Dynamical Properties of Continuous Maps of the Interval

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Abstract: In this paper, we prove that a conjugacy may exist for two different maps only if both maps have a critical point whose image is less than one. In all other cases, no conjugacy can exist. We first provide an "elementary" proof of this. We then assess the possibility of a proof using ergodic theory. On an unrelated topic, we completely characterize the long term behavior of Ricker's model for $p : 2 < p < \sqrt{6}$.

1 Introduction

At their most basic, dynamical systems are functions that map a set I onto itself ($f : I \to I$). These systems operate on a wide variety of sets, but for our purposes we are interested in continuous functions from a topological space onto itself. Specifically, functions for which I is an interval of the real line. First, we define a few notions from calculus important for our discussion.

Definition 1.1 (One-to-one) *Let f be a continuous function $f : I \to J$. The function is one-to-one on I if for every $x, y \in I$, if $x \neq y$, then $f(x) \neq f(y)$.*

Definition 1.2 (Onto) *$f : I \to J$ is onto if for every $y \in J$, there is a $x \in I$ so that $f(x) = y$.*

Definition 1.3 (Homeomorphism) *A function $f : I \to J$ is a homeomorphism if it is one-to-one, onto, and continuous, and f^{-1} is also continuous.*

Of immediate interests in such systems are the orbits of a given point ($\{x, f(x), f^2(x), \dots\}$, where $f^i = \underbrace{f \circ f \circ \dots f(x)}_{i}$). For a given dynamical systems there are certain seed values of x whose orbits have some regularity. Perhaps there is a point where the orbit is constant, $\{x, x, x, \dots\}$. Or maybe a point x has a repeating orbit. Such an orbit would be $\{x, f(x), \dots, f^{k-1}(x), x, f(x), \dots\}$. We classify a few of these special points below.

Definition 1.4 *A point x of a dynamical system f is a fixed point if $f(x) = x$.*

Definition 1.5 *A point x is periodic of period k if $f^k(x) = x$. The smallest k for which this holds is the prime period of x.*

Definition 1.6 *A point x is eventually periodic or eventually fixed if $\exists k > 0$ so that $f^k(x)$ is periodic or fixed, respectively.*

In certain maps, it may be extremely difficult to understand the long-term behavior of most points. Such maps are in some sense chaotic, and we define chaos here ([**?**]).

Definition 1.7 *Let V be a set. $F : V \to V$ is said to be chaotic on V if*

1) (sensitive dependence) There exists $\delta > 0$ such that for any $x \in V$ and any neighborhood N of x there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

2) (topologically transitive) If for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

3) Periodic points are dense in V.

The following device will become useful later ([**?**]).

Definition 1.8 *Let $f : I \to I$. The map is unimodal if*

1) $f(0) = f(1) = 0$.

2) f has a unique critical point c with $0 < c < 1$.

Definition 1.9 *Let $x \in I$. The itinerary of x under f is the infinite sequence $S(x) = (s_0 s_1 s_2 \dots)$ where*

$$s_j = \begin{cases} 0 & \text{if } f^j < c \\ 1 & \text{if } f^j > c \\ C & \text{if } f^j = c \end{cases}$$

Most important for us will be the itinerary of the critical point.

Definition 1.10 *The Kneading Sequence $K(f)$ of f is the itinerary of $f(c)$.*

Between two dynamical systems a relationship may exist through a function h. We call such a relationship a conjugacy and define it thus:

Definition 1.11 *A conjugacy between two continuous functions $f : I \to I$ and $g : J \to J$ is any function $h : I \to J$ satisfying the following condition:*

$$h \circ f = g \circ h, \quad \forall x \in I, \text{and} \tag{1}$$

$$h \circ f^k = g^k \circ h. \tag{2}$$

Furthermore, this h must be a homoeomorphism. That is, h maps one-to-one, onto, and continuous.

Conjugate pairs (g and f) are of special interest in dynamical systems, because each member mimics the behavior of its conjugate pair through the function h. If x is a fixed point of f then $h(x)$ is a fixed point of g, because $h \circ f(x) = h(x) = g \circ h(x)$. Periodic points are mapped through h as well. If x is a periodic point of f with period k, then $h(x)$ is a periodic point of g also of period k.

2 Conjugacies Between Tent Maps of the Interval

The specific topic of research will be families of maps known as tent maps. These maps have the general form:

$$T_\mu = \begin{cases} \mu x & 0 \le x \le 1/2 \\ \mu x\,(1-x) & 1/2 < x \le 1 \end{cases}$$

While many interesting dynamics happen for $\mu > 2$, we will constrict ourselves to the case $0 \le \mu \le 2$. Clearly, for such $\mu's$ T maps from $[0,1]$ to $[0,1]$. Are there any conjugate pairs in this family at all? An answer either way would be extremely interesting. If there were no conjugacy at all possible, then in a certain sense the dynamical behavior of each T_μ is unique from other maps in the family. If a conjugacy is possible, then we can understand the behavior of T for any one value of μ through the behavior of T for a second value of μ.

2.1 Specific Characteristics of a Conjugacy

Before attacking the problem outright, we want to investigate what a conjugacy would have to look like if it were to exist. Natural points to investigate are $x = 0$ and $x = 1$ and $x = 1/2$. This leads us to our first result.

