x_1) &= \begin{cases} 5, & x_1 < 1 \\ 0, & x_1 \geq 1, x_1 \in Q \\ 1, & \text{otherwise} \end{cases} \\ f_{(2,2)}(x_2) &= -x_2, \\ f_{(3,1)}(x_1) &= e^{x_1}, \text{ and} \\ f_{(3,2)}(x_2) &= (x_2 - 1)^2. \end{aligned}$$

Notice that $f_{(1,1)}$ and $f_{(3,1)}$ are strictly increasing. Thus, they do not contribute any points to H_1. The function $f_{(2,1)}$ is constant over $(-\infty, 1)$, but for $x_1 \geq 1$ there is no point at which $f_{(2,1)}$ is locally strongly monotonic. Thus $f_{(2,1)}$ contributes the entire interval $[1, \infty)$ to H_1. Similar to the construction of H_1, we have that $f_{(1,2)}$ and $f_{(2,2)}$ contribute no points to H_2 because they are strictly monotonic. However, $f_{(3,2)}$ is not locally strongly monotonic at $x_2 = 1$, and hence, $H_2 = \{1\}$.

A component x_j of $x \in R^n$ is **cornered** if $x_j \in H_j$. The following index sets indicate which components are cornered and which are not,

$$\begin{aligned}\nu(x) &= \{j : x_j \text{ is cornered}\}, \text{ and} \\ \beta(x) &= \{j : x_j \text{ is not cornered}\}.\end{aligned}$$

Notice that $\nu(x)$ and $\beta(x)$ are similar to $N(x)$ and $B(x)$, the difference being that N and B indicate which components of x are 0 and not 0, while ν and β indicate which components are cornered and not cornered. The index sets $\nu(x)$ and $\beta(x)$ allow us to define a corner in the following manner.

Definition 2 (Corner Solution) *x is a* **corner** *if the columns of $A_{\beta(x)}$ are linearly independent.*

Notice that a corner is similar to a basic solution. Returning to our previous example for F, suppose that the constraint $Ax = 0$ is included. If $A = [1, 0]$ the feasible region is the x_2 axis. Since $0 \notin H_1$, we cannot corner x_1. Suppose that we corner x_2. This is possible only if $x_2 = 1$, since that is the only point in H_2. Then we have that $\beta(x) = \{2\}$, and $A_{\beta(x)} = [1]$, which is linearly independent. So, $x = (0, 1)^T$ is the only corner. If $A = [1, -1]$, the feasible region is the line $x_1 = x_2$, and we have corners at every point $(x', x')^T$, so long as $x' \geq 1$. If $x' = 1$, $\beta(x', x') = \emptyset$, and $A_{\beta(x',x')}$ is vacuous (and hence its columns are linearly independent). For this case we have that (x', x') is a degenerate corner. If $x' > 1$, $\beta(x', x') = \{2\}$, and $A_{\beta(x',x')} = [-1]$.

Since our goal is to prove a result about the existence of a corner optimal solution, we need to require that corners exists. Hence, we assume that F has the property that $H_j \neq \emptyset$, for all $j \in \{1, 2, \ldots, n\}$. Other than making sure a corner exists, we need for F to have one other property. We now examine what is would mean for a function to be monotonic along arbitrary line segments. In order to describe a line segment in R^n we define

$$\begin{aligned}l(x^1, x^2) &= \{x : x = (1-\theta)x^1 + \theta x^2, \theta \in (0, 1)\}, \text{ and} \\ \bar{l}(x^1, x^2) &= \{x : x = (1-\theta)x^1 + \theta x^2, \theta \in [0, 1]\}.\end{aligned}$$

We state that if $\theta_1 \leq \theta_2$ implies either $f(x(\theta_1)) \geq f(x(\theta_2))$ or $f(x(\theta_1)) \leq f(x(\theta_2))$ for all $\theta_1, \theta_2 \in [0, 1]$, then f is monotonic over $\bar{l}(x^1, x^2)$. Similarly, if $\theta_1 < \theta_2$ implies that $f(x(\theta_1)) < f(x(\theta_2))$ or that $f(x(\theta_1)) > f(x(\theta_2))$, for all $\theta_1, \theta_2 \in (0, 1)$, then f is strictly monotonic over $l(x^1, x^2)$. We use this concept of monotonicity over a line segment in the following definition the explains what it means for a function to be linearly monotonic.

Definition 3 (Linear Monotonicity) *F is* **(strongly) linearly monotonic** *if it is (strongly) monotonic on $\bar{l}(x^1, x^2)$ for all $x^1, x^2 \in R^n$. F is*

(strongly) linearly monotonic over S *if it is (strongly) monotonic over* $\bar{l}(x^1, x^2)$ *for all* $x^1, x^2 \in S$ *and* $\bar{l}(x^1, x^2) \cap S^c = 0$.

