Length Sets and $\mathcal{V}$ Sets of Numerical Monoids

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Abstract

We study $\mathcal{V}$ sets of numerical monoids and find upper and lower bounds on the $\Delta_{\mathcal{V}}$ set based on the delta sets of these monoids. The paper then focuses on numerical monoids generated by an arithmetic progression increasing by a constant. First, we determine exact solutions for length sets and then we use these formulas to enumerate $\mathcal{V}$ sets. Next, we investigate if two such numerical monoids are isomorphic if their $\mathcal{V}$ sets are equal. Finally, we investigate if two such numerical monoids are isomorphic if their length sets are equal.

1 Introduction

If M is a nonempty set and $\cdot$ is a binary operation on M, then the pair $(M, \cdot)$ is a monoid if

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \;\forall\; a, b, c \in M$

2. $\exists\; 1 \in M$ such that $1 \cdot a = a \cdot 1 = a \;\forall\; a \in M$.

A numerical monoid is an additive submonoid of $\mathbb{N} \cup \{0\}$.

The set of elements of S are $x \in S$ such that x can be written in the form

$$x = x_1 a_1 + \cdots + x_t a_t = \sum_{i=1}^{t} x_i a_i$$

for some $x_i \in \mathbb{N} \cup \{0\}$ where $\{a_1, \ldots, a_t\}$ is the generating set of S, often denoted as $S = \langle a_1, \ldots, a_t \rangle$.

Every numerical monoid S has a unique minimal set of generators. S is primitive if $\gcd\{s \mid s \in S\}$. Every numerical monoid S is isomorphic to a unique primitive numerical monoid, so we always assume that S is a primitive numerical monoid.

Definition 1.1. *The set of lengths of $m \in S$ is*

$$\mathcal{L}(m) = \left\{ \sum_{i=1}^{t} x_i \mid x_i \in \mathbb{N} \cup \{0\}, m = \sum_{i=1}^{t} x_i a_i \right\}.$$

If it is not clear which monoid we are considering, we use the notation $\mathcal{L}_S(m)$ to indicate $\mathcal{L}(m)$ of S . Also, define $L(m) = \max \mathcal{L}(m)$ and $l(m) = \min \mathcal{L}(m)$. We define the set of lengths of S as $\mathcal{L}(S) = \{\mathcal{L}(m) \mid m \in S\}$.

Example 1.2. *Let $S = \langle 2, 3 \rangle$. $\mathcal{L}(21) = \{7, 8, 9, 10\}$.*

First, we must list all the factorizations of 21 in S.

$$\begin{aligned} 21 &= 9 \cdot 2 + 1 \cdot 3 \\ &= 6 \cdot 2 + 3 \cdot 3 \\ &= 3 \cdot 2 + 5 \cdot 3 \\ &= 0 \cdot 2 + 7 \cdot 3 \end{aligned}$$

Thus, $\mathcal{L}(21) = \{7, 8, 9, 10\}$, $L(21) = 10$, and $l(21) = 7$.

Definition 1.3. *The elasticity of $m \in S$, denoted $\rho(m)$, is*

$$\rho(m) = \frac{L(m)}{l(m)}.$$

The elasticity of S is then defined as $\rho(S) = \sup\{\rho(m) \mid m \in S\}$. For example, in $S = \langle 2, 3 \rangle$, $\rho(21) = \frac{10}{7}$.

Definition 1.4. *Let $m \in S$ and suppose $\mathcal{L}(m) = \{n_1, \ldots, n_t\}$ with the n_i's listed in increasing order. The delta set of m is*

$$\Delta(m) = \{n_i - n_{i-1} \mid 2 \leq i \leq t\}.$$

For example, in $S = \langle 2, 3 \rangle$, $\Delta(21) = \{1\}$.

The delta set of S is

$$\Delta(S) = \bigcup_{m \in S} \Delta(m).$$

Definition 1.5. $\mathcal{W}(n) = \{m \in S \mid n \in \mathcal{L}(m)\}$.

Example 1.6. *Let $S = \langle 2, 3 \rangle$. $\mathcal{W}(7) = \{14, 15, 16, 17, 18, 19, 20, 21\}$.*

Let us investigate which length sets contain 7 as an element.

$$\begin{aligned}
\mathcal{L}(14) &= \{5, 6, 7\} \\
\mathcal{L}(15) &= \{5, 6, 7\} \\
\mathcal{L}(16) &= \{6, 7, 8\} \\
\mathcal{L}(17) &= \{6, 7, 8\} \\
\mathcal{L}(18) &= \{6, 7, 8, 9\} \\
\mathcal{L}(19) &= \{7, 8, 9\} \\
\mathcal{L}(20) &= \{7, 8, 9, 10\} \\
\mathcal{L}(21) &= \{7, 8, 9, 10\}
\end{aligned}$$

Thus, $\mathcal{W}(7) = \{14, 15, 16, 17, 18, 19, 20, 21\}$.

Definition 1.7. $\mathcal{V}(n) = \displaystyle\bigcup_{m \in \mathcal{W}(n)} \mathcal{L}(m)$.

For example, in $S = \langle 2, 3 \rangle$, $\mathcal{V}(7) = \{5, 6, 7, 8, 9, 10\}$.

If it is not clear which monoid we are considering, we use the notation $\mathcal{V}_S(n)$ to indicate $\mathcal{V}(n)$ of S.