Lemma 2.1 *h(0)=0.*

Proof: By definition of conjugacy, $h \circ T_{\mu_1}(0) = T_{\mu_2} \circ h(0) \Rightarrow h(0) = T_{\mu_2} \circ h(0)$. Thus, $h(0)$ is a fixed point of T_{μ_2}. However, T_{μ_2} only has two fixed points $x = 0$ and $x = \frac{\mu_2}{\mu_2+1}$. If $h(0) = \frac{\mu_2}{\mu_2+1}$, then by the onto condition there must be $0 < x_1 < x_2 < 1$ such that $h(x_1) = 0$ and $h(x_2) = 1$. Then by the intermediate value theorem, $\exists \bar{x}$ where $x_1 < \bar{x} < x_2$ so that $h(\bar{x}) = \frac{\mu_2}{\mu_2+1}$. This violates the one-to-one condition on h. Therefore, $h(0) = 0$. □

Lemma 2.2 *h(1)=1.*

Proof: By definition of conjugacy, $h \circ T_{\mu_1}(1) = T_{\mu_2} \circ h(1) \Rightarrow h(0) = T_{\mu_2} \circ h(1)$. This means that $h(1)$ is eventually fixed under T_{μ_1}. There are four eventually fixed points of T_{μ_1} : $0, \frac{1}{\mu_2+1}, \frac{\mu_2}{\mu_2+1}, 1$. Because of the above, $h(1) \neq 0$, and a similar argument with the intermediate value theorem shows that $h(1) \neq \frac{1}{\mu_2+1}$ and $h(1) \neq \frac{\mu_2}{\mu_2+1}$. Therefore, $h(1) = 1$. □

Lemma 2.3 *h is strictly increasing.*

Proof: First, assume that h is not strictly increasing. Then on some interval $[a,b] \subset [0,1]$ h must be decreasing. In other words, $a < b$ and $h(a) > h(b)$. By the intermediate value theorem there must be a $h(b) < d < h(a)$ so that there is a $c : a < c < b$ and a $c' : 0 < c' < a$ so that $h(c) = h(c') = d$. This violates the one-to-one condition on h. □

Lemma 2.4 *h(1/2)=1/2.*

Proof: Let $x \in [0, 1/2]$, then because $T(1-x) = T(x)$

$$\mu_2 h(x) = T_{\mu_2} \circ h(x) = h \circ T_{\mu_1}(x) = h \circ T_{\mu_1}(1-x) = T_{\mu_2} \circ h(1-x) = \mu_2 - \mu_2 h(1-x)$$

Cancelling out the μ_2's leaves $h(x) = 1 - h(1-x)$. We simply let $x = 1/2$ giving us $h(1/2) = 1 - h(1/2) \Rightarrow h(1/2) = 1/2$. $\square$

These results suggest that a conjugacy might be possible, as they do not place any bizarre restrictions on h. On the other hand, these restrictions are rather specific. Perhaps, they can be easily violated. Indeed, the weak hypothesis of the following theorem suggests exactly that.

Theorem 2.5 *If there is a $k > 0$ so that $T^k_{\mu_1}(1/2) < 1/2 < T^k_{\mu_2}(1/2)$ or $T^k_{\mu_2}(1/2) < 1/2 < T^k_{\mu_1}(1/2)$, then there can be no conjugacy.*

Proof: Under the first case, the following would have to hold.

$$h \circ T^k_{\mu_1}(1/2) = T^k_{\mu_2} \circ h(1/2) = T^k_{\mu_2}(1/2)$$

Because $T^k_{\mu_2}(1/2) > 1/2$ and $h(0) = 0$, by the intermediate value theorem there must be an $a : 0 < a < T^k_{\mu_1}(1/2) < 1/2$ so that $h(a) = 1/2$. However, a is clearly not equal to 1/2. This violates the one-to-one condition on h. A similar argument holds if $T^k_{\mu_2}(1/2) < 1/2 < T^k_{\mu_1}(1/2)$. $\square$

This theorem simply says that if the iterations of $x = 1/2$ under T_{μ_1} and T_{μ_2} fall on opposite sides of 1/2, then we cannot find a conjugacy. This hypothesis is not especially strong. Because $\mu_1 \neq \mu_2$, it is easy to imagine that iterations of 1/2 might eventually separate by enough to fall on opposite sides of 1/2.

3 Several Results on Conjugacies

Theorem 3.1 *For $\mu_1 \neq \mu_2$ and $0 < \mu_1 < \mu_2 < 1$, conjugacies can be constructed between two maps T_{μ_1} and T_{μ_2}.*

Proof: In this case, we prove conjugacy between T_{μ_1} and T_{μ_2} by explicitly constructing a map h which satisfies all the conditions of a conjugacy. We construct h starting with the interval $[0, \frac{1}{2}]$. Proceed by induction, where $q = 1$ or 2. Let $h_{|I_0} : I_{0_{\mu_1}} \to I_{0_{\mu_2}}$ where $I_{0_{\mu_q}} = (\frac{1}{2\mu_q}, \frac{1}{2})$ we only require that $h_{|I_0}$ be a homeomorphism (with of course $h(\frac{1}{2}) = \frac{1}{2}$).