We define the set

$$\begin{aligned} Q(x) &= \{x + \alpha q : q \in \text{ Nul}(A), \nu(x + \alpha q) = \nu(x), \alpha > 0\} \text{ and} \\ Q &= \bigcup_{x \in \mathcal{O} \setminus \mathcal{C}} \overline{Q}(x) \end{aligned}$$

where $\mathcal{C}$ is the set of corners and $\mathcal{O}$ is the set of optimal solutions. We assume that the objective functions we deal with have the property that they are strongly linearly monotonic over Q.

The following lemma shows that so long as an optimal solution is not a corner, we find another optimal solution which has more cornered compoents.

Lemma 2 *If* x^0 *is an optimal solution that is not a corner, there exists* x^1 *such that*

(i) $x^1_{\nu(x^0)} = x^0_{\nu(x^0)}$,

(ii) $\nu(x^0) \subset \nu(x^1)$, *and*

(iii) x^1 *is optimal.*

Proof: Note that since x^0 is not a corner, the columns of $A_{\beta(x^0)}$ are linearly dependent. It follows that $\text{Nul}(A_{\beta(x^0)}) \neq \{0\}$. Choose a direction vector

$$q = \left[\frac{q_{\beta(x^0)}}{q_{\nu(x^0)}}\right], \text{ where } q_{\beta(x^0)} \in \text{ Nul}(A_{\beta(x^0)}) \setminus \{0\} \text{ and } q_{\nu(x^0)} = 0.$$

Let

$$x(\alpha) = x^0 + \alpha q = \left[\frac{x^0_{\beta(x^0)} + \alpha q_{\beta(x^0)}}{x^0_{\nu(x^0)}}\right], \text{ where } \alpha \geq 0.$$

We have that $x(\alpha)$ is feasible because

$$\begin{aligned} Ax(\alpha) &= A(x^0 + \alpha q) \\ &= A_{\beta(x^0)}[x^0_{\beta(x^0)} + \alpha q_{\beta(x^0)}] + A_{\nu(x^0)} x^0_{\nu(x^0)} \\ &= A_{\beta(x^0)} x^0_{\beta(x^0)} + A_{\nu(x^0)} x^0_{\nu(x^0)} + \alpha A_{\beta(x^0)} q_{\beta(x^0)} \\ &= Ax^0 + \alpha A_{\beta(x^0)} q_{\beta(x^0)} \\ &= b. \end{aligned}$$

We now find the smallest α that will corner x^0_j for some $j \in \beta(x^0)$. However, we first need to ensure that q directs x^0_j towards $h \in H_j$ for some $j \in \beta(x^0)$. Suppose that for all $j \in \beta(x^0)$, either $x^0_j > \sup\{H_j\}$ and $q_j > 0$, or $x^0_j < \inf\{H_j\}$ and $q_j < 0$. Then for all $j \in \beta(x^0)$ and $\alpha \geq 0$, $x^0_j + \alpha q \notin H_j$. Thus it is impossible to corner x^0_j for $j \in \beta(x^0)$. However, since $q_{\beta(x^0)} \in \text{Nul}(A_{\beta(x^0)})$, we also know that $-q_{\beta(x^0)} \in \text{Nul}(A_{\beta(x^0)})$. So, if q does not direct x^0_j towards $h \in H_j$, then $-q$ does. Hence, we may always choose $q \in \text{Nul}(A)$, with $q_{\beta(x^0)} \in \text{Nul}(A_{\beta(x^0)})$ and $q_{\nu(x^0)} = 0$, such that q directs x^0 towards a corner, and we assume throughout that q has this property.

Define

$$\begin{aligned}\hat{\alpha}_+ &= \inf\left\{\frac{h-x_j^0}{q_j} : j\in\beta(x^0), h\in H_j, q_j>0, h>x_j^0\right\}, \text{ and}\\ \hat{\alpha}_- &= \inf\left\{\frac{h-x_j^0}{q_j} : j\in\beta(x^0), h\in H_j, q_j<0, h<x_j^0\right\}.\end{aligned}$$

Now let $\hat{\alpha} = \inf\{\hat{\alpha}_+, \hat{\alpha}_-\}$. Fix $x^1 = x(\hat{\alpha}) = x^0 + \hat{\alpha}q$. Observe that for all $j \in \nu(x^0)$, we have that $x_j^1 = x_j^0 + \hat{\alpha}q_j = x_j^0$. Therefore we conclude that $x^1_{\nu(x^0)} = x^0_{\nu(x^0)}$, which proves statement (i).

By the definition of x^1, we know there must exist $j \in \beta(x^0)$ such that $x_j^1 \in H_j$ and $x_j^0 \notin H_j$. So, $j \in \nu(x^1)$, and because $j \notin \nu(x^0)$ and $x^1_{\nu(x^0)} = x^0_{\nu(x^0)}$, we have that $\nu(x^0) \subset \nu(x^1)$. Hence, statement (ii) is proven.