Definition 1.8. $\Phi(n) = |\mathcal{V}(n)|$.

Definition 1.9. *Suppose $\mathcal{V}(n) = \{v_{1,n}, \ldots, v_{t,n}\}$ The delta set of $\mathcal{V}(n)$ is*

$$\Delta_{\mathcal{V}}(n) = \{v_{i,n} - v_{i-1,n} \mid 2 \leq i \leq t\}.$$

For example, in $S = \langle 2, 3 \rangle$, $\Delta_{\mathcal{V}}(7) = \{1\}$.

Also, define $\Delta_{\mathcal{V}}(S) = \displaystyle\bigcup_{n \in \mathbb{N}} \Delta_{\mathcal{V}}(n)$. Let $\mathcal{V}^* = \max \Delta_{\mathcal{V}}(S)$ and $\mathcal{V}_* = \min \Delta_{\mathcal{V}}(S)$.

2 $\mathcal{V}$ Sets and $\Delta_{\mathcal{V}}$ Sets of Numerical Monoids

Let $S = \langle a_1, a_2, \ldots, a_t \rangle$ where $\{a_1, \ldots, a_t\}$ is the minimal set of generators.

By [1], there exists a method for calculating $\max \Delta(S)$ in finite time and

$$\min \Delta(S) = \gcd \{a_i - a_{i-1} \mid i \in \{2, 3, \ldots, t\}\} = d.$$

Thus $\Delta(S) \subseteq \{d, 2d, \ldots, qd\}$ for $q \in \mathbb{N}$ [2].

Let $\Delta(S) = \{b_1, b_2, \ldots, b_k\}$ with $\max \Delta(S) = b_k = qd$.

Lemma 2.1. $\mathcal{V}_* = \min \Delta(S)$.

Proof. $\exists\ n$ such that $\mathcal{V}_* = v_{i,n} - v_{i-1,n}$ where $v_{i,n}$ and $v_{i-1,n}$ are consecutive elements in $\mathcal{V}(n)$. Then, $\exists\ x,\ y \in \mathcal{W}(n)$ such that $\{v_{i,n}, n\} \subseteq \mathcal{L}(w_1)$ and $\{v_{i-1,n}, n\} \subseteq \mathcal{L}(w_2)$.

$$n - v_{i,n} = \sum_{i=1}^{k} b_i x_i$$

$$n - v_{i-1,n} = \sum_{i=1}^{k} b_i y_i$$

Thus,

$$\mathcal{V}_* = (n - v_{i-1,n}) - (n - v_{i,n}) = \sum_{i=1}^{k} b_i (y_i - x_i).$$

So,

$$\mathcal{V}_* \geq \gcd\{b_1, b_2, \ldots, b_k\} = \gcd(\Delta(S)) = \min \Delta(S).$$

Assume $\mathcal{V}_* > \min \Delta(S) = d$. Then, $\forall\ i$, $v_{i,n} - v_{i-1,n} > d$. Since $\min \Delta(S) = d$, $\exists\ u \in S$ such that $\mathcal{L}(u) = \{l_1, \ldots, l_r\}$ and $l_p - l_{p-1} = d$. Let $n \in \mathcal{L}(u)$, so $\{l_1, \ldots, l_r\} \subseteq \mathcal{V}(n)$.

Then, since $v_{i,n} - v_{i-1,n} > d\ \forall\ i$, l_p and l_{p-1} are not consecutive elements in $\mathcal{V}(n)$. Thus, $\exists\ z \in \mathcal{V}(n)$ such that $l_{b-1} < z < l_b$ and $l_b - z \in \Delta_{\mathcal{V}}(S)$, but $d = l_p - l_{p-1} > l_p - z$, which is a contradiction.

Therefore, $\mathcal{V}_* = \min \Delta(S)$. □

Lemma 2.2. $\mathcal{V}^* \leq \max \Delta(S) = b_k$.

Proof. $\exists\ n$ such that $\mathcal{V}^* = v_{i,n} - v_{i-1,n}$ where $v_{i,n}$ and $v_{i-1,n}$ are consecutive elements in $\mathcal{V}(n)$.

Then, $\exists\ x \in \mathcal{W}(n)$ such that $\mathcal{L}(x) \supseteq \{v_{i,n}, n\}$. Let $\mathcal{L}(x) = \{n_1, \ldots, n_j\}$ with $v_{i,n} = n_t$.

Thus, $\mathcal{V}(n) \supseteq \{n_1, \ldots, n_j\}$ and since $\max \Delta(S) = b_k$, then $v_{i,n} - v_{i-1,n} \leq n_t - n_{t-1} \leq b_k$ for $2 \leq t \leq j$.

For $t = 1$, then $v_{i,n} = n_1$. If $n_1 = \min \mathcal{V}(n)$, then $v_{i-1,n}$ does not exist, so there must be an $m = v_{i-1,n} \in \mathcal{V}(n)$. Thus, $\exists\ y \in \mathcal{W}(n)$ such that $\mathcal{L}(y) \supseteq \{m, n\}$. Let $\mathcal{L}(y) = \{m_1, \ldots, m_p\}$ with $v_{i-1,n} = m_q$.