In general we introduce, for $i > 0$, $h_{|I_i} : I_{i_{\mu_1}} \to I_{i_{\mu_2}}$ where $I_{i_{\mu_q}} = (\frac{1}{2\mu_q^{i+1}}, \frac{1}{2\mu_q^i}]$. All $h_{|I_i}$ are defined recursively by $h_{|I_i}(x) = \mu_2 h_{|I_{i-1}}(\frac{1}{\mu_1})$ To show that $h_{|I_i}$ is one-to-one, onto, and continuous use induction. We will solely demonstrate one-to-oneness, but the other properties can be proven in a similar spirit. Note the base case is satisfied by definition. So we, assume that it is true for $i \leq n$

Suppose $\exists\ x, x' \in I_{(n+1)_{\mu_1}}$ such that $h_{|I_{n+1}}(x) = h_{|I_{n+1}}(x')$. Then by the recursive definition of $h_{|I_{n+1}}$ we have $\mu_2 h_{|I_n}(\frac{x}{\mu_1}) = \mu_2 h_{|I_n}(\frac{x'}{\mu_1}) \Rightarrow h_{|I_n}(\frac{x}{\mu_1}) =$

$h_{|I_n}(\frac{x'}{\mu_1})$ since the intervals $I_{i_{\mu_q}}$ have been chosen so that $x \in I_{i+1_{\mu_q}} \Rightarrow \frac{x}{\mu_q} \in I_{i_{\mu_q}}$. But by the induction hypothesis, $h_{|I_n}$ is one-to-one, so $h_{|I_n}(\frac{x}{\mu_1}) = h_{|I_n}(\frac{x'}{\mu_1}) \Rightarrow \frac{x}{\mu_1} = \frac{x'}{\mu_1} \Rightarrow x = x'$.

Now we are ready to define h on $[0, \frac{1}{2}]$. For $x \in I_i$, $h(x) = h_{|I_i}(x)$. Since $[0, \frac{1}{2}] = \cup_{i=0}^{\infty} I_i \cup \{0\}$ and $I_i \cap I_j = \emptyset$, $i \neq j$, $h(x)$ is well-defined on $[0, \frac{1}{2}]$. This said, we define h on $(\frac{1}{2}, 1]$ by $h(x) = 1 - h(1-x)$.

At this point, it should be fairly clear that our h is, in fact, a conjugacy. Take an $x \in [0, \frac{1}{2}]$. Then $x \in I_i$ and $h \circ T_{\mu_1}(x) = h(\mu_1 x) = h_{|I_{i+1}}(\mu_1) = \mu_2 h_{|I_i}(\frac{\mu_1 x}{\mu_1}) = T_{\mu_2} \circ h(x)$. $\Box$

Corollary 3.2 *There can be no conjugacy between two maps T_{μ_1} and T_{μ_2}, where $\mu_1 < 1 < \mu_2$.*

This is easily seen several ways. **Proof1:** First, one calculates $T_{\mu_1}(1/2) = \mu_1/2 < 1/2$ and $T_{\mu_2}(1/2) = \mu_2/2 > 1/2$, and by Theorem 2.5 there is no conjugacy. **Proof2:** One may also think about it thus. The function T_{μ_2} has two fixed points ($x = 0$ and $x = \mu_2/\mu_2 + 1$), while T_{μ_1} only has one ($x = 0$). $\Box$

Theorem 3.3 *There can be no conjugacy between two maps T_{μ_1} and T_{μ_2} where $1 < \mu_1 < \mu_2 < 2$.*

Proof: We proceed by contradiction. Let $\mu_1 < \mu_2$, in particular $\mu_2 = \mu_1 + \epsilon$. Assume that two such maps do indeed have a conjugacy. Then the kth iteration of $x = 1/2$ will be a polynomial in terms of μ that will have the same form for both T_{μ_i}'s. Another implication of conjugacy is that $T^i_{\mu_1}(1/2) < T^k_{\mu_1}(1/2) < T^j_{\mu_1}(1/2) \Leftrightarrow T^i_{\mu_2}(1/2) < T^k_{\mu_2}(1/2) < T^j_{\mu_2}(1/2)$. Using this and the fact that the orbit of $x = 1/2$ is dense in our invariant interval implies that there are kth iterations of μ's k arbitrarily large that are arbitrarily close to $\frac{1}{2}$. For notation, we denote $p_\mu(n)$ to be the polynomial expansion of the nth iteration of $\frac{1}{2}$. So from the above statements we conclude there are arbitrarily large n's such that $|p_{\mu_2}(n) - p_{\mu_1}(n)| = a > 0$ where $a \approx 0$. Let $L_\mu(n)$ be the leading term of $p_\mu(n)$. Consider then $|p_{\mu_2}(n+1) - p_{\mu_1}(n+1)|$. If greater than $\frac{1}{2}$, then we are done. If not, then note that since we can choose n large so $|L_{\mu_1}(n+1) - L_{\mu_2}(n+1)| = L$ and we would have to have $|p_{\mu_2}(n+1) - p_{\mu_1}(n+1) - L_{\mu_2}(n+1) + L_{\mu_1}(n+1)| \approx L$. But this would imply $|D(b_0) - D(b_1) \pm ... \mp D(b_j)| = L$ where $D(b_i) = [\mu_2^{b_i} - \mu_1^{b_i} - \mu_2^{b_i+1} + \mu_1^{b_i+1}]$. The b_i are the exponents "missing" from $p_\mu(n)$. But substituting $\mu_1 + \epsilon$ for μ_2 in the expressions $|L_{\mu_2}(n+1) - L_{\mu_1}(n+1)| \approx \sum_{i=0}^{j} D(b_i)$ we see that for large n, the expression on the left is a polynomial all terms positive whose leading term is $\epsilon\mu^n$ whereas the polynomial on the right is alternating and the leading term is at most $\epsilon^2\mu^{n-1}$ $\rightarrow\leftarrow$. So $|p_{\mu_2}(n+1) - p_{\mu_1}(n+1) - L_{\mu_2}(n+1) + L_{\mu_1}(n+1)| << k$ $\Rightarrow |p_{\mu_2}(n+1) - p_{\mu_1}(n+1)| > \frac{1}{2}$. $\Box$