Notice that by the definition of $\hat{\alpha}$, for $\alpha \in [0, \hat{\alpha})$, we have that $x(\alpha) \notin H_j$. This implies that for $\alpha \in [0, \hat{\alpha}]$, $x(\alpha) \in Q$. Since F is strongly linearly monotonic over Q, we have that $F(x(\alpha))$ is monotonic over $[0, \hat{\alpha}]$. Additionally, since x^0 is not a corner, for $j \in \beta(x^0)$ there exists some neighborhood $(x_j^0 - \epsilon_j, x_j^0 + \epsilon_j)$ around x_j^0 such that $h \notin (x_j^0 - \epsilon_j, x_j^0 + \epsilon_j)$, for all $h \in H_j$. Let $\bar{\alpha} = \inf\left\{\frac{\epsilon_j}{|q_j|} : j \in \beta(x^0)\right\}$. Then for all $\alpha \in (-\bar{\alpha}, \bar{\alpha})$, and all $j \in \beta(x^0)$, $x(\alpha) \notin H_j$. This implies that for $\alpha \in (-\bar{\alpha}, \bar{\alpha})$, $x(\alpha) \in Q$. Since F is srtrongly linearly monotonic over Q, we have that $F(x(\alpha))$ is monotonic over $(-\bar{\alpha}, \bar{\alpha})$. Then we have that $F(x(\alpha))$ is monotonic over $(-\bar{\alpha}, \hat{\alpha}]$.

Since x^0 is optimal, we conclude that $F(x^0) \leq_L F(x^1)$. Suppose that $F(x^0) <_L F(x^1)$. Let $k_1 \in \{1, 2, \ldots, p\}$ be the smallest index such that $F_{k_1}(x^0) < F_{k_1}(x^1)$. Since $F_{k_1}(x(\alpha))$ is monotonic over $(-\bar{\alpha}, \hat{\alpha}]$, this implies that $F_{k_1}(x^0) > F_{k_1}(x^0 - \bar{\alpha}q)$, which contradicts the fact that x^0 is optimal. Thus $F_{k_1}(x^0) = F_{k_1}(x^1)$. Now let $k_2 \in \{1, 2, \ldots, p\} \setminus \{y\}$ be the next smallest element such that $F_{k_2}(x^0) < F_{k_2}(x^1)$. We again show this contradicts the fact that x^0 is optimal. Continuing in this fashion, we conclude that for all $i \in \{1, 2, \ldots, p\}$, $F_i(x^0) = F_i(x^1)$. We conclude that $F(x^0) = F(x^1)$. This implies that x^1 is optimal, which proves statement (iii). ■

Theorem 3 *If the* $min\{F(x) : Ax = b\}$ *has a solution, then it has a corner solution.*

Proof: Let x^0 be an optimal solution. If x^0 is a corner, then the proof is complete. If x^0 is not a corner, the previous lemma implies that there exists an optimal solution x^1 such that $x^1_{\nu(x^0)} = x^0_{\nu(x^0)}$ and $\nu(x^0) \subset \nu(x^1)$. Then there exists some $j \in \beta(x^0)$ such that $j \notin B(x^1)$. This implies that there are fewer columns in $A_{\beta(x^0)}$ than in $A_{B(x^1)}$. There are two possible cases. First, suppose that the columns of $A_{B(x^1)}$ are linearly independent. Then x^1 is a corner optimal solution. Second, suppose that the columns of $A_{B(x^1)}$ are linearly dependent. Then there exists an optimal solution x^2 such that $x^2_{\nu(x^0)} = x^1_{\nu(x^0)}$ and $\nu(x^1) \subset N(x^2)$. Again, there are two possible cases. If the columns of $A_{B(x^2)}$ are linearly independent, x^2

is a corner. If the columns of $A_{B(x^2)}$ are linearly dependent, there exists another optimal solution x^3 such that $x^3_{\nu(x^0)} = x^2_{\nu(x^0)}$ and $N(x^2) \subset N(x^3)$. Continuing in this fashion, we find an optimal solution $\hat{x}$ such that the columns of $A_{B(\hat{x})}$ are linearly independent. Thus $\hat{x}$ is a corner optimal solution. ■

We now show that the fundamental theorem of linear programming follows from Thoerem 3. We make the assumption that $(lp) \min\{c^T x : Ax = b, x \geq 0\}$ has an optimal solution. For $j = 1, 2, \ldots, n$, define $d_j : \mathbb{R}^n \to \mathbb{R}$ by

$$d_j(x) = \begin{cases} -x_j, & x < 0 \\ 0, & x \geq 0, \end{cases}$$

and set $D(x) = \sum_{j=1}^{n} d_j(x)$. Notice that D has a minimum value of zero over $\mathbb{R}^n_+$. So, the definition of D, the fact that (lp) is feasible, and the lexicographic ordering implies that

$$\operatorname{argmin}\left\{\begin{pmatrix} D(x) \\ c^T x \end{pmatrix} : Ax = b\right\} = \operatorname{argmin}\left\{c^T x : Ax = b, x \geq 0\right\}. \quad (1)$$

