P-adic Upper Half Planes and Representation Numbers of Quadratic Forms

Benjamin Ellison
Case Western Reserve University, Cleveland, OH.
James B. Wilson
Portland State University, Portland, OR.[1]

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Recent work on representation numbers for certain quadratic forms has revealed their solution lies within the structure of $(p+1)$–regular trees. Specifically we study denumeration methods for directed paths at a relative distance to a central path. Certain specific formulas are given as well as some theory concerning the structure and derivation of such denumeration formulas. Some attention is placed on computational methods of determining these formulas.

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1 Introduction

Representation numbers of quadratic forms can be described so simply that their often deceptive derivation can be baffling. The modern approaches to determine certain representation numbers often bring together large areas of mathematics in what may seem like magical connections at first. True to form, the approaches explored in this article rely on quaternion algebras, p-adic numbers, graph theory, and many other fields in between. Due to the magnitude of background theory and the limited scope of this article, the reader is assumed to have a general knowledge of p-adic fields and abstract algebra.

One of the fundamental properties of quaternion algebras is the norm of an element in the algebra. Given a quaternion algebra with an order, an interesting problem is attempting to find how many elements of the order have the property that their norm is equal to a power of some prime p. These elements are called the *representation numbers* of p^k. Interestingly, it is possible to determine the number of representation numbers for powers of primes using quaternion algebras as well as the topology of the p-adic upper half planes.

The representation numbers that will be obtained will come from a class of quadratic forms that can be constructed through the norm functions of certain quaternion algebras over the rationals. Due to the nature of the constructed quaternion algebras, certain maximal subrings can be placed in a one to one correspondence with a $(p+1)-$ regular tree. A distance formula is defined on the quaternion algebra and generalized to the abstract tree. This established, a special subgroup is chosen to act on the vertices of the tree resulting in a partition of the vertices.

The partition is a critical step in connecting the denumeration of paths in the tree with the original norm functions whose representation numbers we seek. Fortunately the group is chosen to act as an isometry with respect to the generalized distance described earlier. This means the distance between vertices and directed paths is not modified by the group action. However this property only guarantees the distance is preserved. When creating the formulas which will denumerate the paths at a given distance from another, careful attention must be paid to ensure that the formula can be partitioned by the action. Specifically this implies that simple combinatorical arguments which result in the appropriate cardinalities will more than likely fail when the group action is applied. Thus closed forms in general cannot be found that satisfy the specifications. More detail on the exact requirements will be given later.

2 Quaternion Algebras

Algebras behave in part as vector spaces and in part also as rings having a defined multiplication between the "vectors". Quaternion algebras extend this requirement by imposing a rule on the dimension of the vector space and requiring certain rules be observed between specific vectors. The construction of the quaternion algebras is such that they can be described, up to isomorphism, as certain 2×2 matrices, a point which we will return to later. First the formal definition for quaternions as well as some needed operations are given.

Definition 2.1 *Let $\mathbb{F}$ be a field of characteristic not equal to 2, and $a, b \in \mathbb{F}^{\times}$. A quaternion algebra over $\mathbb{F}$ is a four dimensional vector space over $\mathbb{F}$ with vector space basis $\{1, i, j, k\}$ which satisfies the relations $i^2 = a$, $j^2 = b$, $ij = -ji = k$. Such an algebra will be denoted $\left(\frac{a,b}{\mathbb{F}}\right)$.*

Observe that the condition $ij = -ji$ implies these algebras are non-commutative.

Example 2.2 *The classic example is Hamilton's quaternions, $\left(\frac{-1,-1}{\mathbb{Q}}\right)$.*

Definition 2.3 *Let $\mathbb{A} = \left(\frac{a,b}{\mathbb{F}}\right)$ be a general quaternion algebra and also let $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{A}$. The following operations are defined on x as follows:*

- *Conjugate of x*
$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k$$
- *Norm of x*
$$N(x) = x\overline{x} = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$$
- *Trace of x*
$$Tr(x) = x + \overline{x} = 2x_1$$

Notice also that the norm function describes one quadratic form in four variables. Various other quaternion algebras result in norm functions with other quadratic forms. We will restrict our consideration to the following quaternion algebras. For prime p, define:

$$\mathbb{A} = \begin{cases} \left(\frac{-1,-1}{\mathbb{Q}}\right), & p = 2 \\ \left(\frac{-1,-p}{\mathbb{Q}}\right), & p \equiv 3 \bmod 4 \\ \left(\frac{-2,-p}{\mathbb{Q}}\right), & p \equiv 5 \bmod 8 \\ \left(\frac{-p,-r}{\mathbb{Q}}\right), & p \equiv 1 \bmod 8, \end{cases} \tag{1}$$

where r is a prime which satisfies $r \equiv 3 \bmod 4$ and $\left(\frac{p}{r}\right) = -1$.

Additionally we will make use of a specific substructure of these quaternions.

Definition 2.4 *An order of $\mathbb{A}$ is a free $\mathbb{Z}$-submodule of $\mathbb{A}$ of rank 4 which is also a subring of $\mathbb{A}$ containing 1.*

This means that if $\mathcal{O}$ is an order of $\mathbb{A}$, there exists a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\mathcal{O} = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4.$$

Theorem 2.5 *([Rei75]) If $x \in \mathcal{O}$, then $N(x) \in \mathbb{Z}$ and $Tr(x) \in \mathbb{Z}$.*

In fact, let $x \in \mathcal{O} = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$. It can easily be verified then that

$$N(x) = \sum_{i=1}^{4} x_i^2 N(e_i) + \sum_{i<j} x_i x_j Tr(e_i \overline{e_j}).$$

Thus these norms when restricted to these orders produce new quadratic forms with integer coefficients as desired. While not all quadratic forms of 4 variables can be given such a representation, the ones that do can be used to solve for their representation numbers.