$\mathcal{V}(n) \supseteq \{m_1, \ldots, m_p\}$ and since $\max \Delta(S) = b_k$, then $m_{q+1} - m_q \leq b_k$. So, if $n_1 = m_{q+1}$, then $v_{i,n} - v_{i-1,n} \leq m_{q+1} - m_q \leq b_k$. If $m_q < n_1 < m_{q+1}$, then $v_{i,n} - v_{i-1,n} < m_{q+1} - m_q \leq b_k$. □

Lemma 2.3. $\Delta_{\mathcal{V}}(S) \subseteq \{d, 2d, \ldots, qd\}$ *for some* $q \in \mathbb{N}$.

Proof. From Lemma 2.1, $\mathcal{V}_* = d$ and from Lemma 2.2, $\mathcal{V}^* \leq b_k = qd$ for some $q \in \mathbb{N}$. Suppose $\exists\ j$ where $d \leq j \leq qd$ but $d \nmid j$ and $j \in \Delta_{\mathcal{V}}(S)$.

$\exists\ n$ such that $j \in \Delta_{\mathcal{V}}(n)$ and so $\exists\ i, i-1$ such that $j = v_{i,n} - v_{i-1,n}$.

$\exists\ x$ such that $\mathcal{L}(x) \supseteq \{v_{i,n}, n\}$ where

$$n - v_{i,n} = \sum_{i=1}^{k} b_i x_i.$$

$\exists\ y$ such that $\mathcal{L}(y) \supseteq \{v_{i-1,n}, n\}$ where

$$n - v_{i-1,n} = \sum_{i=1}^{k} b_i y_i.$$

Then, $j = v_{i,n} - v_{i-1,n} = (n - v_{i-1,n}) - (n - v_{i,n}) = \sum_{i=1}^{k} b_i(y_i - x_i)$. Since $d | b_i$ $\forall\ i$, thus $d | j$, which is a contradiction. □

Corollary 2.4. *If* $\Delta(S) = \{g\}$*, then* $\Delta_{\mathcal{V}}(S) = \{g\}$*.*

Proof. Since $\Delta(S) = \{g\}$, $\min \Delta(S) = \max \Delta(S) = g$. From Lemmas 2.1 and 2.2, $\mathcal{V}_* = \mathcal{V}^* = g$ and therefore $\Delta_{\mathcal{V}}(S) = \{g\}$. □

Theorem 2.5. $\frac{a_t^2 - a_1^2}{\mathcal{V}^* a_1 a_t} \leq \lim\limits_{n\to\infty} \inf \frac{\Phi(n)}{n} \leq \lim\limits_{n\to\infty} \sup \frac{\Phi(n)}{n} \leq \frac{a_t^2 - a_1^2}{\mathcal{V}_* a_1 a_t}$

Proof. Let $\sum_{i=1}^{t} x_i a_i = \sum_{i=1}^{t} y_i a_i \in S$. Let $\sum_{i=1}^{t} x_i = n, \sum_{i=1}^{t} y_i = m$. From [3], $\rho(S) = \frac{a_t}{a_1}$.

It follows that

$$\frac{a_1}{a_t} \leq \frac{m}{n} \leq \frac{a_t}{a_1} \Rightarrow \frac{a_1 n}{a_t} \leq m \leq \frac{a_t n}{a_1}.$$

Therefore we can bound the size of $\mathcal{V}(n)$. So,

$$\frac{(\frac{a_t}{a_1} - \frac{a_1}{a_t})n}{\mathcal{V}^*} + 1 \leq \Phi(n) \leq \frac{(\frac{a_t}{a_1} - \frac{a_1}{a_t})n}{\mathcal{V}_*} + 1$$

Thus,

$$\frac{a_t^2 - a_1^2}{\mathcal{V}^* a_1 a_t} n + 1 \leq \Phi(n) \leq \frac{a_t^2 - a_1^2}{\mathcal{V}_* a_1 a_t} n + 1$$

By taking the limit, we get that

$$\frac{a_t^2 - a_1^2}{\mathcal{V}^* a_1 a_t} \leq \lim_{n\to\infty} \inf \frac{\Phi(n)}{n} \leq \lim_{n\to\infty} \sup \frac{\Phi(n)}{n} \leq \frac{a_t^2 - a_1^2}{\mathcal{V}_* a_1 a_t}.$$

□

Corollary 2.6. *If* $\Delta_{\mathcal{V}}(S) = \{g\}$*, then* $\lim\limits_{n\to\infty} \frac{\Phi(n)}{n} = \frac{a_t^2 - a_1^2}{g a_1 a_2}$*.*

Proof. $\Delta_{\mathcal{V}}(S) = \{g\}$ implies $\mathcal{V}_* = \mathcal{V}^* = g$. By Theorem 2.5,

$$\frac{a_t^2 - a_1^2}{g a_1 a_t} \leq \lim_{n\to\infty} \inf \frac{\Phi(n)}{n} \leq \lim_{n\to\infty} \sup \frac{\Phi(n)}{n} \leq \frac{a_t^2 - a_1^2}{g a_1 a_t}$$

Therefore,

$$\lim_{n\to\infty} \frac{\Phi(n)}{n} = \frac{a_t^2 - a_1^2}{g a_1 a_t}.$$

□

3 Numerical Monoids Generated by an Interval

Lemma 3.1. *Let $S = \langle a, a+k, \dots, a+wk \rangle$ where $w \leq (a-1)$.*

$$\mathcal{W}(n) = \{an, an+k, \dots, an+nwk\}.$$

Proof. Assume $\exists r \in \mathcal{W}(n)$ such that $r < an$. Then,

$$r = \alpha_0 \cdot a + \cdots + \alpha_w \cdot (a+wk) < an.$$

Since $a < a+k < \cdots < a+wk$,

$$\alpha_0 \cdot a + \alpha_1 \cdot a + \cdots + \alpha_w \cdot a < an.$$

Thus,

$$\alpha_0 + \alpha_1 + \cdots + \alpha_w < n.$$

There is a contradiction and therefore $r \geq an$. Clearly $an \in \mathcal{W}(n)$, so $\min \mathcal{W}(n) = an$.