We denote the multiple objective program on the left-hand side of the equality in 1 by MOP. Since the objective functions in MOP are (piecewise) linear, they meet the conditions of Theorem 3, and hence MOP has a corner optimal solution. Let x be a corner optimal solution to MOP. Since the second objective in MOP is linear, we have that $H_j = \{0\}$ for $j = 1, 2, \ldots n$. This means that x has the following properties: 1) $N(x) = \nu(x)$, 2) $x_N = x_\nu = 0$, and 3) the columns of $A_{B(x)} = A_{\beta(x)}$ are linearly independent. The third observation, together with the fact that x is optimal to (lp) from (1), implies that x is a basic optimal solution to (lp). Hence, every corner optimal solution to MOP is a basic optimal solution to (lp), and Theorem **??** implies the fundamental theorem of linear programming.

3 Connection with Linear Programming

In this section, we investigate the relationship between a corner optimal solution and a basic optimal solution. First, we formulate a linear program that provides an optimal set that is equipotent to the optimal set for the GLPwLgP problem with two objectives. Set

$$\begin{aligned} O_1 &= \operatorname{argmin}\left\{\begin{pmatrix} D_0 \\ D_1 \end{pmatrix} : Ax = b\right\} \text{ and} \\ O_2 &= \operatorname{argmin}\{(c^-)^T\gamma + (c^+)^T\rho : Ax = b, x + \gamma \geq h^-, x - \rho \leq h^+, \\ &\qquad x \geq 0, \gamma \geq 0, \rho \geq 0\}. \end{aligned}$$

where

$$c^- = \begin{pmatrix} c^-_{(1,1)} \\ c^-_{(1,2)} \\ \vdots \\ c^-_{(1,n)} \end{pmatrix}, c^+ = \begin{pmatrix} c^+_{(1,1)} \\ c^+_{(1,2)} \\ \vdots \\ c^+_{(1,n)} \end{pmatrix}, h^- = \begin{pmatrix} h^-_{(1,1)} \\ h^-_{(1,2)} \\ \vdots \\ h^-_{(1,n)} \end{pmatrix}, \text{ and } h^+ = \begin{pmatrix} h^+_{(1,1)} \\ h^+_{(1,2)} \\ \vdots \\ h^+_{(1,n)} \end{pmatrix}.$$

Figure 4 is an extremely simplified example where

$$O_1 = \text{argmin}\left\{ \begin{pmatrix} D_0 \\ D_1 \end{pmatrix} : x = 3 \right\},$$

and $D_1 = d_{(1,1)}(x|1,2,1,5)$. Because the example is so simple, $x = 3$ is the only feasible point, and therefore the optimal point and corner.

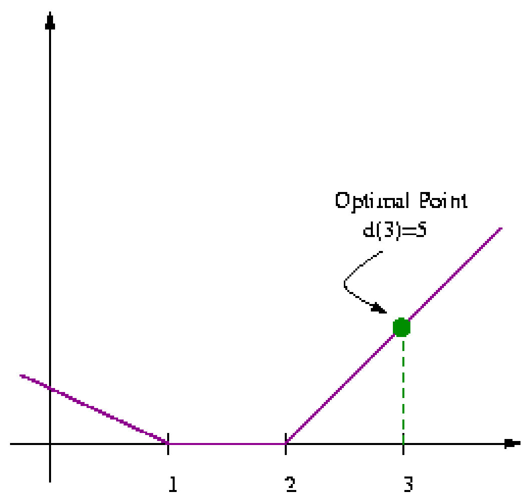

Figure 4: Example of O_1

In Theorem 4 we show that $O_1 \cong O_2$. For this example,

$$O_2 = \text{argmin}\{\gamma + 5\rho : x = 3, x + \gamma \geq 1, x - \rho \leq 2, x \geq 0, \gamma \geq 0, \rho \geq 0\},$$

and Figure 5 depicts the geometry describing O_2. The feasible region for O_2 is the part of the plane $x = 3$ that exists in the positive orthant and $\rho \geq 1$. O_2 contains only the point $(x, \gamma, \rho) = (3, 0, 1)$, which is an extreme point (and hence basic).