2.1 Quaternion Algebras over $\mathbb{Q}_p$

Let $\mathbb{Q}_p$ be the field of p-adic numbers; so as a set,

$$\mathbb{Q}_p = \{a_{-n}p^{-n} + \ldots + a_0 + a_1 p + \ldots \mid a_i \in \mathbb{Z}/p\mathbb{Z}\}.$$

The "ring of integers" of $\mathbb{Q}_p$ is the p-adic integers:

$$\mathbb{Z}_p = \{a_0 + a_1 p + \ldots a_n p^n + \ldots \mid a_i \in \mathbb{Z}/p\mathbb{Z}\}.$$

For $x = \sum_{i=-n}^{\infty} a_i p^i \in \mathbb{Q}_p$, we define

$$\begin{aligned} \mathrm{ord}_p(x) &= t, \text{ if } p^t | x, p^{t+1} \nmid x \\ \mathrm{ord}_p(0) &= \infty. \end{aligned}$$

For a detailed introduction to the p-adic numbers, the interested reader should consult [Gou97].

For each prime q, let $\mathbb{A}_q$ be the corresponding quaternion algebra over $\mathbb{Q}_q$ of the original ones given in the Equation 1. That is,

$$\mathbb{A}_q = \begin{cases} \left(\frac{-1,-1}{\mathbb{Q}_q}\right), & p = 2 \\ \left(\frac{-1,-p}{\mathbb{Q}_q}\right), & p \equiv 3 \bmod 4 \\ \left(\frac{-2,-p}{\mathbb{Q}_q}\right), & p \equiv 5 \bmod 8 \\ \left(\frac{-p,-r}{\mathbb{Q}_q}\right), & p \equiv 1 \bmod 8, \end{cases} \tag{2}$$

Theorem 2.6 *([Piz80], Proposition 5.1) For $p \neq q$, $\mathbb{A}_q \cong M_2(\mathbb{Q}_q)$.*

The analog of the Definition 2.4 comes by replacing the integers with the p-adic integers.

Definition 2.7 *An order of $\mathbb{A}_q$ is a free $\mathbb{Z}_q$-submodule of $\mathbb{A}_q$ of rank 4 that is also a subring with a 1.*

We now further restrict the orders with properties provided to us from the p-adic integers. Given that they can be represented as 2×2 matrices, the orders will be expressed in terms of their matrix equivalent. From here we obtain the following theorem.

Theorem 2.8 *([Rei75], Chapter 5, Theorem 17.3) For $p \neq q$, the maximal orders of $\mathbb{A}_q$ are $\{xM_2(\mathbb{Z}_q)x^{-1}, |x \in \mathbb{A}_q^{\times}\}$.*

Definition 2.9 *The discriminant of an order is defined as*

$$\det(Tr(e_i\overline{e_j})).$$

We now define special orders in $\mathbb{A}$ by specifying what they look like for each prime p. Let $\mathcal{O} = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$ be an order in one of the quaternion algebras listed in Equation 1. We say $\mathcal{O}$ has "level q^n" if the following conditions hold:

- $\mathrm{disc}(\mathcal{O}) = (pq^n)^2$
- $\mathcal{O}_q = \mathbb{Z}_q e_1 + \mathbb{Z}_q e_2 + \mathbb{Z}_q e_3 + \mathbb{Z}_q e_4$ is isomorphic to $\begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ q^n\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix}$
- $\mathcal{O}_r = \mathbb{Z}_r e_1 + \mathbb{Z}_r e_2 + \mathbb{Z}_r e_3 + \mathbb{Z}_r e_4$ is a maximal order of $\mathbb{A}_r$ for $r \neq q$.

2.2 The Tree

One can define a "distance" between any two maximal orders of $\mathbb{A}_q$:

$$d(xM_2(\mathbb{Z}_q)x^{-1}, yM_2(\mathbb{Z}_q)y^{-1}) = \mathrm{ord}_q(\det(x^{-1}y)) - 2 \cdot \min\{\mathrm{ord}_q(x^{-1}y)_{ij}\}.$$

Clearly this distance behaves differently from a formal metric since certain elements may produce a negative distance. However we stick with the label of distance as it will relate to a more metric like operation when abstracted to the tree we are constructing.

Theorem 2.10 *([Vig80], Chapter II, Corollary 2.6) For $q \neq p$, one can associate to $\mathbb{A}_q$ a graph. The vertices are the maximal orders of $\mathbb{A}_q$, and an edge is placed between any two vertices at distance 1. The graph is a $(q+1)$-regular tree.*

This tree can also be identified with $SL_2(\mathbb{Q}_p)/SL_2(\mathbb{Z}_P)$, so it can be called a p-adic upper half-plane. We let $\mathcal{O}[\frac{1}{q}]^{\times}$ denote the invertible elements of the ring of polynomials in $\frac{1}{q}$ with coefficients in $\mathcal{O}$.

Theorem 2.11 *([Vig80], Chapter 5, Proposition 3.3) Let $\mathcal{O}$ be an order of level q^0 in $\mathbb{A}$. Then $\mathcal{O}[\frac{1}{q}]^{\times}$ acts on the vertices of the tree by conjugation, and partitions the vertices into a finite number of orbits.*

Define the matrix $D^{(k)} = (D_{ij}^{(k)})$ by letting $D_{ij}^{(k)}$ be the number of vertices in orbit j at distance k from a fixed vertex in orbit i.

These matrices have the operations of multiplication, addition and scaling between them. We wish to relate these operations to the graphical representation so that a formula derived from the graph can be converted to these desired matrices.

Definition 2.12 *Let P_0 be the set of all vertices in the tree. Given a fixed vertex $\tau \in P_0$, define*

$$P_0^d \tau = \{\alpha \in P_0 : d(\tau, \alpha) = d\}$$

so that $P_0^d \tau$ represents all vertices a distance d from the central vertex τ. Since the central vertex is arbitrary to the cardinality and structure of these sets, in certain instances the τ may be omitted.

Definition 2.13 *Given P_0^d and $P_0^{d'}$ the product $P^d P^{d'}$ is defined as*

$$P_0^d P_0^{d'} = \overset{\circ}{\bigcup_{\alpha \in P_0^d(\tau)}} P_0^{d'}(\alpha)$$

where each $P_0^d(\alpha)$ is considered disjoint from the others. So $P_0^d P_0^{d'}$ is considered a family of vertices, possibly with multiple copies of the same vertex.