Assume $\exists r \in \mathcal{W}(n)$ such that $r > (a+wk)n$. Then,

$$r = \beta_0 \cdot a + \cdots + \beta_w \cdot (a+wk) > (a+wk)n.$$

Since $a+wk > a+(w-1)k > \cdots > a+k > a$,

$$\beta_0 \cdot (a+wk) + \beta_1 \cdot (a+wk) + \beta_2 \cdot (a+wk) + \cdots + \beta_w \cdot (a+wk) > (a+wk)n.$$

Thus,

$$\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_w > n.$$

There is a contradiction and therefore $r \leq (a+wk)n$. Clearly $(a+wk)n \in \mathcal{W}(n)$, so $\max \mathcal{W}(n) = (a+wk)n$.

Therefore, $\mathcal{W}(n) = \{an, an+k, \dots, an+nwk\}$. □

Lemma 3.2. *If $n \in S$, then $n = c_1 a + c_2 k$ with $c_1, c_2 \in \mathbb{N}$ and $0 \leq c_2 < a$.*

Proof. If $n \in S$, then

$$\begin{aligned} n &= \sum_{i=0}^{w} b_i(a+ik) \\ &= a\sum_{i=0}^{w} b_i + k\sum_{i=0}^{w} ib_i \\ &= ad_1 + kd_2. \end{aligned}$$

Let $d_2 = pa + c_2$ with $0 \le c_2 < a$.

$$\begin{aligned} n &= ad_1 + (pa+q)k \\ &= a(d_1 + pk) + c_2k \ . \\ &= ac_1 + kc_2 \end{aligned}$$

□

Theorem 3.3. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$. With $0 \le c_2 < a$,*

$$\mathcal{L}(c_1a + c_2k) = \{c_1 + k\left\lceil \frac{c_2 - c_1w}{a+wk} \right\rceil, c_1 + k\left\lceil \frac{c_2 - c_1w}{a+wk} \right\rceil + k, \ldots, c_1\}$$

Proof. Let $n = c_1a + c_2k \in S$ with $0 \le c_2 < a$. Let $x \in \mathcal{L}(n)$.

Let $n = b_0a + b_1(a+k) + \cdots + b_w(a+wk)$ and then $x = \sum_{i=0}^{w} b_i$. So,

$$n = a\sum_{i=0}^{w} b_i + k\sum_{i=0}^{w} ib_i = ax + k\sum_{i=0}^{w} ib_i.$$

Thus, $n \equiv xa \equiv c_1a \mod k$. Since $gcd(a,k) = 1$, then, $x \equiv c_1 \mod k$.

Then, $\mathcal{L}(n) \subset c_1 + k\mathbb{N}_0$. Let $c_1 + kd \in \mathcal{L}(n)$. Then, we know that

$$a(c_1 + kd) \le n \le (a+wk)(c_1 + kd).$$

It follows that

$$\left\lceil \frac{\frac{n}{a+wk} - c_1}{k} \right\rceil \le d \le \left\lfloor \frac{\frac{n}{a} - c_1}{k} \right\rfloor.$$

So,

$$\min \mathcal{L}(n) = c_1 + k \left\lceil \frac{\frac{n}{a+wk} - c_1}{k} \right\rceil = c_1 + k \left\lceil \frac{c_2 - c_1 w}{a + wk} \right\rceil.$$

Also,

$$\max \mathcal{L}(n) = c_1 + k \left\lfloor \frac{\frac{n}{a} - c_1}{k} \right\rfloor = c_1.$$

Thus, $\mathcal{L}(n) \subset \{c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil, c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil + k, \ldots, c_1\}$.

Now we need to show that $c_1 + dk \in \mathcal{L}(n)$.

Since $a(c_1 + dk) \leq n \leq (a + wk)(c_1 + dk)$ and $n \equiv a(c_1 + dk) \mod k$, then

$$n - a(c_1 + dk) = pk.$$

We have that

$$a(c_1 + dk) - a(c_1 + dk) \leq n - a(c_1 + dk) \leq (a + wk)(c_1 + dk) - a(c_1 + dk).$$

Then

$$0 \leq pk \leq wk(c_1 + dk) \Rightarrow 0 \leq p \leq w(c_1 + dk).$$

So, $n = a(c_1 + dk) + kp$. Let $p = w \left\lfloor \frac{p}{w} \right\rfloor + q$ with $0 \leq q < w$. Thus,

$$n = \left\lfloor \frac{p}{w} \right\rfloor (a + wk) + (a + qk) + (c_1 + dk - 1 - \left\lfloor \frac{p}{w} \right\rfloor)a.$$

Then, there exists a factorization of n with length

$$\left\lfloor \frac{p}{w} \right\rfloor + 1 + \left(c_1 + dk - 1 - \left\lfloor \frac{p}{w} \right\rfloor\right) = c_1 + dk.$$

So, $c_1 + dk \in \mathcal{L}(n)$.