4 A Different Approach

4.1 Ergodic Theory

We may assault the problem of conjugacy from a different direction, Ergodic Theory. Let us first make the following observation: that when examining the kneading sequences of the critical point, the frequency of zeros depends on μ. In particular, the frequency of zeros seems inversely proportional to μ. First, we define the following intervals.

$$I_a = [1 - T_\mu^{-2}(1/2), T_\mu^0(1/2)]$$
$$I_b = [T_\mu^2(1/2), T_\mu^0(1/2)]$$

Then, for $x \in I_a, T^2(x)$ is to the left of the critical point (0 in the Kneading Sequence); for $x \notin I_a, x \in I_b, T^2(x)$ is to the right of the critical point(1 in the Kneading Sequence). Define $R(\mu) = \frac{|I_a|}{|I_b|}$ where $||$ denotes length, then we note that for $1 < \mu < \sqrt{2}$, R is strictly decreasing. Heuristically speaking, we might conclude that for points to the left of the critical point, the "chance" that after two iterations the point returns to the left, is in a one-to-one relationship to μ. Or, in terms of the kneading sequence, for every 0 in the sequence, the probability that the digit two places over is 0 is unique to the value of μ. But if $T_{\mu_1} \sim T_{\mu_2}$ then $K(T_{\mu_1}) = K(T_{\mu_2})$ so the frequency that for any 0 in the kneading sequence a 0 follows two digits later must be identical for μ's who are conjugate. But we have reason to suspect that in fact the frequency is unique for each μ. It is this "probabilistic" obstruction to conjugacy which will motivate our further discussions.

Unfortunately, the above informal argument is rather impoverished and unqualified; to construct a solid mathematical proof, we must make a foray into the realm of Ergodic Theory. The main falsity in the informal proof lies in the assumption that $R(\mu)$=the "probability" that for $x \in I_b, T_\mu^2(x) \in I_b$. Thus, we must abandon that crude argument, but not without trying to capture its spirit in another more sophisticated form which involves borrowing a big gun from Ergodic Theory, namely Birkhoff's theorem.

Let us consider I and a measure m on it. A function $\tau : I \to I$ is m-Ergodic provided that:
i)τ is measure preserving i.e $m([a, b]) = m(\tau^{-1}[a, b])$ and
ii)if $I' \subset I$ is τ-invariant, then $m(I') = 0$ or 1.

Theorem 4.1 (Birkhoff's Theorem) *Let τ be ergodic on I. Then for $A \subset I$ and almost every $x \in I$ we have,*

$$\mu(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} a_i \quad a_i = \begin{cases} 1 & \tau^i(x) \in A \\ 0 & \tau^i(x) \notin A \end{cases}$$

Armed with this tool the outline of our proof is as follows. Assume $T_{\mu_1} \sim T_{\mu_2}$. We seek probability measures, m_1 and m_2, absolutely continuous with respect to Lebesgue measure (the standard way of measuring intervals in $\mathbb{R}$) with respect to which T_{μ_1} and T_{μ_2} are ergodic. We also note that if there is a conjugacy then Birkhoff's Theorem tells us if there are corresponding intervals $I_{a_{\mu_1}}$ and $I_{a_{\mu_2}}$ under conjugacy (where $T_{\mu_1}^k \in I_{a_{\mu_1}} \Leftrightarrow T_{\mu_2}^k \in I_{a_{\mu_2}}$) then $m_1(I_{a_{\mu_1}}) = m_2(I_{a_{\mu_2}})$. If an absolutely continuous T-invariant measure, $\phi_m(x)$ can be found which we can integrate over the I_{a_μ}'s, and if

$$\int_{I_{a_{\mu_1}}} \phi_{\mu_1} d\lambda \neq \int_{I_{a_{\mu_2}}} \phi_{\mu_2} d\lambda \Longleftrightarrow m_1(I_{a_{\mu_1}}) \neq m_2(I_{a_{\mu_2}}) \rightarrow\leftarrow$$

This is sufficient to show non-conjugacy.

4.2 Measure Theory

We wish to find, if it exists, an absolutely continuous invariant measure (henceforth referred to as an *acim*). To do so, we enlist the aid of the $Frobenius-Perron\ Operator$, an operator which describes the effect of a transformation τ (in our case T) has on a probability density function.