Note the sum of two families of vertex is simply their disjoint union. Since this is a disjoint union it is easy to see that

$$|P_0^d P_0^{d'}| = \sum_{\alpha \in P_0^d(\tau)} |P_0^{d'}(\alpha)| = \left(\sum_{\alpha \in P_0^d(\tau)} \right) |P_0^{d'}| = |P_0^d||P_0^{d'}|$$

For convenience we define nP_0^d as the sum n times of P_0^d, where $n \in \mathbb{N}$. We now assert $D_{i,j}^{(n)} = |P_0^n(x_i) \cap \mathcal{O}X_j|$, where x_i is an arbitrary fixed vertex in orbit i and x_j is an element in orbit j so that $\mathcal{O}x_j$ represents the entire orbit j.[2] In general if we have an expression relating families of P_0^d vertices as products and sums, the induced partition can be applied uniformly to the expression to determine the corresponding matrix expression.

[2] Although the partitioned objects are families, the intersection here is defined to admit only elements in j whether or not there are multiple copies of a given element. That is to say the intersection acts as a set intersection and not as an analog to subtraction. However if both objects in the intersection contain an particular element, the resulting object will contain the minimum number of copies of this element found in either of the two objects.

2.3 Formula for Paths of Length 0

Theorem 2.14 *([Pay91], Theorem 2.5) The matrices $D^{(k)}$ satisfy:*

$$\begin{aligned} D^{(2)} &= D^{(1)}D^{(1)} - (q+1)I \\ D^{(k+1)} &= D^{(k)}D^{(1)} - qD^{(k-1)}, \qquad k \geq 2. \end{aligned}$$

We note that this is the same recursion formula that the Hecke operator for modular forms on $\Gamma_0(1) = SL_2(\mathbb{Z})$.

Theorem 2.15 *([Pay91], Theorem 2.2)*

$$\#\{x \in \mathcal{O} - q\mathcal{O} \mid N(x) = q^k\} = |\mathcal{O}^\times| D^{(k)}_{1,1}$$

Example 2.16 *We now present an extended example of how to compute the representation numbers of the corresponding norm form. For the quaternion algebra we choose $A = \left(\frac{-1,-23}{\mathbb{Q}}\right)$, and thus for the respective quaternion algebra over the $\mathbb{Q}$ we have $\left(\frac{-1,-23}{\mathbb{Q}}\right)$. The order we use is $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}(\frac{i+j}{2}) + \mathbb{Z}(\frac{1+k}{2})$. One can show that the the norm of x is*

$$N(x) = x_1^2 + x_1x_2 + 6x_2^2 + x_3^2 + x_3x_4 + x_4^2$$

From the previous theorem, we know that

$$\#\{x \in \mathcal{O} - q\mathcal{O} \mid N(x) = q^k\} = |\mathcal{O}^\times| D^{(k)}_{1,1}$$

One can show [Pay91] that $|\mathcal{O}^\times| = 4$. If we choose $p = 2$, one can also show that $D^{(1)} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}$. Using generating functions, we get that

$$\sum_{n=0}^{\infty} D^{(n)}x^n = [I - xD^{(1)} + px^2I]^{-1}$$

This results in

$$\begin{aligned} \sum_{n=0}^{\infty} D^{(n)}x^n &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} x & 2x & 0 \\ x & x & x \\ 0 & 3x & 0 \end{pmatrix} + \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 2x^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - x + 2x^2 & -2x & 0 \\ -x & 1 - x + 2x^2 & -x \\ 0 & -3x & 1 + 2x^2 \end{pmatrix} \end{aligned}$$

Using Maple to find the inverse of this matrix, we get that the rational polynomial in the $(1,1)$ position is $\frac{1-x+x^2-2x^3+4x^4}{1-2x+2x^2-5x^3+4x^4-8x^5+8x^6}$. Again, using Maple and expanding the

Taylor Series of this rational polynomial about 0, we get $1 + x + x^2 + 3x^3 + 13x^4 + 33x^5 + \dots$ *Therefore, we get*

$$\#\{x \in \mathcal{O} - q\mathcal{O} \,|\, N(x) = 2^k\} = |\mathcal{O}^\times| D_{1,1}^{(k)} = 4(1 + x + x^2 + 3x^3 + 13x^4 + 33x^5 + \dots)$$

Thus, for example, the number of representation numbers which equals 2^k *is given by* $4 \cdot a_k$*, where* a_k *is the kth coefficient in* $1+x+x^2+3x^3+13x^4+33x^5+\dots$ *Therefore, the number which equals* $2^0 = 1$ *is* $4 \cdot 1$*. Similarly, with* 2^1 *and* 2^2 *is also* 4*. With* $2^3 = 8$*, the number of representation numbers is* $4 \cdot 3 = 12$*. And with* $2^4 = 16$ *we have* $4 \cdot 13 = 52$*. From this, we can then get all the representation for all values of* 2^k*.*

2.4 Formula for Paths of Length 1

Our goal is to extend these results. Let g be a path of length n on the tree: $\alpha = (a_0, a_2, \dots, a_n)$. Since the graph is a tree, we observe that α is uniquely determined by its initial and terminal vertices: $\alpha = (a_0, a_n)$. One can show (see [Ros]) that every path of length n can be written in the form

$$\left(x \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ \mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} x^{-1}, x \begin{pmatrix} 0 & 1 \\ q^n & 0 \end{pmatrix} \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ \mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q^n & 0 \end{pmatrix}^{-1} x^{-1} \right),$$

where $x \in \mathbb{A}_q^\times$.

Additionally we extend the definition of distance to define a distance between to paths of the same length as follows.