Thus, $\mathcal{L}(n) \supset \{c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil, c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil + k, \ldots, c_1\}$ and therefore $\mathcal{L}(n) = \{c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil, c_1 + k \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil + k, \ldots, c_1\}$. □

Corollary 3.4. *Let $S = \langle a, a + k, \ldots, a + wk \rangle$. $\Delta(S) = \{k\}$.*

Proof. Since $\mathcal{L}_S(c_1 a + c_2 k) = \left\{c_1 + k \left\lceil \frac{c_2 - c_1 w}{a + wk} \right\rceil, c_1 + k \left\lceil \frac{c_2 - c_1 w}{a + wk} \right\rceil + k, \ldots, c_1 \right\}$, $\Delta_S(c_1 a + c_2 k) = \{k\}$ and therefore $\Delta(S) = \{k\}$. □

Lemma 3.5. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$. Given $x, x+k \in S$, either $l(x) = l(x+k)$ or $l(x)+k = l(x+k)$ and either $L(x) = L(x+k)$ or $L(x)+k = L(x+k)$.*

Proof. Let $x = c_1a+c_2k$ where $0 \leq c_2 < a$. If $c_2 = a-1$, then $x+k = c_1a+(a-1)k+k = (c_1+k)a$. If $0 \leq c_2 < a_1$, then $x+k = c_1a+c_2k+k = c_1a+(c_2+1)k$.

Case 1: If $c_2 = a-1$, then

$$\begin{aligned} l(x+k) - l(x) &= \left(c_1 + k - k\left\lfloor \tfrac{(c_1+k)w}{a+wk} \right\rfloor\right) - \left(c_1 - k\left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor\right) \\ &= k\left(1 - \left\lfloor \tfrac{(c_1+k)w}{a+wk} \right\rfloor + \left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor\right). \end{aligned}$$

We see that

$$\begin{aligned} k\left(\tfrac{c_1w-(a-1)}{a+wk} - \tfrac{c_1w+wk}{a+wk}\right) &< k\left(1 - \left\lfloor \tfrac{(c_1+k)w}{a+wk} \right\rfloor + \left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor\right) \\ &< k\left(\tfrac{c_1w-(a-1)}{a+wk} \tfrac{c_1w+wk}{a+wk} + 2\right). \end{aligned}$$

Thus

$$k\left(-1 + \frac{1}{a+wk}\right) < k\left(1 - \left\lfloor \frac{(c_1+k)w}{a+wk} \right\rfloor + \left\lfloor \frac{c_1w-(a-1)}{a+wk} \right\rfloor\right) < k\left(1 + \frac{1}{a+wk}\right).$$

Since $k\left(1 - \left\lfloor \tfrac{(c_1+k)w}{a+wk} \right\rfloor + \left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor\right) \in \mathbb{Z}$, thus $l(x+k) - l(x) = 0$ or k.

Also, $L(x+k) - L(x) = (c_1+k) - c_1 = k$

Case 2: If $c_2 < a-1$, then

$$\begin{aligned} l(x+k) - l(x) &= \left(c_1 - k\left\lfloor \tfrac{c_1w-(a-1)-1}{a+wk} \right\rfloor\right) - \left(c_1 - k\left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor\right) \\ &= k\left(\left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor - \left\lfloor \tfrac{c_1w-(a-1)-1}{a+wk} \right\rfloor\right). \end{aligned}$$

We see that

$$k\left(\frac{1}{a+wk} - 1\right) < k\left(\left\lfloor \frac{c_1w-(a-1)}{a+wk} \right\rfloor - \left\lfloor \frac{c_1w-(a-1)-1}{a+wk} \right\rfloor\right) < k\left(\frac{1}{a+wk} + 1\right).$$

Since $k\left(\left\lfloor \tfrac{c_1w-(a-1)}{a+wk} \right\rfloor - \left\lfloor \tfrac{c_1w-(a-1)-1}{a+wk} \right\rfloor\right) \in \mathbb{Z}$, thus $l(x+k) - l(x) = 0$ or k.

Also, $L(x+k) - L(x) = c_1 - c_1 = 0$. □

Theorem 3.6. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$.*

$$\mathcal{V}(n) = \{n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor, n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor + k, \ldots, n + k\left\lfloor \frac{nw}{a} \right\rfloor\}.$$

Proof. Let $i = ap + q$ where $p = \left\lfloor \frac{i}{a} \right\rfloor$ and $0 \leq q < a$. From Theorem 3.3,

$$l(an + ik) = l(a(n + pk) + qk) = n + kp - k\left\lfloor \frac{(n+kp)w - q}{a+wk} \right\rfloor,$$

$$L(an + ik) = L(a(n + pk) + qk) = n + kp.$$

For $0 \leq i \leq nw - 1$, from Lemma 3.5, $l(an + ik) \leq l(an + (i+1)k)$ and since $\min \mathcal{W}(n) = an$, then

$$l(an + ik) = l(an) = n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor.$$