Let $\mathcal{X}$ be a random variable on the space $J = [a, b]$ having the probability density function f. Then for any measurable set $J' \subset J$, $Prob\{\mathcal{X} \in \mathcal{J}'\} = \int_{J'} f d\lambda$ where λ is the normalized Lebesgue measure on J. Let $\tau : J \to J$ be a transformation. Then $\tau(\mathcal{X})$ is a random variable on the space J. How do we find $\tau(\mathcal{X})$?

$$Prob\{\tau(\mathcal{X}) \in J'\} = Prob\{\in \tau^{-1}(J)\} = \int_{\tau^{-1}J'} f d\lambda$$

More specifically we want a probability density function for $\tau(\mathcal{X})$, so we let

$$\int_{\tau^{-1}J'} f d\lambda = \int_{J'} \psi d\lambda$$

for some function ψ. Note that if ψ exists, it will depend on f and on τ. Let

$$m(A) = \int_{\tau^{-1}A} f d\lambda$$

where $f \in \mathcal{L}^1$ and A an arbitrary measurable set. By the Radon-Nikodym Theorem, $\exists \psi \in \mathcal{L}^1$ such that for all measurable sets A,

$$m(A) = \int_A \psi d\lambda.$$

Moreover, ψ is unique and depends on τ and f. Let $P_\tau f = \psi$. P_τ is an operator from the space of probability density functions on J into itself. P_τ is called the $Frobenius - Perron\ operator\ associated\ with\ \tau$. Because $f \in \mathcal{L}^1$, $P_\tau f \in \mathcal{L}^1$. Thus $P_\tau : \mathcal{L}^1 \to \mathcal{L}^1$ is well defined. If we let $A = [a, x] \subset J$, then

$$\int_a^x P_\tau f d\lambda = \int_{\tau^{-1}[a,x]} f d\lambda.$$

Upon differentiation with respect to x we get

$$P_\tau f(x) = \frac{d}{dx}\int_{\tau^{-1}[a,x]} f d\lambda.$$

Because we are interested in invariant measures, of particular interest are f's such that $P_\tau f = f$. Certain theorems exist which proscribe what properties such a function f should have in the case it exists.

Definition 4.1 *Let $I = [a,b]$. The transformation $\tau : I \to I$ is called* ***piecewise monotonic*** *if there exists a partition of I, $a = a_0 < a_1 < ... < a_q = b$, and a number $r \geq 1$ such that i) $\tau_{|(a_{i-1},a_i)}$ is a C^r function, $i = 1, ..., q$ which can be extended to a C^r function on $[a_{i-1}, a_i]$, $i = 1, ..., q$, and ii) $\mid \tau'(x) \mid > 0$ on (a_{i-1}, a_i), $i = 1, ..., q$.*

If τ is piecewise monotonic then

$$P_\tau f(x) = \sum_{z\in\{\tau^{-1}(x)\}} \frac{f(z)}{\mid \tau'(z) \mid}.$$

Note that T_μ is in fact piecewise monotonic. A probability density function which is fixed under P_{T_μ} must satisfy the following relation:

$$f(x) = P_{T_\mu} f(x) = \sum_{z\in\{\tau^{-1}(x)\}} \frac{f(z)}{\mid \tau'(z) \mid}.$$

Or more explicitly:

$$f(x) = P_{T_\mu} f(x) = \frac{1}{\mu} f(1 - \frac{x}{\mu}) + \frac{1}{\mu} f(\frac{x}{\mu}) \mathcal{X}_{[T_\mu^3(\frac{1}{2}), T_\mu^1(\frac{1}{2})]}$$

where $\mathcal{X}$ is the characteristic function.

In 1957, Renyi defined a class of transformations that have an acim. Since then, Lasota and Yorke proved a generalization of Renyi's result. It is this result we state here (in slight variation) that guarantees our map has an acim.

Definition 4.2 *Consider the interval $I = [a,b]$ with normalized Lebesgue measure λ on I. Let $\mathcal{T}$ denote the class of transformations $\tau : I \to I$ that satisfy the following conditions: i) τ is piecewise expanding, i.e., there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, ..., q\}$ of I such that $\tau_{|I_i}$ is C^1, and $\mid \tau_i'(x) \mid \geq \alpha > 1$ for any i and for all $x \in (a_{i-1}, a)$; ii) $g(x) \equiv \frac{1}{|\tau'(x)|}$ is a function of bounded variation, where $\tau'(x)$ is the appropriate one-sided derivative at the endpoints of $\mathcal{P}$. For each $n \geq 1$, we define the partition $\mathcal{P}^{(n)}$ as follows:*

$$\mathcal{P}^{(n)} \equiv \vee_{k=0}^{n-1} \tau^{-k}(\mathcal{P}) \equiv \{I_{i_0} \cap \tau^{-1}(I_{i_1}) \cap ... \cap \tau^{-n+1}(I_{i_n}) : I_{i_j} \in \mathcal{P}\}$$

So τ^n is piecewise expanding on $\mathcal{P}^n$ if τ is piecewise expanding on $\mathcal{P}$.

Theorem 4.2 *Let $\tau \in \mathcal{T}(I)$. Then it admits an absolutely continuous invariant measure whose density is of bounded variation.*

Note that $T_\mu \in \mathcal{T}(I)$, so it must have an acim.