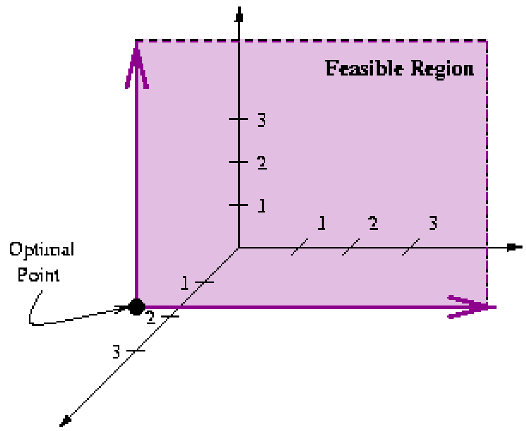

Figure 5: Example of O_2

We show a slightly more complex example of O_1 in Figure 6, where

$$O_1 = \text{argmin} \left\{ \begin{pmatrix} D_0 \\ D_1 \end{pmatrix} : Ax = 0 \right\},$$

$$A = [1\ -1], \text{ and } D_1 = d_{(1,1)}(x_1|1,2,1,1) + d_{(1,2)}(x_2|3,4,1,5).$$

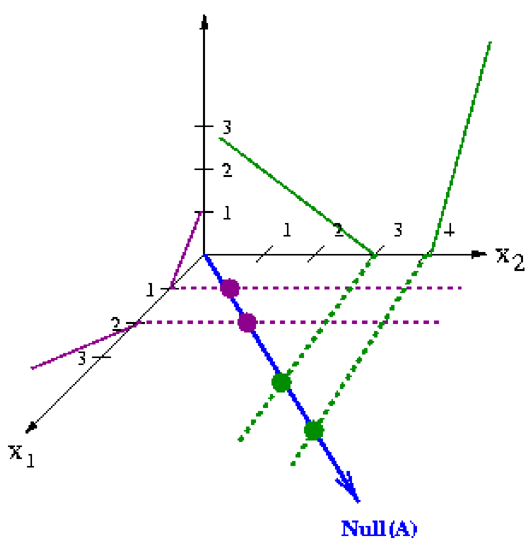

Figure 6: Another example of O_1

The feasible region for O_1 is the line in the positive orthant that represents Null(A). Although the points on Null(A) are not extreme points, they are corners by definition. From Theorem 3 we know that at least one of these corners is optimal.

The example in Figure 6 only has one more variable than that in Figure 4, but it is already impossible to draw a representation of the linear program that is equipotent to Figure 6 because the LP has six decision variables.

The goal of this section is to show that there exists a bijection between O_1 and O_2 such that a corner in O_1 is mapped to a basic optimal solution in O_2. We begin by showing that the optimal sets of O_1 and O_2 are equipotent.

Theorem 4 *We have that* $O_1 \cong O_2$*, provided* $O_2 \neq \emptyset$

Proof: Let:

$$\begin{aligned}
\mathcal{F}^1 &= \{x : Ax = b\} \\
\mathcal{F}^1_+ &= \{x : Ax = b, x \geq 0\}, \text{ and} \\
\mathcal{F}^2 &= \{(x, \gamma, \rho) : Ax = b, x + \gamma \geq h^-, x - \rho \leq h^+, x \geq 0, \gamma \geq 0, \rho \geq 0\}.
\end{aligned}$$

Define P: $\mathcal{F}^1_+ \mapsto \mathcal{F}^2 : x \mapsto (x, \gamma(x), \rho(x))$, where

$$\gamma_j(x) = \begin{cases} h^-_{(1,j)} - x_j & \text{if } x_j < h^-_{(1,j)} \\ 0 & \text{if } x_j \geq h^-_{(1,j)} \end{cases}$$

and

$$\rho_j(x) = \begin{cases} 0 & \text{if } x_j \leq h^+_{(1,j)} \\ x_j - h^+_{(1,j)} & \text{if } x_j > h^+_{(1,j)}. \end{cases}$$

Let $\bar{x} \in O_1$. Then, $O_2 \neq \emptyset \Rightarrow \exists\ x$ such that $x \geq 0$ and $Ax = b$. So, $D_0(\bar{x}) = 0$. Thus, $\bar{x} \in \mathcal{F}^1_+$. Notice that $P(x) = (x, \gamma(x), \rho(x)) \in \mathcal{F}^2$ — i.e. $P(x)$ is feasible to the math program in O_2. Furthermore, notice that the definition of $\gamma(x)$ and $\rho(x)$ imply that $D_1(x) = (c^-)^T\gamma(x) + (c^+)^T\rho(x)$. We show that $P(x)$ is optimal. Let $(\hat{x}, \hat{\gamma}, \hat{\rho}) \in \mathcal{F}^2$. Then,

$$D_1(\bar{x}) = (c^-)^T\gamma(\bar{x}) + (c^+)^T\rho(\bar{x}) \leq (c^-)^T\gamma(\hat{x}) + (c^+)^T\rho(\hat{x}) = D_1(\hat{x}).$$

By definition of $\gamma(x)$ and $\rho(x)$, we have that

$$\begin{aligned}
\gamma(\hat{x}) &\leq \hat{\gamma}, \text{ and} \\
\rho(\hat{x}) &\leq \hat{\rho}.
\end{aligned}$$