Also, from Lemma 3.5, $L(an + ik) \leq L(an + (i+1)k)$ and since $\max \mathcal{W}(n) = an + nwk$, then

$$L(x) = L(an + nwk) = n + k\left\lfloor \frac{nw}{a} \right\rfloor.$$

From Corollary 3.4, $\Delta(S) = k$ and then from Corollary 2.4, $\Delta_{\mathcal{V}}(S) = k$. Since $\mathcal{V}(n) = \bigcup_{m \in w(n)} \mathcal{L}(m)$, therefore

$$\mathcal{V}(n) = \left\{n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor, n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor + k, \ldots, n + k\left\lfloor \frac{nw}{a} \right\rfloor\right\}.$$

□

4 Equality of $\mathcal{V}$ Sets and Length Sets

Theorem 4.1. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$ where $w \leq a - 1$ and $S' = \langle c, c+t, \ldots, c+vt \rangle$ where $v \leq c - 1$ and $S \ncong S'$. $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$ if and only if $k = t$ and $\frac{c}{a} = \frac{v}{w}$.*

Proof. $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)$ implies that

$$\min \mathcal{V}_S(n) = \min \mathcal{V}_{S'}(n) \Rightarrow n - k \left\lfloor \frac{nw}{a+wk} \right\rfloor = n - t \left\lfloor \frac{nv}{c+vt} \right\rfloor.$$

Let $n = (a+wk)(c+vt)$. So, $kw(c+vt) = tv(a+wk)$ and thus $avt = cwk$.

However, $\Delta(S) = k$ and $\Delta(S') = t$, so from Corollary 2.4, $\Delta_{\mathcal{V}}(S) = k$ and $\Delta\mathcal{V}(S') = t$, thus $k = t$ and therefore $\frac{c}{a} = \frac{v}{w}$.

Given $k = t$ and $\frac{c}{a} = \frac{v}{w}$, show that $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)$. Since $k = t$, thus $\Delta_{\mathcal{V}}(S) = \Delta_{\mathcal{V}}(S')$. Since $\frac{c}{a} = \frac{v}{w}$, thus $c = la$ and $v = lw$ for some $l \in \mathbb{Q}$.

$$\min \mathcal{V}_{S'}(n) = n - t \left\lfloor \frac{nv}{c+vt} \right\rfloor = n - k \left\lfloor \frac{nlw}{la+lwk} \right\rfloor = n - k \left\lfloor \frac{nw}{a+wk} \right\rfloor = \min \mathcal{V}_S(n).$$

$$\max \mathcal{V}_{S'}(n) = n + t \left\lfloor \frac{nv}{c} \right\rfloor = n + k \left\lfloor \frac{nlw}{la} \right\rfloor = n + k \left\lfloor \frac{nw}{a} \right\rfloor = \max \mathcal{V}_S(n).$$

Therefore, $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$. □

Definition 4.2. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$. A set of lengths has a jump if $\exists\ x, x+k \in S$ such that $l(x) + k = l(x+k)$ and $L(x) + k = L(x+k)$.*

Lemma 4.3. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$. $\mathcal{L}(S)$ has a jump if and only if* $\gcd(a, w) = 1$.

Proof. Given that $\mathcal{L}(S)$ has a jump, show that $\gcd(a, w) = 1$. Let $l(x) = m$, so $l(x+k) = l(x)+k = m+k$. Let $x = d_0 a + \cdots + d_w(a+wk)$ where $\sum_{i=0}^{w} d_i = m$.

If $x \neq (a+wk)m$, then $d_w < m$, so select some $i < k$ such that $d_i \neq 0$. $x+k = d_0 a + \cdots + (d_i - 1)(a+ik) + (d_{i+1}+1)(a+(i+1)k) + \cdots + d_w(a+wk)$ where $\sum_{i=0}^{w} d_i = m$. So, this is a factorization of $x+k$ of length m, but $l(x+k) = m+k$, so there is a contradiction. Thus, $x = (a+wk)m$.

Let $L(x+k) = m + nk$, so $L(x) = L(x+k) - k = m + (n-1)k$. Say that $x + k = d_0 a + \cdots + d_w(a+wk)$ where $\sum_{i=0}^{w} d_i = m + nk$.

If $x+k \neq a(m+nk)$, then $d_0 < m+nk$, so select $i > 0$ such that $d_i \neq 0$. $x = d_0 a + \cdots + (d_{i-1}+1)(a+(i-1)k) + (d_i - 1)(a+ik) + \cdots + d_w(a+wk)$ where $\sum_{i=0}^{w} d_i = m+nk$. So, this is a factorization of x of length $m+nk$, but $L(x) = m+(n-1)k$, so there is a contradiction. Thus, $x+k = a(m+nk)$.

$x = (a+wk)m$ implies $x - am = wmk$ and $x+k = a(m+nk)$ implies $x - am = ank - k$. So, $wmk = ank - k$ or equivalently $na - mw = 1$. Therefore, there are positive integral solutions for m, n if and only if $\gcd(a,w) = 1$.