The major difficulty actually lies in arriving at an explicit formula for such an invariant measure. For simpler maps such as $l(x) = (ax)mod1 \mid a \mid > 1$, the invariant measure takes the form of

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\mu^n}$$

where

$$a_n = \begin{cases} 1 & x < l^n(0) \\ 0 & otherwise \end{cases}$$

We would hope that a similar formula might be used for T_μ. Unfortunately, after many failed attempts, no correct measure has been found so far. When defining f, it seems necessary to include the μ^n's in the denominator to respect the functional relationship which has a factor of $\frac{1}{\mu}$. We also want an expression for $a_n(x)$ such that

$$a_n(x) = \sum_{y, T_\mu(y)=x} a_{n-1}(T^{-1}(y).$$

In the above example, the map $l(x)$ preserves less-than and greater than. Or in terms of the kneading sequence, it preserves the kneading sequence's ordering. Unfortunately, this is not the case with our map, so a simple formula for a_n will not suffice. To further complicate things, when we consider what Birkhoff's Theorem means, we come to the conclusion that the interval $D = [T_\mu^4(\frac{1}{2}), T_\mu^3(\frac{1}{2})]$ should have measure 0. The interval D is a "dead-zone" so to speak. All points in D are eventually mapped outside of D, (except for the fixed point), and there are no points outside of D which are mapped into it. Birkhoff's Theorem states that the invariant measure of an interval is equal to the asymptotic frequency of entrances of interations of a point x into that interval. Clearly for D, the asymptotic frequency approaches 0. Or in mathematical terms:

$$m(d) = \int_d f(x) d\lambda = 0.$$

where $d \subset D$. The obvious way to deal with this is to simply set $f(x) = 0$ for all $x \in D$. But this leads to serious problems. Consider points in the interval $[T_\mu^3(\frac{1}{2}), T_\mu^5(\frac{1}{2})]$. By the functional relation derived form the $Frobenius-PerronOperator$, $f(x) = \frac{1}{\mu} f(\frac{x}{\mu}) + \frac{1}{\mu} f(1 - \frac{x}{\mu})$, but $f(1 - \frac{x}{\mu}) = 0$ so we get $f(x) = \frac{1}{\mu} f(\frac{x}{\mu})$. But this in turn will "redetermine" more values of $f(x)$ and we run into the problem of recursively having to redefine our function on certain intervals. Lastly, we have the problem that we would like $a_0(x) = 0$, but not neccessarily $\sum_{y,T_{mu}(y)} a_0(y)$=0. This in itself poses a problem. One possible solution may be to let $f(x)$ have a form of $\sum_{n=-\infty}^{\infty}$. Once again, adopting this characterization of $f(x)$ may lead to unforseen problems. Of course, the answer may be to adopt an expression of $f(x)$ that is much different than the one studied in this paper. However such a formula has not conveniently presented itself. As of this paper, no known invariant measure has been found.

5 Ricker's Model

Our topic in this section is a simplified version of Ricker's population model. Specifically, we were interested in the long-term behavior of its orbits. Because the model is intended to model population, we restrict our attention to only meaningful values for x ($x > 0$). We define the function thus:

$$f(x) = xe^{p-x}$$

We state some easily verifiable facts:

First, the reader can see that f has only two fixed points ($x = 0$ and $x = p$).

$$f(0) = 0 \tag{3}$$
$$f(p) = p \tag{4}$$

These facts about the derivatives of f will become useful later.

$$f'(x) = e^{p-x}(1-x) \tag{5}$$
$$f''(x) = e^{p-x}(x-2) \tag{6}$$
$$f'(1) = 0 \quad \text{and } f(1) \text{ is the maximum of } f \tag{7}$$

By a calculation, we know that for $2 < p < \sqrt{6}$, f has a two-cycle (a point $\bar{x}$ such that $f^2(\bar{x}) = \bar{x}$. Moreover, this two-cycle is attracting, because $|[f^2(\bar{x})]'| \leq 1$.

For instructive purposes, we include a proof that there are at least two periodic points of f (which then make up the two-cycle).

Theorem 5.1 *The function f has at least one periodic point $\bar{x}, \bar{x} \neq 0$ or p.*

Proof: We notice that:

$$[f^2(0)]' = e^{2p} > 1$$

$$[f^2(p)]' = (1-p)^2 > 1$$

For small $\epsilon > 0$, we can say $f^2(0+\epsilon) \approx f^2(0) + \epsilon[f^2(0)]' > \epsilon \Rightarrow f^2(\epsilon) > \epsilon$. We call such an ϵ, a. By a similar argument there is a $b < p$ so that $f^2(b) < b$. We next define the function $g(x) = f^2(x) - x$. Therefore, $g(a) > 0$ and $g(b) < 0$. By the intermediate value theorem, there is a $\bar{x}$ so that $g(\bar{x}) = 0$, then $f^2(\bar{x}) = \bar{x}$. □

Corollary 5.2 *As a consequence, $f(\bar{x})$ is also a periodic point of period 2, because $f^2 \circ f(\bar{x}) = f \circ f^2(\bar{x}) = f(\bar{x})$.* □

This result means that $\bar{x}$ has this orbit $\{\bar{x}, f(\bar{x}), \bar{x}, f(\bar{x}), \dots\}$.