Thus,

$$(c^-)^T\gamma(\hat{x}) + (c^+)^T\rho(\hat{x}) \leq (c^-)^T\hat{\gamma} + (c^+)^T\hat{\rho}.$$

We now have that:

$$\begin{aligned}(c^-)^T\gamma(\bar{x}) + (c^+)^T\rho(\bar{x}) &\leq (c^-)^T\gamma(\hat{x}) + (c^+)^T\rho(\hat{x}) \\ &\leq (c^-)^T\hat{\gamma} + (c^+)^T\hat{\rho}.\end{aligned}$$

This shows that for $x \in O_1$, $P(x) = (x, \gamma(x), \rho(x)) \in \mathcal{F}^2$ is optimal. So, $x \in O_1$ implies $(x, \gamma(x), \rho(x)) \in O_2$.

Let $x^1, x^2 \in O_1$ be such that $P(x^1) = P(x^2)$. Then,

$$P(x^1) = (x^1, \gamma(x^1), \rho(x^1)) = (x^2, \gamma(x^2), \rho(x^2)) = P(x^2).$$

Thus, $x^1 = x^2$, and P is one-to-one.

Let $(\bar{x}, \bar{\gamma}, \bar{\rho}) \in O_2$. Notice that $\bar{x} \in \mathcal{F}^1_+$, and hence, $D_0(\bar{x}) = 0$. So,

$$\operatorname{argmin}\left\{\begin{pmatrix} D_0 \\ D_1 \end{pmatrix} : Ax = b\right\} = \operatorname{argmin}\{D_1(x) : Ax = b, x \geq 0\}.$$

By the definition of $\gamma(x)$ and $\rho(x)$,

$$D_1(\bar{x}) = (c^-)^T\gamma(\bar{x}) + (c^+)^T\rho(\bar{x}) \leq (c^-)^T\bar{\gamma} + (c^+)^T\bar{\rho}.$$

Since $(\bar{x}, \bar{\gamma}, \bar{\rho})$ are optimal, nothing can be less than $(c^-)^T\bar{\gamma} + (c^+)^T\bar{\rho}$. We conclude that

$$D_1(\bar{x}) = (c^-)^T\bar{\gamma} + (c^+)^T\bar{\rho}.$$

Suppose that $\hat{x} \in \mathcal{F}^1_+$ has the property that $D_1(\hat{x}) < D_1(\bar{x})$. Then,

$$D_1(\hat{x}) = (c^-)^T\gamma(\hat{x}) + (c^+)^T\rho(\hat{x}) < (c^-)^T\gamma(\bar{x}) + (c^+)^T\rho(\bar{x}) = D_1(\bar{x}).$$

However, as was shown above, $D_1(\bar{x}) = (c^-)^T\bar{\gamma} + (c^+)^T\bar{\rho}$. Thus,

$$(c^-)^T\gamma(\hat{x}) + (c^+)^T\rho(\hat{x}) < (c^-)^T\bar{\gamma} + (c^+)^T\bar{\rho}.$$

Since $\bar{\gamma}$ and $\bar{\rho}$ are optimal, the above inequality is a contradiction. Thus, $D_1(\hat{x}) \geq D_1(\bar{x})$ which implies that $\bar{x} \in O_1$.

We know that $\bar{\gamma} \geq \gamma(\bar{x})$. Suppose $\exists\ i$ such that $\bar{\gamma}_i > \gamma_i(\bar{x})$. Since $\bar{\gamma}_i$ is optimal, we have reached a contradiction because $\gamma(x)$ would lead to a smaller objective value. Thus, $\bar{\gamma}_i = \gamma_i(\bar{x})$. A similar argument shows that $\bar{\rho}_i = \rho_i(\bar{x})$. Then, $\bar{x} \in O_1$ such that

$$\begin{aligned}P(\bar{x}) &= (\bar{x}, \gamma(\bar{x}), \rho(\bar{x})) \\ &= (\bar{x}, \bar{\gamma}, \bar{\rho}).\end{aligned}$$

Thus, P is onto O_2 and $O_1 \cong O_2$. ■

We have shown that there exists a bijection between O_1 and O_2. Now we consider the relationship between optimal corners for the GLPwLgP in O_1 and basic optimal solutions for the LP in O_2. We show that $P(x)$ maps these corners to basic optimal solutions.