Definition 2.17 *Let* $\alpha = (a_0, a_n)$ *and* $\beta = (b_0, b_n)$ *be two arbitrary paths of length* n*. The* distance *between* α *and* β *is given as* $d(\alpha, \beta) = d$ *when* $d(a_0, b_0) = d$ *and* $d(a_n, b_n) \leq d$*.*

Example 2.18 *The path* $\alpha = (a_0, a_n)$ *has a distance* l *from its reverse path* $\overline{\alpha} = (a_n, a_0)$ *since* $d(a_0, a_n) = d(a_n, a_0)$*.*

However the distance of α *to a path* $\beta = (a_0, b_n)$ *where* $b_n \neq a_n$ *is not defined since* $d(a_0, a_0) = 0$ *but* $d(a_n, b_n) > 0$*.*

Let $\mathcal{O}$ be an order of level q^n in $\mathbb{A}$. Then $\mathcal{O}[\frac{1}{q}]^\times$ acts on paths of length n on the tree as follows. If $a \in \mathcal{O}[\frac{1}{q}]^\times$, then

$$a \cdot g = \left(ax \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ \mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} x^{-1} a^{-1}, ax \begin{pmatrix} 0 & 1 \\ q^n & 0 \end{pmatrix} \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q \\ \mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q^n & 0 \end{pmatrix}^{-1} x^{-1} a^{-1} \right).$$

Theorem 2.19 *([Ros], Theorem 2.6)* $\mathcal{O}[\frac{1}{q}]^\times$ *acts on the paths of length* n *by conjugation, and partitions the paths into a finite number of orbits.*

Now define the matrix $(D_n^{(k)})_{i,j}$ by letting the ij^{th} entry of $D_n^{(k)}$ be the number of paths of length n in orbit j at distance k from a fixed path in orbit i. Once again we define their graphical counterparts.

Definition 2.20 *Let P_n be the set of all directed paths of length n. Given a directed path $\tau \in P_n$, define*

$$P_n^k(\tau) = \{\alpha \in P_n : d(\tau, \alpha) = d\}$$

so that $P_n^k(\tau)$ represents all directed paths a distance d from the central path τ. Since the central path is arbitrary to the cardinality and structure of these sets, in certain instances the τ may be omitted.

Definition 2.21 *Given P_n^k and $P_n^{d'}$ the product $P_n^k P_n^{d'}$ is defined as*

$$P_n^k P_n^{d'} = \overset{\circ}{\bigcup_{\alpha \in P_n^k d(\tau)}} P_n^{d'}(\alpha)$$

where each $P_n^{d'}(\alpha)$ is considered disjoint from the others. So $P_n^k P_n^{d'}$ is considered a family of paths, possibly with multiple copies of the same path.

It is noted then that the case for vertices is simply the case for paths of length 0.

Example 2.22 *Refer to Figure 1 to see the graphical representation of $P_2^2 P_2^1(\tau)$.*

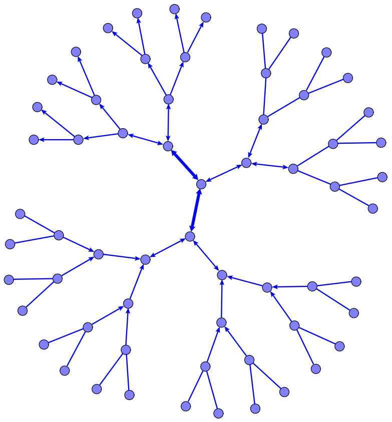

Figure 1

As before the partition is applied through the intersection with the families so that $(D_n^{(k)})_{i,j} = |P_n^k(x_i) \cap \mathcal{O}x_j|$, where x_i is an arbitrary fixed path in orbit i, and x_j is an element of orbit j so that $\mathcal{O}x_j$ represents the entire orbit of j.

We may now begin to compute the recursive formula for paths of length 1. We let $\tau = (t_0, t_1)$ be an arbitrary fixed path of length 1. We begin by considering the distance 0 once again. Clearly as shown in the Example 2.22 the central path is a distance 0 from itself and also we see that any path of the same length that begins at a distance 0 from τ must end at t_1 thus $P_1^0 = \{\tau\}$. This property is true for any length path by the same argument.

We now consider the less trivial cases for greater distances from τ. Refer to Figure 2 for an illustration of all paths in the set $P_1^1(\tau)$. The reader is left to verify all this does represent all of $P_1^1(\tau)$. Additionally refer to Figure 3 which depicts $P_1^3(\tau)$. When we take the product $P_1^3(\tau)P_1^1(\tau)$ we generate the covering illustrated in Figure 4.

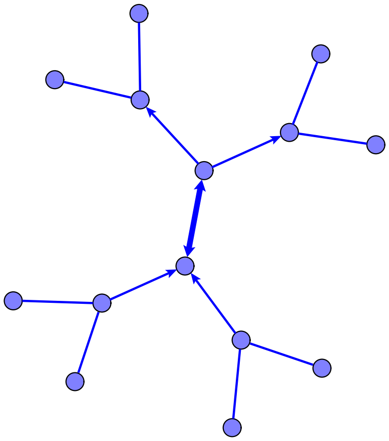

Figure 2

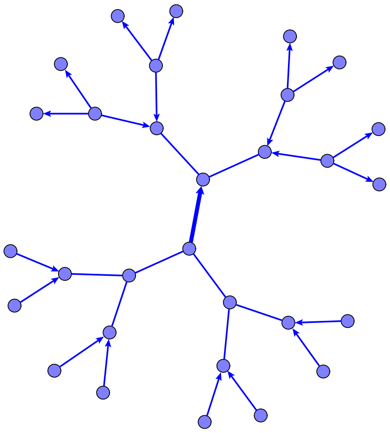

Figure 3

We now proceed to categorize and regroup all the paths in this newly generated cover. Observe that $k = 3$ in our illustrated examples. The case for distance 3 is sufficient for length 1 paths as will be demonstrated in general in section 3. One checks that

$$P_1^k P_1^1 = P_1^{k+1}(\tau) + pP_1^{(k-1)}(\tau) + p\overline{P_1^k(\tau)},$$

where each component is illustrated in Figures 4 and 5. Notice that the elements of $\overline{P_1^k(\tau)}$ are a full copy of P_1^k only their orientation is reversed. Thus to account for this reversal we require a device to treat these paths in their reversed orientation. Such a device is allowable when we work with the expression as matrices.

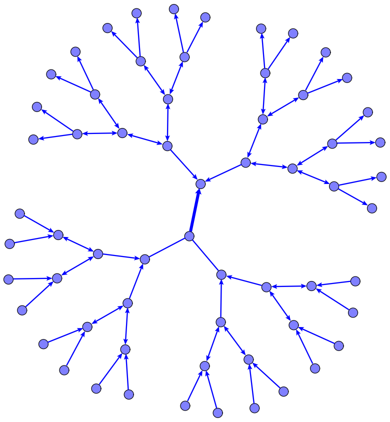

Figure 4

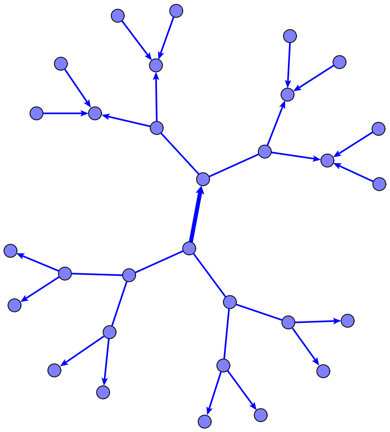

Figure 5

Let $F_1^{(k)}$ be the matrix whose ij^{th} entry is the number of paths in $\overline{P_1^k}$ in orbit j whose reversed path is at distance k from a fixed path in orbit i. We now determine the relationship between $F^{(k)}$ and $D^{(k)}$.