Because f has periodic points with period 2, we can learn much by understanding the structure of f^2. The following four results establish the existence of an invariant region of f^2.

Lemma 5.3 *For $2 < p < \sqrt(6)$, $f(1) < f_r^{-2}(p)$, where r designates the greater choice between pre-images of $f^{-1}(p)$.*

Proof: For reasons that will become clear, we want to establish some bounds on $f^{-1}(p)$. Because f has a maximum at $x = 1$ (remember that $f'(1) = 0$) and $p > 1$, it must be the case that $f^{-1}(p) < 1$. The reader can verify that for $2 < p < \sqrt{6}$, $f(1/5) < p$ and $f(1/2) > p$. Clearly, the following is true, $f(1/2) < f(f^{-1}(p)) = p < f(1/5)$. Because f is increasing, $1/5 < f^{-1}(p) < 1/2$. It is easy to see with some calculation, that $f^2(1) > 1/2$. Therefore, $f^{-1}(p) < f^2(1)$. On the interval (p, ∞), $f'(x) < 1$ and f is strictly decreasing. Therefore, $f^{-1}(p) < f^2(1) \Rightarrow f_r^{-2} > f(1)$. □

Theorem 5.4 *For any $x > 0$, there is a k so that $f^k(x) \in (f^{-1}(p), f_r^{-2}(p))$.*

Proof: The preceding fact allows everything to work out nicely. Consider the following two sets of intervals, where subscript l designates the left choice of inverses and r the right choice:

$$L_1 = (f^{-1}(p), p), \quad L_i = (f_l^{-i}(p), f^{-i+1}(p))$$
$$R_1 = (p, f_r^{-2}(p)), \quad R_i = (f_l^{-i}(p), f^{-i-1}(p))$$

One can see that these intervals cover the entire positive region of the x-axis except for the two fixed points ($x = 0$ and $x = p$) and their inverses. Without much difficulty, one sees that $f(R_i) = L_i$. Also, $f(L_i) = L_{i-1}$. Therefore, for $i > 1$, there is a k so that $f^k(L_i) = L_1$ and a k so that $f^k(R_i) = L_1$. The above lemma allows us to say that $f(L_1) = (p, f(1)) \subset R_1$. Any $x > 0$ is either in L_i or R_i, or is a pre-image of p. Therefore, there is a k such that for all $x > 0$ $f^k(x) \in L_1 \cup R_1 = (f^{-1}(p), f_r^{-2}(p))$. □

Corollary 5.5 *The interval $L_1 \cup R_1$ is invariant under f, because $f(L_1) \subset R_1$ and $f(R_1) = L_1$.* □

Corollary 5.6 *The interval L_1 is invariant under f^2, because $f(L_1) \subset R_1$ and $f(R_1) = L_1 \Rightarrow f^2(L_1) \subset L_1$.* □

This result allows us to see more intuitively why any periodic point of f must have an even period, because the iterates of any x under f will eventually end up alternating between L_1 and R_1. Moreover, one can see that $\bar{x}$ must be in L_1.

The following theorem is included in any book on dynamical systems ([**?**]), and we adapt it here for instructive purposes.

Theorem 5.7 *If f^2 is C^1 (as our function clearly is) and $\bar{x}$ is an attracting fixed point (note that $\bar{x}$ is not a fixed point of f but it is one of f^2) , then there is a region $(\bar{x}-\epsilon, \bar{x}+\epsilon)$ such that for any x in this region,*

$$\lim_{2k\to\infty}(x) = \bar{x}.$$

Proof: Because f^2 is C^1 there is a region $(\bar{x}-\epsilon, \bar{x}+\epsilon)$ such that for any x in this region $|[f^2(x)]'| < A < 1$. By the Mean Value theorem there is a $x < c < \bar{x}$ so that,

$$\frac{|f^2(x)-\bar{x}|}{|x-\bar{x}|} = [f^2(c)]'$$

However, for every x in $(\bar{x}-\epsilon, \bar{x}+\epsilon)$, $|[f^2(x)]'| < A$; and c is one such x. Thus:

$$\begin{aligned} \frac{|f^2(x)-\bar{x}|}{|x-\bar{x}|} &= [f^2(c)]' \\ &< A, \quad \text{then} \\ |f^2(x)-\bar{x}| &< A|x-\bar{x}| \end{aligned}$$

We notice that $f^2(x)$ must be in $(\bar{x}-\epsilon, \bar{x}+\epsilon)$ as well, so we can do the same thing again. Thus,

$$|f^{2k}(x)-\bar{x}| < A^{2k}|x-\bar{x}|.$$

If we take the limit, then:

$$\lim_{k\to\infty} f^{2k}(x) = \bar{x}. \quad \square$$

Notice that Lemma 5.7 guarantees that any point inside a region around $\bar{x}$ (defined as the region where $|[f^2(x)]'| < 1$ for every point) will converge to $\bar{x}$ upon repeated iterations of f^2.

The following series of results proves that the two-cycle attracts every point that is not a pre-image of the fixed point.