Theorem 5 *A corner*

$$x \in \operatorname{argmin}\left\{\left(\begin{array}{c} D_0(x) \\ D_1(x) \end{array}\right) : Ax = b\right\}$$

has the property that $P(x)$ *is a basic optimal solution to*

$$\min\left\{(c^-)^T\gamma + (c^+)^T\rho : Ax = b, x + \gamma \geq h^-, x - \rho \leq h^+, x, \gamma, \rho \geq 0\right\}.$$

Proof: Let

$$x \in \operatorname{argmin}\left\{\left(\begin{array}{c} D_0(x) \\ D_1(x) \end{array}\right) : Ax = b\right\}$$

be a corner. We define the following index sets:

$$\begin{array}{rcl} L & = & \{i \in \beta : x_i < h_i^-\} \\ M & = & \{i \in \beta : h_i^- < x_i < h_i^+\} \\ R & = & \{i \in \beta : x_i > h_i^+\} \\ H^- & = & \{i \in \nu : x_i = h_i^-\}\text{and} \\ H^+ & = & \{i \in \nu : x_i = h_i^+\}. \end{array}$$

Notice $L \cup M \cup R = \beta$, $H^- \cup H^+ = \nu$, and

$$x = \left(\begin{array}{c} x_L \\ x_M \\ x_R \\ \hline x_{H-} \\ x_{H+} \end{array}\right) = \left(\begin{array}{c} x_\beta \\ \hline x_\nu \end{array}\right)$$

.

Now let

$$U = \left[\begin{array}{c|c|c|c|c} A & 0 & 0 & 0 & 0 \\ \hline I & I & 0 & -I & 0 \\ \hline I & 0 & -I & 0 & I \end{array}\right]$$

and

$$y = \left(\begin{array}{c} x \\ \hline \gamma \\ \hline \rho \\ \hline su \\ \hline sl \end{array}\right)$$

where su is a surplus vector such that $x + \gamma - su = h^-$, and sl is a slack vector such that $x - \rho + sl = h^+$. We partition the vector y as follows:

$$y_x = \left(\begin{array}{c} x_L \\ x_M \\ x_R \\ x_{H-} \\ x_{H+} \end{array}\right) y_\gamma = \left(\begin{array}{c} \gamma_{\beta_L} \\ \gamma_{\beta_M} \\ \gamma_{\beta_R} \\ \gamma_{\beta_{H-}} \\ \gamma_{\beta_{H+}} \end{array}\right) y_\rho = \left(\begin{array}{c} \rho_{\beta_L} \\ \rho_{\beta_M} \\ \rho_{\beta_R} \\ \rho_{\beta_{H-}} \\ \rho_{\beta_{H+}} \end{array}\right) y_{su} = \left(\begin{array}{c} su_{\beta_L} \\ su_{\beta_M} \\ su_{\beta_R} \\ su_{\beta_{H-}} \\ su_{\beta_{H+}} \end{array}\right) y_{sl} = \left(\begin{array}{c} sl_{\beta_L} \\ sl_{\beta_M} \\ sl_{\beta_R} \\ sl_{\beta_{H-}} \\ sl_{\beta_{H+}} \end{array}\right),$$

where γ, ρ, su, and sl are subdivided so the system

$$Uy = \left(\begin{array}{l} Ay_x + 0y_\gamma + 0y_\rho + 0y_{su} + 0y_{sl} \\ Iy_x + Iy_\gamma + 0y_\rho - Iy_{su} + 0y_{sl} \\ Iy_x + 0y_\gamma - Iy_\rho + 0y_{su} + Iy_{sl} \end{array} \right) = \left(\begin{array}{l} b \\ h^- \\ h^+ \end{array} \right)$$

can be written as

$$\begin{aligned}
A_{\beta_L} x_L + A_{\beta_M} x_M + A_{\beta_R} x_R + A_{H-} x_{H-} + A_{H+} x_{H+} &= b, && (2)\\
x_L + \gamma_{\beta_L} - su_{\beta_L} &= h^-_{\beta_L}, && (3)\\
x_M + \gamma_{\beta_M} - su_{\beta_M} &= h^-_{\beta_M}, && (4)\\
x_R + \gamma_{\beta_R} - su_{\beta_R} &= h^-_{\beta_R}, && (5)\\
x_{H-} + \gamma_{H-} - su_{H-} &= h^-_{H-}, && (6)\\
x_{H+} + \gamma_{H+} - su_{H+} &= h^-_{H+}, && (7)\\
x_L - \rho_{\beta_L} + sl_{\beta_L} &= h^+_{\beta_L}, && (8)\\
x_M - \rho_{\beta_M} + sl_{\beta_M} &= h^+_{\beta_M}, && (9)\\
x_R - \rho_{\beta_R} + sl_{\beta_R} &= h^+_{\beta_R}, && (10)\\
x_{H-} - \rho_{H-} + sl_{H-} &= h^+_{H-}, && (11)\\
x_{H+} - \rho_{H+} + sl_{H+} &= h^+_{H+}. && (12)
\end{aligned}$$

Our goal is to subdivide the vector y into y_B and y_N, where $y_B > 0$ and $y_N = 0$. In Equation (6) $x_{H-} = h^-_{H-}$, and thus we know that $\gamma_{H-} = su_{H-} = 0$. In Equation (12) $x_{H+} = h^+_{H+}$ so that $\rho_{H+} = sl_{H+} = 0$. In Equation (4) $x_M > h^- \Rightarrow \gamma_{\beta_M} = 0$. Using similar logic and the eqations indicated below, the following are zero:

0	Equation	0	Equation	0	Equation	0	Equation
γ_{β_M}	(3)	ρ_{β_M}	(8)	su_{β_L}	(2)	sl_{β_R}	(9)
γ_{β_R}	(4)	ρ_{β_L}	(7)	su_{H-}	(6)	sl_{H+}	(11)
γ_{H-}	(5)	ρ_{H-}	(10)				
γ_{H+}	(6)	ρ_{H+}	(11)				

We define

$$\begin{array}{c} \eta = \gamma_{\beta_M} \cup \gamma_{\beta_R} \cup \gamma_{H-} \cup \gamma_{H+} \cup \rho_{\beta_M} \cup \rho_{\beta_L} \cup \rho_{H-} \cup \\ \rho_{H+} \cup su_{\beta_L} \cup su_{H-} \cup sl_{\beta_R} \cup sl_{H+}. \end{array}$$

Then, $y_\eta = 0$, and hence, y_η is a subvector of y_N and η is a subset of N.

We define

$$\begin{array}{c} \xi = x_L \cup x_M \cup x_R \cup x_{H-} \cup x_{H+} \cup \gamma_{\beta_L} \cup \rho_{\beta_R} \cup \\ su_{\beta_M} \cup su_{\beta_R} \cup su_{H+} \cup sl_{\beta_L} \cup sl_{\beta_M} \cup sl_{H-}. \end{array}$$

Then,

$$
y_\xi = \left(\begin{array}{c} x_L \\ x_M \\ x_R \\ x_{H-} \\ x_{H+} \\ \hline \gamma_{\beta_L} \\ \hline \rho_{\beta_R} \\ \hline su_{\beta_M} \\ su_{\beta_R} \\ su_{H+} \\ \hline sl_{\beta_L} \\ sl_{\beta_M} \\ sl_{H-} \end{array} \right).
$$

Notice that y_B is a subvector of y_ξ and B is a subset of ξ.

Now let

$$
U_\xi = \left[\begin{array}{rrrrr|r|r|rrr|rrr}
A_{\beta_L} & A_{\beta_M} & A_{\beta_R} & A_{H-} & A_{H+} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ \hline
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & I & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right]
$$

so that U_B is a submatrix of U_ξ. We show the columns of U_ξ are linearly independent.

Let z_ξ be a vector partioned the same as y_ξ. Suppose $U_\xi z_\xi = 0$. Then we get

$$
\begin{aligned}
A_{\beta_L} z_L + A_{\beta_M} z_M + A_{\beta_R} z_R + A_{H^-} z_{H^-} + A_{H^+} z_{H^+} &= 0, & (13)\\
z_L + \gamma_{\beta_L} &= 0, & (14)\\
z_M - su_{\beta_M} &= 0, & (15)\\
z_R - su_{\beta_R} &= 0, & (16)\\
z_{H^-} &= 0, & (17)\\
z_{H^+} - su_{H^+} &= 0, & (18)\\
z_L + sl_{\beta_L} &= 0, & (19)\\
z_M + sl_{\beta_M} &= 0, & (20)\\
z_R - \rho\beta_R &= 0, & (21)\\
z_{H^-} + sl_{H^-} &= 0, & (22)\\
z_{H^+} &= 0. & (23)
\end{aligned}
$$

By equations (16) and (22) $z_{H^-} = 0$ and $z_{H^+} = 0$, and we rewrite equation (12) as

$$A_{\beta_L} z_L + A_{\beta_M} z_M + A_{\beta_R} z_R = 0.$$

Since our original vector x is a corner, we know the columns of the matrix A_β are linearly indepedent. Therefore,

$$z_L = 0, z_M = 0, and z_R = 0.$$

Using that z_{H^-}, z_{H^+}, z_L, z_M, and z_R are zero, it is easy to see that $z_\xi = 0$. Thus we conclude that x is a basic optimal solution of the set

$$\operatorname{argmin}\left\{(c^-)^T\gamma + (c^+)^T\rho : Ax = b, x + \gamma \geq h^-, x - \rho \leq h^+, x, \gamma, \rho \geq 0\right\}.$$

We have shown that the fundamental theorem of linear programming can be extended to include a much weaker set of functions that include nonlinear and non-continuous functions. We have also shown that there exists a bijection that maps corner optimal solutions for the GLPwLgP to basic optimal solutions for an LP. This is very helpful because instead of having to solve a complicated GLPwLgP with two objectives, we can simply formulate the corresponding LP and solve it. Avenues for furthur research include finding a correspondingLP for GLPwLgP problems with more than two objectives, and determining if basic optimal solutions for LP problems are mapped into corner optimal solutions for GLPwLgP problems.

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