Let g and h be a paths in the tree of length 1. Additionally let g and h be paths of the same orbit j. By our construction both can be represented as:

$$g = \left(xM_2(\mathbb{Z}_q)x^{-1}, x\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} M_2(\mathbb{Z}_q)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}^{-1} x^{-1} \right)$$

$$h = \left(yM_2(\mathbb{Z}_q)y^{-1}, y\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} M_2(\mathbb{Z}_q)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}^{-1} y^{-1} \right).$$

But since both are in the same orbit there exists an element $\lambda \in \mathcal{O}[\frac{1}{q}]^\times$ such that $y = \lambda x$ and so we substitute appropriately leaving

$$h = \left((\lambda x)M_2(\mathbb{Z}_q)(\lambda x)^{-1}, (\lambda x)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} M_2(\mathbb{Z}_q)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}^{-1} (\lambda x)^{-1} \right).$$

We denote the reverse of a path as $\overline{g}$. Now we assume that $\overline{g}$ lines in an orbit r and we will show that $\overline{h}$ must therefore also lie in orbit r. Given

$$\overline{g} = \left(x\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} M_2(\mathbb{Z}_q)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}^{-1} x^{-1}, xM_2(\mathbb{Z}_q)x^{-1} \right)$$

it follows for any $a \in \mathcal{O}[\frac{1}{q}]^\times$

$$a\overline{g} = \left((ax)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} M_2(\mathbb{Z}_q)\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}^{-1} (ax)^{-1}, (ax)M_2(\mathbb{Z}_q)(ax)^{-1} \right) = \overline{ag}.$$

Therefore $\lambda\overline{g} = \overline{\lambda g} = \overline{h}$. Since $(a\overline{g})$ is by definition in the orbit r it follows $\overline{h}$ is in the orbit r.

Therefore we now conclude $(F_1^{(k)})_{ij} = (D_1^{(k)})_{ir}$, so that the columns of F are a permutation of the columns of D. Hence $(F_1^{(k)})_{ij} = (D_1^{(k)})E$, where E is permutation matrix that permutes the columns, and thus the orbits, of $D_1^{(k)}$. So we obtain the following theorem.

Theorem 2.23 *The paths of length 1 are related recursively as:*

$$D_1^{(n+1)} = D_1^{(n)}D^{(1)} - pD_1^{(n)}E - pD_1^{(n-1)}, n \geq 3.$$

Note once again that this takes the form of certain Hecke operators on modular forms on $\Gamma_0(p)$.

Theorem 2.24 *([Ros], Theorem 4.6) Let $\mathcal{O}$ be an order of level q^n in $\mathbb{A}$. Then*

$$\#\{x \in \mathcal{O} - q\mathcal{O} \mid N(x) = q^k\} = |\mathcal{O}^\times|(D_n^{(k)})_{1,1}.$$

Therefore finding recursion formulas for the matrices $D_n^{(k)}$ give the representation numbers of an even larger class of quadratic forms. Given that we know $D_1^{(1)}$ and $D_1^{(2)}$ and having determined the formula for paths of length 1 we can compute the representation numbers for norms of orders of level p.

3 Stability

Examples of the various path sets of the $(p+1)$-regular tree suggest that the growth of a path sets is very erratic with small distances but slowly begins to grow in a regular manner. In other words, given a fixed path length, the graph of the path set with distance $(k+1)$ looks very similar to the graph with distance k, only with the distance between the paths and the central path increased by 1. Having thus presented a sense of what this property means in the graph, we now present the definition.

Definition 3.1 (Stability) *Let $P_n^k(\tau)$ be the set of all paths of length n at distance k from τ, where $k \geq n$ and $\tau = (\tau_0, \tau_n)$. Then $P_n^k(\tau)$ is* stable *if and only if P_n^{k+1} contains p copies of the set of paths $P_n^k(\tau)$.*

Graphically, this definition relates the path set P_n^{k+1} to $P_n^k(\tau)$. What this means is that when the distance from the central vector increases, no "new" paths are developed. Instead, the paths in $P_n^k(\tau)$ simply slide out to a further distance. Since there are p points to which these points can slide, that means there are p times as many paths.

Example 3.2 *With a (3)-regular tree, with distance $k = 1$ and length $n = 2$, the number of paths is just 4. However, simply increasing the distance to 2 results in a total of 12 paths. If one looks at the next case (distance equals 3) then the total number of paths is 24, which is $p = 2$ times as many paths in the lower distance. This means that with distance $k = 3$ the path set is stable. Further, $k = 4$ has 48, $p = 2$ times as many paths as $k = 3$. After this point, as distance continues to increase, the number of paths simply increases by p each time. Therefore, it becomes useful to define a sequence of paths to be stable.*

Definition 3.3 *Let n be a fixed length. A sequence of path sets $\{P_n^k(\tau)\}_{k \geq a}$, for some $a \in \mathbb{N}$, is* stable *when every element of the sequence is stable. The* stability point *for this length n is the smallest a where $\{P_n^k(\tau)\}_{k \geq a}$ is stable.*

As it turns out, for each length n, there are definite places after which all path sets are stable. With regards to the example given above, it could not begin at where $k = n$, since the number of paths increases by a factor greater than p from $k = n - 1$ to $k = n$. However, in the example it does increase by $p = 2$ from $k = n$ to $k = n + 1$. In fact, this is the exact place where it always begins.