Lemma 5.8 *For $1 < x < 2$, $[f^2(x)]'$ is strictly increasing.*

Proof: We state some results from equations (5) and (6).

$$\begin{aligned} [f^2(x)]' &= f'(f(x)) \cdot f'(x) \\ [f^2(x)]'' &= f''(f(x)) \cdot f'(x) \cdot f'(x) + f''(x) \cdot f'(f(x)) \\ &= e^{p-f}(f-2) \cdot e^{2(p-x)}(1-x)^2 + e^{p-x}(x-2) \cdot e^{p-f}(1-f) \end{aligned}$$

For any $1 < x < 2$, $f(x) - 2 > 0$ and $(1-x)^2 > 0$. Therefore the first term above is positive. Moreover, $x - 2 < 0$ and $1 - f(x) < 0$. Thus, the second term is positive as well. So:

$$[f^2(x)]'' > 0. \quad \square$$

Theorem 5.9 *For any $p : 2 < p < \sqrt{6}$, $\bar{x} < 2$.*

Proof: To prove this, we just show that $f^2(2) < 2$. If we let $p = 2 + \epsilon$, then for our values of p, $0 < \epsilon < 1/2$ and:

$$2p - 2 - 2e^{p-2} = 2\epsilon - 2e^{\epsilon} < 0 \Rightarrow e^{2p-2-2e^{p-2}} < 1 \Rightarrow f^2(2) < 2$$

Now, by an intermediate value argument similar to the one used earlier, we know that $\bar{x} < 2$. $\square$

Corollary 5.10 *If $\bar{x} > 1$, then for any $x : 1 < x < \bar{x}$, $\lim_{k\to\infty} f^{2k} = \bar{x}$.*

Proof: Because $f'(1) = 0$, then $[f^2(1)]' = 0$. Also, $|[f^2(\bar{x})]'|< 1$. Because f^2 is strictly increasing on $[1, 2]$ and $[1, \bar{x}] \subset [1, 2]$, then it must be true that for every $x \in [1, \bar{x}]$, $|[f^2(x)]'|< 1$. By Theorem 5.7, every $x \in [1, \bar{x}]$ is attracted to $\bar{x}$. $\square$

Lemma 5.11 *For any $x : \bar{x} < x < p$, $\lim_{k\to\infty} f^{2k}(x) = \bar{x}$.*

Proof: It is a simple consequence of the above results that $f^2(x) < x$ for $\bar{x} < x < p$.

Case 1, $f^2(1) \geq 1$: Because $|[f^2(p)]'|> 1$, by the intermediate value thoerem $1 < \bar{x} < p$ (this is just a refinement of a our proof for the existence of $\bar{x}$). Because f^2 is strictly increasing for $1 < x < p$ and $f^2(x) < x$ for $\bar{x} < x < p$, it must be true that $\bar{x} < f^2(x) < p$ (in other words, $f^2(\bar{x}, p) = (\bar{x}, p)$). In this case, for any x one can iterate it enough times to get into the region around $\bar{x}$ outlined in Theorem 5.7. At this point we know that $\lim_{k\to\infty} f^{2k}(x) = \bar{x}$.

Case 2, $f^2(1) < 1$: We leave it to the reader to show that for $|[f^2(x_0)]'|> |[f^2(x_1)]'|$ where $x_0 < 1$ and x_1 is chosen so that $f(x_1) = f(x_0)$. This can be done knowing that f' increases faster on $(f^-1(p), 1)$ than on $(1, p)$. After this, we know that for any $x : \bar{x} < x < p$ there is a k so that $f^{2k}(x) \in (\bar{x}, a)$ where $1 < a < p$ and $f^2(a) = \bar{x}$. At this point we are in the region around $\bar{x}$ described in Theorem 5.7, and we know $\lim_{k\to\infty} f^{2k}(x) = \bar{x}$. $\square$

Theorem 5.12 *If x satisfies the following conditions, then $\lim_{k\to\infty} f^{2k}(x) = \bar{x}$.*

$$x \neq f^{-i}(p),\ i \in \mathbb{Z}$$
$$x \neq p$$
$$x \neq 0$$

Proof: We already know that the interval L_1 is invariant under f^2 and $\bar{x} \in L_1$. Moreover, one can see that f^2 has a critical point at $x = 1$ and this point is the global minimum on the interval L_1. The reader can verify that $f^2(1) < p$. Now, either $f^2(1) < 1$ or $f^2(1) > 1$. If $f^2(1) \leq 1$, then $f^2((f^-1(p), \bar{x})) \subset (\bar{x}, p)$ and then the above Corollary holds. If $f^2(1) > 1$, then $f^2((f^-1(p), 1)) \subset (1, p)$ and the above holds again. In either case $\lim_{k\to\infty} f^{2k}(x) = \bar{x}$. $\square$

Corollary 5.13 *Because f is continuous, as points approach $\bar{x}$ under iteration of f^2 they will also approach $f(\bar{x})$ under one additional iteration of f.* $\square$

Corollary 5.14 *There are no periodic points of any prime period other than 2.*

Proof: Clearly any periodic point must be in the interval $L_1 \cup R_1$. However, everything in this interval is either an inverse of the fixed point, or a periodic point, or is attracted to a periodic point (Theorem 5.12). So there is no possibility of a point that is of any period other than 2. □

We have now completely characterized the behavior of Ricker's model (for $2 < p < \sqrt{6}$) under iteration. Clearly, the set of pre-images of p form a countable set in $\mathbb{R}$. Therefore, in almost every case, repeated iterations of Ricker's model will approach the attracting two-cycle.