Given $\gcd(a,w) = 1$, show that $\mathcal{L}(S)$ has a jump. $\exists\ v_1, v_2$ such that $av_1 + wv_2 = 1$. Let $x = -v_2(a+wk)$, then $x = ((v_1-1)k - v_2)a + (a-1)k$. Then, $c_1 = ((v_1-1)k - v_2)$ and $c_2 = a-1$.

$$\begin{aligned}
l(x+k) - (l(x)+k) &= \left(c_1 + k + k\left\lfloor \tfrac{(c_1+k)w}{a+wk} \right\rfloor\right) - \left(c_1 + k\left\lfloor \tfrac{c_1 w - c_2}{a+wk} \right\rfloor + k\right) \\
&= k\left(\left\lfloor \tfrac{((v_1-1)k-v_2)w+wk}{a+wk} \right\rfloor - \left\lfloor \tfrac{((v_1-1)k-v_2)w-a+1}{a+wk} \right\rfloor\right) \\
&= k\left(\left\lfloor \tfrac{v_1(a+wk)-1}{a+wk} \right\rfloor - \left\lfloor \tfrac{(v_1-1)(a+wk)}{a+wk} \right\rfloor\right) \\
&= k((v_1-1) - (v_1-1)) \\
&= 0.
\end{aligned}$$

$L(x+k) - (L(x)+k) = (c_1+k) - (c_1+k) = 0.$

Since $l(x)+k = l(x+k)$ and $L(x)+k = L(x+k)$, $\mathcal{L}(S)$ has a jump. $\square$

Definition 4.4. *Let $S = \langle a, a+k, \ldots, a+wk \rangle$. Let $i = ap+q$ with $0 \leq q < a$ for $i \geq a$. $f : \mathbb{N} \to S$ such that $f(i) = f(ap+q) = ap+qk \in S$.*

Theorem 4.5. *Let $S = \langle a, a+k, \ldots a+wk \rangle$ where $w \leq a-1$ and $S' = \langle c, c+t, \ldots c+vt \rangle$ where $v \leq c-1$ and $S \not\cong S'$. $\mathcal{L}(S) = \mathcal{L}(S')$ if and only if $k = t$, $\frac{c}{a} = \frac{v}{w}$, $\gcd(a,w) \geq 2$, and $\gcd(c,v) \geq 2$.*

Proof. Given $\mathcal{L}(S) = \mathcal{L}(S')$, we know that $\rho(S) = \rho(S')$, which implies $\frac{a+wk}{a} = \frac{c+vt}{c}$, so $\frac{c}{a} = \frac{vt}{wk}$. Also, we know that $\Delta(S) = \Delta(S')$, so $k = t$ and thus $\frac{c}{a} = \frac{v}{w}$.

Without loss of generality, let $c > a$. Assume $\gcd(a, w) = 1$. From Lemma 4.3, we know that S has a jump and we need S' to have the same jump.

However, $\frac{c}{a} = \frac{v}{w}$ implies $cw = av$. Since $\gcd(a, w) = 1$, thus $a|c$. Then, $c = ja$ and $v = jw$ where $j \in \mathbb{N}$ with $j \geq 2$ since $c > a$. So,

$$\gcd(c, v) = \gcd(ja, jw) = j \geq 2.$$

Therefore, from Lemma 4.3, there are no jumps in S' and thus $\mathcal{L}(S) \neq \mathcal{L}(S')$. Thus, $\gcd(a, w) \geq 2$ and $\gcd(c, v) \geq 2$.

Given $k = t$, $\frac{c}{a} = \frac{v}{w}$, $\gcd(a, w) \geq 2$ and $\gcd(c, v) \geq 2$, show that $\mathcal{L}(S) = \mathcal{L}(S')$.

Let $i = ap + q$ where $0 \leq q < a$. Also, we know $\frac{c}{a} = \frac{v}{w} = \frac{c+vk}{a+wk}$.

Case 1: Assume $\frac{cq}{a} \in \mathbb{Z}$, then $0 \leq \frac{cq}{a} < c$. Therefore,

$$\begin{aligned}
l\left(f\left(\left\lceil \tfrac{ci}{a} \right\rceil\right)\right) &= l\left(cp + \tfrac{cq}{a}k\right) \\
&= p + k\left\lceil \frac{\frac{cq}{a} - vp}{c+vk} \right\rceil \\
&= p + k\left\lceil \frac{v(\frac{cq}{av} - p)}{c+vk} \right\rceil \\
&= p + k\left\lceil \frac{\frac{cwq}{av} - pw}{a+wk} \right\rceil \\
&= p + k\left\lceil \frac{q - pw}{a+wk} \right\rceil \\
&= l(ap + qk) \\
&= l(f(i)).
\end{aligned}$$

Also,

$$\begin{aligned} L\left(f\left(\left\lceil \tfrac{ci}{a} \right\rceil\right)\right) &= L\left(cp + \tfrac{cq}{a}k\right) \\ &= p \\ &= L(ap+qk) \\ &= L(f(i)). \end{aligned}$$

Case 2: Assume $\frac{cq}{a} \notin \mathbb{Z}$. Then, since $0 \leq q < a$,

$$0 \leq \left\lceil \frac{cq}{a} \right\rceil \leq \left\lceil \frac{c(a-1)}{a} \right\rceil \leq c.$$

Case A: Assume $0 \leq \left\lceil \frac{cq}{a} \right\rceil \leq c-1$. Then,

$$\begin{aligned} l(f(\lceil \tfrac{ci}{a} \rceil)) &= l(cp + \lceil \tfrac{cq}{a} \rceil k) \\ &= p + k\left\lceil \frac{\lceil \frac{cq}{a} \rceil - vp}{c+vk} \right\rceil \\ &= p + k\left\lceil \frac{\frac{cq}{a} - vp}{c+vk} \right\rceil \\ &= p + k\left\lceil \frac{\frac{wc}{av}q - wp}{a+wk} \right\rceil \\ &= p + k\left\lceil \frac{q-wp}{a+wk} \right\rceil \\ &= l(ap+qk) \\ &= l(f(i)). \end{aligned}$$

Also, $L(f(\lceil \frac{ci}{a} \rceil)) = L(cp + \lceil \frac{cq}{a} \rceil k) = p = L(ap+qk) = L(f(i))$.