Theorem 3.4 *The sequence $\{P_n^k(\tau)\}_{k \geq a}$ is stable iff $a \geq n + 1$. The stability point of $P_n^k(\tau)$ is $n + 1, \forall n \geq 1$ and 2 when $n = 0$.*

Proof:

($\Rightarrow$). Now, if $n = 0$, note that $|P_0^0| = 1$, $|P_0^1| = p + 1$, and $|P_0^a| = p^{a-1}(p + 1)$ if $a \geq 2$. If $n = 1$, note that $|P_1^0| = 1$, $|P_1^1| = 2p+1$, and for all $a \geq 2$, $|P_1^a| = p^{a-1}(2p+1)$.

So $2 \leq k \leq n-1$. Then there exists a starting point s which begins on the central vector. Let n be the number of paths which began at s and hit τ_n. When distance increases by 1 then there will still be a path which begins on the central path. Let s' be this starting point on the central vector. Then it is obvious that the number of paths out of s' which hit τ_n is at least n. However, because $d(s', \tau_n) = d(s, \tau_n) - 1$, there more paths beginning at s'. In other words, because the starting point is closer to τ_n, it will have more paths because the paths themselves can "hit" τ_n quicker than before. Thus, more paths will come out of this point than with smaller distances. Therefore, there will be more than p times as many paths $P_n^{k+1}(\tau)$ if $k < n$. Thus, if $P_n^{k+1}(\tau)$ is stable, then k must be greater than equal to n.

($\Leftarrow$)Suppose $k \geq n+1$. Certainly $P_n^{k+1}(\tau)$ contains at least p times as many paths as $P_n^k(\tau)$. However, if it were to contain more, that means a starting point would have had to "gain" paths, as the points on the central path did in the previous paragraph. That means that the starting point would have to be moving closer to τ_n, for if it were moving away, then it could not create any "new" paths.[3] However, because $k \geq n+1$, the starting points are moving farther away. Therefore they cannot gain any new paths out. Thus, $|P_n^{k+1}(\tau)| = p|P_n^k(\tau)|$, completing the proof.

□

Example 3.5 *An example of how this works can be found in P_3^1, P_3^2 and P_3^3. In comparing P_3^1 and P_3^2 note that while the number of staring points only increase by $p = 2$, the number of paths coming out of the starting point on the central vector doubles, thereby increasing the number of paths by more than a factor of p. Similarly, comparing P_3^2 and P_3^3, again we note that the starting point on the central path has an increase in the number of paths staring at it. Therefore, it is not stable before $k = n+1$. However, looking at P_3^4 we notice that it has p times as many paths as in P_3^3. Thus, P_3^4 is the first stable case.*

3.1 Stability and Recursive Formulas

We will now discuss the reasons for these definitions. The first motivation, which has already been discussed, is to note when the growth of the path sets begin to normalize. The second motivation is quite different. As we attempt to develop recursive formulas for these different graphs, we notice that once graphs become stable, the recursive formulas all have the same form. Before we discuss this formally, we present a lemma.

Lemma 3.6 *Let $P_n^k(\tau)$ with distance k. Then in the recursive formula for $D^{(k)} \cdot D^{(1)}$ the smallest k such that $D^{(k)}$ appears is $k = n-1$.*

It should be fairly easy to see why this is true, given a bit of explanation. Graphically, $D^{(k)} \cdot D^{(1)}$ means that you take the paths of distance 1 about the paths about

[3] This was alluded to in the proof of the converse of this. The reason the starting point on the central path gained "new" paths is because it was moving closer to τ_n.

distance k. Since the smallest distance that paths in $D^{(k)} \cdot D^{(1)}$ can come is $k-1$. Therefore, the smallest $D^{(k)}$ which can appear is $D^{(k-1)}$.

We now present an important proposition.

Proposition 3.7 *Let $P_n^k(\tau)$ be the path set where $k = n+1, n \geq 1$ that is, $P_n^k(\tau)$ is stable and length is not equal to* 0*. Then if we take $D^{(k)} \cdot D^{(1)}$ where $k \geq n$, then the formula will be similar to the formula for $D^{(k)} \cdot D^{(1)}$. More precisely, the formulas will have the same number of terms, and letting $k = n + l$ for some $a \in \mathbb{N}$ then if $D^{(k)}$ is in the recursive formula for $D^{(k)} \cdot D^{(1)}$, then in the recursive formula for $D^{(k)} \cdot D^{(1)}$, $D^{(k)}$ is replaced with $D^{(k+a)}$.*

Before discussing why this is true, it seems best to further explain what this proposition means via examples so that the reader has a better understanding of what the theorem says.

Example 3.8 *For this formula, the length is set at* 2*, the tildes represent just part of that set, while the line over the matrices represents that the set is taken about the reverse central path. The actual formulas we get are:*

$$D^{(3)} \cdot D^{(1)} = \widetilde{D^{(4)}} + pD^{(2)} + \frac{1}{4}D^{(3)} + \widetilde{\widetilde{D^{(4)}}} + \overline{\widetilde{D^{(4)}}} + p(\text{leftovers})$$

and

$$D^{(4)} \cdot D^{(1)} = \widetilde{D^{(5)}} + pD^{(3)} + \frac{1}{4}D^{(3)} + \widetilde{\widetilde{D^{(5)}}} + \overline{\widetilde{D^{(5)}}} + p(\text{leftovers})$$

This example shows that the formula in fact does work for this case.

Here is now an inductive explanation for the proposition. Suppose we have an equation for $D^{(k)} \cdot D^{(1)}$ where $k = n+1$ (k is distance and n is length as always). Now suppose we were to look at $D^{(k+1)} \cdot D^{(1)}$. If it were to have any different terms in it, then that means that either new paths would have been introduced or the original paths cannot be arranged as before. However, because in $D^{(k)} \cdot D^{(1)}$ the lowest term can be $D^{(k-1)}$, no "new" paths can be created, so that cannot have changed the formula. So, the paths must have been arranged in a different manner. This should be intuitively contradictory, however, since the set of paths at $k+1$ have the same form as the paths at distance k (provided $k \geq n+1$). Thus, the set of paths at $k+1$ can be grouped in exactly the same way as those at distance k. Therefore, the paths can be grouped similarly. Thus, the equations will be very much the same.

The power of Proposition 3.7 is that it shows that in order to get all the recursive formulas, one must calculate at most $n+1$ different equations. That is, they must find the recursive formulas for $D^{(1)}, \ldots, D^{(n)}$, plus one more $D^{(k)}$ for $k \geq n+1$, which will be similar to all the other stable cases. At that point one can derive from the single stable formula all the rest. This is perhaps the most important property of stability: it shows the upper bound of recursive formulas that one would have to calculate.

4 Conclusion

The main attributes we require of our formulas are:

- the formulas will be given as recursive forms.
- the formulas must involve a product of form $D^{(s)}D^{(t)}$ on one side of the equation. This property is motivated by the Hecke operator.

Thus we seek formulas which can be expressed in the form:

$$\sum_{i=0}^{l} E_i D^{(s)} D^{(1)} = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} a_{ijk} E_i D^{(j)} F_k$$

where E_i represents the permutation matrices that allow for the use of multiple central paths and F_i corresponds the permutation matrices used to account for reverse paths and similar forms. Yet this can be factored as a product of matrices as follows. We let i, j and k be defined on the rows u and columns v as

$$u = v = i(m+1)(n+1) + j(n+1) + k.$$

Then we see

$$\begin{aligned} A_{u,v} &= \begin{cases} a_{i,j,k} I & u = v \\ 0 & u \neq v \end{cases} \\ E_{u,v} &= \begin{cases} E_i & u = v \\ 0 & u \neq v \end{cases} \\ D_{u,v} &= \begin{cases} D^{(j)} & u = v \\ 0 & u \neq v \end{cases} \\ F_{u,v} &= \begin{cases} F_k & u = v \\ 0 & u \neq v \end{cases} . \end{aligned}$$

Additionally the left hand side can be expanded by replacing each occurrence of $D^{(s)}$ with the appropriate equivalent $D^{(s-1)}D^{(1)}$ recursion formula until all terms are expressed as $D^{(1)}$. With some manipulation this resulting sum can be expressed as diagonal matrix of $D^{(1)}$'s defined as $[D^{(1)}]$.

Since all the matrices are diagonal matrices, the following equation is equivalent to the original.

$$Tr(E[D^{(1)}]) = Tr(AEDF)$$

where Tr is the trace of the matrix. This equivalence motivates the following conjecture.

Conjecture 4.1

$$E[D^{(1)}] = AEDF$$

If this conjecture is provable then solving these recursive formulas amounts to solving this linear equation. In general for a given formula E, $[D^{(1)}]$, D, and F are given or presumed and A is the desired unknown. The trace of the resulting matrices would then return the desired formula.

4.1 Open problems in Stability

Perhaps the most intriguing open problem presented with this stability is improving Proposition 3.7. In the case of $n = 1$, there was actually only 1 recursion formula which applied for all distances. This shows that the converse cannot be true, i.e., that one may need less than $n + 1$ formulas to find all the recursion relations. Since we were unable to determine any formulas for $n \neq 0, 1$, it is unknown whether 1 is simply a special case or if fewer than $n + 1$ formulas are needed for all the different values of the lengths. It seems more likely that the former is correct, because with $n = 1$, there is only one unstable case so that nearly all the formulas use stable path sets. For large values of n where there are many unstable cases, it would appear that many different formulas would be required for these unstable cases, but would eventually level out once the stability point was reached. Once more recursive formulas are found, an interesting problem would be to find if the upper bound of $n + 1$ is a least upper bound for every length n, or if $n + 1$ is a simply an upper bound, but can in fact be improved.

Another interesting problem would be relevant only if it is true that stable path sets and unstable path sets have different recursive formulas. While it is true that in unstable cases the path sets do not grow by a set factor of p, they may grow in different "stable" manner which can be classified. This property is somewhat alluded to in the proof of Theorem 3.4. In our limited study of these path sets, it as noticed that in the unstable cases, they only starting point which "gained" new paths was the point which was moving toward the end of the central vector. So while it does not grow as nicely as stable cases, unstable cases may in fact grow in a fairly stable manner. If this is true, it may be that the set of unstable points has one recursive formula which relates only to it, and the stable path sets have a different recursive formula. Perhaps there are different sets of unstable paths which have different formulas. Again, this would be an interesting problem which could lead to an enlightening discovery.

5 Appendix: Algorithms

5.1 Regular Tree Coordinates

In order to create and manipulate the tree and its paths we need to establish a system for representing these objects as software elements. We do this by establishing a coordinate system for an abstract regular tree and defining the required formulas with respect to this coordinate system. Fix a positive integer n which will result in the production of an $(n+1)$-regular tree.[4]

Definition 5.1 *Given (a,b) and $(c,d) \in \mathbb{N}^2$, $(a,b) \sim (c,d)$ when $a = c$ and if $a \neq 0$ then also $b \equiv d \pmod{(n+1)n^{a-1}}$.*

This relation can be shown to be an equivalence relation and therefore it partitions the set $\mathbb{N}^2$. We denote each equivalence class as $[a,b]$ and refer to it as a vertex.

Definition 5.2 *The* shift down operator *is an action on the vertices defined as*

$$(k\downarrow)[a,b] = \begin{cases} \left[a-k, \left\lfloor \frac{b}{n^k} \right\rfloor\right] & 0 \le k < a \\ [0,0] & a \le k. \end{cases}$$

Proposition 5.3 *The shift down operator is well-defined.*

Proof:
Let $[a,b] \in \mathbb{N}^2/\sim$. By applying the division algorithm we observe that

$$\begin{aligned} \left\lfloor \tfrac{b}{n^k} \right\rfloor &\equiv \left\lfloor \tfrac{m(n+1)n^{a-1}+r}{n^k} \right\rfloor & \pmod{(n+1)n^{a-(k+1)}} \\ &\equiv m(n+1)n^{a-(k+1)} + \left\lfloor \tfrac{r}{n^k} \right\rfloor \\ &\equiv \left\lfloor \tfrac{r}{n^k} \right\rfloor. \end{aligned}$$

Therefore $(k\downarrow)[a,b] = (k\downarrow)[a,r]$ for any $b \equiv r \pmod{(n+1)n^{a-1}}$ and so the shift down operator is well-defined. □ With this device in hand we can now define a partial ordering on the partitioned set which will result in a regular tree structure.