Case B: Assume $\lceil \frac{cq}{a} \rceil = c$. Since $\frac{cq}{a} \notin \mathbb{Z}$, then $\lfloor \frac{cq}{a} \rfloor = c-1$.

Assume $l(f(\lfloor \frac{ci}{a} \rfloor)) \neq l(f(i))$. It follows that $l(cp + \lfloor \frac{cq}{a} \rfloor) \neq l(ap+q)$. So,

$$p + k\left\lceil \frac{\lfloor \frac{cq}{a} \rfloor - pv}{c+vk} \right\rceil \neq p + k\left\lceil \frac{q-pw}{a+wk} \right\rceil.$$

Thus,

$$\left\lceil \frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk} \right\rceil \neq \left\lceil \frac{q - pw}{a+wk} \right\rceil.$$

Then,

$$\left\lceil \frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk} \right\rceil \neq \left\lceil \frac{\frac{cq}{a} - \frac{pcw}{a}}{\frac{c}{a}(a+wk)} \right\rceil.$$

Therefore,

$$\left\lceil \frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk} \right\rceil \neq \left\lceil \frac{\frac{cq}{a} - pv}{c+vk} \right\rceil.$$

Thus, there must be some integer contained in the interval $\left[\frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk}, \frac{\frac{cq}{a} - pv}{c+vk} \right)$.
However,

$$\left| \left(\frac{\frac{cq}{a} - pv}{c+vk} \right) - \left(\frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk} \right) \right| = \left| \frac{\frac{cq}{a} - \left\lfloor \frac{cq}{a} \right\rfloor}{c+vk} \right| < \frac{1}{c+vk}.$$

So, there is at most one element in the interval in $\frac{1}{c+vk}\mathbb{Z}$. In fact, the lower bound is the only possibility for an integer in the interval and therefore

$$\frac{\left\lfloor \frac{cq}{a} \right\rfloor - pv}{c+vk} \in \mathbb{Z}.$$

Since $\left\lfloor \frac{cq}{a} \right\rfloor = c - 1$, then $(c+vk) | (c - 1 - pv)$.

Let $d = \gcd(c, v)$. Then, $d|c$ and $d|v$, so $d|(c+vk)$. Thus, $d|(c-1-pv)$, so $d|1$, but that is a contradiction since $d \neq 1$.

Then,

$$l\left(f\left(\left\lfloor \frac{ci}{a} \right\rfloor\right)\right) = l(f(i)).$$

Also,

$$L\left(f\left(\left\lfloor \frac{ci}{a} \right\rfloor\right)\right) = L(cp+qk) = p = L(ap+qk) = L(f(i)).$$

Thus, $\mathcal{L}(S) \subseteq \mathcal{L}(S')$ and since we did not assume any conditions on S or S', then we also have that $\mathcal{L}(S') \subseteq \mathcal{L}(S)$. Therefore, $\mathcal{L}(S) = \mathcal{L}(S')$. □

Corollary 4.6. *Let* $S = \langle a, a+k, \ldots, a+wk\rangle$ *where* $w \le a-1$ *and* $S' = \langle c, c+t, \ldots, c+vt\rangle$ *where* $v \le c-1$ *and* $S \not\cong S'$. *If* $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$, *then* $\mathcal{L}(S) = \mathcal{L}(S')$ *if and only if* $\gcd(a, w) \ge 2$ *and* $\gcd(c, v) \ge 2$.

Proof. From Theorem 4.1, $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)$ implies $k = t$ and $\frac{c}{a} = \frac{v}{w}$, so from Theorem 4.5, we also need $\gcd(a, w) \ge 2$ and $\gcd(c, v) \ge 2$ in order for $\mathcal{L}(S) = \mathcal{L}(S')$. Given $\gcd(a, w) = 1$ or $\gcd(c, v) = 1$, from Theorem 4.5, $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$ implies $\mathcal{L}(S) \ne \mathcal{L}(S')$. □

Corollary 4.7. *Let* $S = \langle a, a+k, \ldots, a+wk\rangle$ *where* $w \le a-1$ *and* $S' = \langle c, c+t, \ldots, c+vt\rangle$ *where* $v \le c-1$ *and* $S \not\cong S'$. *If* $\mathcal{L}(S) = \mathcal{L}(S')$, *then* $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$.

Proof. From Theorem 4.5, $\mathcal{L}(S) = \mathcal{L}(S')$ implies that $k = t$ and $\frac{c}{a} = \frac{v}{w}$, so from Theorem 4.1, $\mathcal{V}_S(n) = \mathcal{V}_{S'}(n)\ \forall\ n \in \mathbb{N}$. □

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