Definition 5.4 *Given $\alpha, \beta \in \mathbb{N}^2/\sim$, $\alpha \le \beta$ if and only if there exists a $k \in \mathbb{N}$ such that $\alpha = (k\downarrow)\beta$.*

Note therefore $[0,0]$ is the least element of the partition and that, unless $\alpha = [0,0]$, the k is unique. This is summed up in the following lemma.

Lemma 5.5 *The set of all lower bounds of a vertex forms a finite chain, specifically*

$$[0,0] \le ((a-1)\downarrow)[a,b] \le \cdots \le\downarrow [a,b] \le [a,b].$$

[4] For compatibility with software applications we assume $0 \in \mathbb{N}$.

Theorem 5.6 *The graph of the partial ordering of $\mathbb{N}^2/\sim$ is an $(n+1)$-regular tree. Therefore we take a connection between two vertices α and β to exist when either $\alpha =\downarrow \beta$ or $\beta =\downarrow \alpha$.*

Proof:
Given that $\downarrow [0,0] = [0,0]$, all vertices connected to $[0,0]$ must be of the form $[1,b]$. Since $\downarrow [1,b] = [0,0]$ all $n+1$ vertices of that form are connected to $[0,0]$.

Given any vertex $[a,b] \neq [0,0]$ there exists a distinct vertex $\downarrow [a,b]$, which by definition is connected to $[a,b]$. For all vertices $[a+1,c]$ such that $\downarrow [a+1,c] = [a,b]$ we once again apply the division algorithm and note

$$\left\lfloor \frac{c}{n} \right\rfloor = \left\lfloor \frac{mn+r}{n} \right\rfloor = m$$

and by assumption we know $b \equiv m \pmod{(n+1)n^{a-1}}$. Therefore all vertices connected to $[a,b]$ from above are of the form $[a+1, bn+r], 0 \leq r < n$. These vertices are clearly n distinct vertices since no two are equivalent mod $(n+1)n^a$. Therefore there are exactly $n+1$ distinct vertices connected to $[a,b]$. Therefore all vertices are connected to exactly $n+1$ distinct vertices so the graph of the partial ordering is an $(n+1)$-regular tree. □

The importance of this theorem lies in its simple generation and representation.[5] Effectively this specific $(n+1)$-regular tree can be used as a coordinate system for any $(n+1)$-regular tree. The graph of this tree is easily drawn by considering a vertex $[a,b]$ as a form of polar coordinates. The first coordinate representing the radius, the second the fraction of rotation.[6]

5.2 Distance

Lemma 5.5 ensures us that the elements of the tree have lower bounds. An important extension to this result is the existence of greatest lower bounds. We denote the greatest lower bound between two vertices α and β as $\alpha \Downarrow \beta$.

Proposition 5.7 *Every non-empty set of vertices has a greatest lower bound.*

Proof:
It is sufficient to show given any two vertices α and β there exists a greatest lower bound. Let L_α and L_β denote the set of all lower bounds of α and β respectively. We know their intersection is non-empty since $[0,0]$ is a lower bound to every element.

[5] In computerized integer arithmetic, the division operation automatically floors the result and the modular arithmetic is equally efficient. Additionally the vertices can easily be iterated over and traversed with simple index loops.

[6] Specifically the map $[a,b] \mapsto (a\cos\theta, a\sin\theta)$ where $\theta = \frac{2\pi}{(n+1)n^{a-1}}(b - \frac{1}{n})$ serves to map the coordinates to the Cartesian plane.

Additionally their intersection is a finite subchain of both L_α and L_β and so it has a top element δ. Therefore δ is the greatest lower bound. □

We can compute the greatest lower bound between two vertices with the following algorithm.

```
$\Downarrow$: ($\mathbb{N}^2/\sim$: $[a,b]$, $[c,d]$) : $\mathbb{N}^2/\sim$
begin
    Let $\gamma, \delta \in \mathbb{N}^2/\sim$;
    // Select the vertex $\gamma$ to be the closest to [0,0] for efficiency.
    if $(a \leq c)$ then
    begin
        $\gamma \leftarrow [a,b]$;
        $\delta \leftarrow [c,d]$;
    end
    else
    begin
        $\gamma \leftarrow [c,d]$;
        $\delta \leftarrow [a,b]$;
    end
    // Traverse down the chain for $\gamma$ until it intersects the chain for $\delta$.
    while $(\gamma \not\leq \delta)$ do
        $\gamma \leftarrow \downarrow \gamma$;

    // The current $\gamma$ is the greatest lower bound.
    return $\gamma$;
end
```

We turn our attention now to the notion of distance within the tree.

Definition 5.8 *Let $\alpha = [a,b]$ and $\beta = [c,d]$ be two comparable vertices. The distance $\partial(\alpha,\beta)$ is defined as $|a - c|$.*

Definition 5.9 *The distance between vertices α and β is the sum of their respective distances to the greatest lower bound. That is,*

$$d(\alpha,\beta) = \partial(\alpha, \alpha \Downarrow \beta) + \partial(\alpha \Downarrow \beta, \beta).$$

Since the greatest lower bound is always comparable with both α and β, it follows this generalized definition of distance is well-defined.

5.3 Path Generation

Proposition 5.10

$$P_0^m(\tau) = U^m(\tau) \cup \left(\bigcup_{1 \leq k \leq m} \big(U^m((k\downarrow)\tau) - U^{m-k}(\tau) \big) \right)$$

where $U^j(\tau)$ is the set of all elements a distance m from τ that are also greater than or equal to τ.

Therefore in order to generate $P_0^m(\tau)$ we need only define an algorithm for $U^j(\tau)$ which we do as follows.

```
$U^j(\mathbb{N}^2/\sim: [a,b]) : \mathcal{P}\mathbb{N}^2/\sim$
begin
    if $(j = 0)$ then
        return {[a,b]};
    if $(a = 0)$ then
        return $\{[j,i] \mid 0 \leq i < (n+1)n^{j-1}\}$;
    else
        return $\{[a+j, bn^j + i] \mid 0 \leq i < n^j\}$;
end
```

Finally the generation of the set for arbitrary length paths follows from the following formula.

Proposition 5.11 $P_l^m(\tau) = \{(\alpha,\beta) \mid \alpha \in P_0^m(\tau), \beta \in P_0^l(\alpha) \text{ and } d(\tau,\beta) \leq m\}$.

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