Characterization of Jump Systems and Other Results

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Abstract

A jump system is a set of lattice points satisfying a certain existence axiom. The main result in this paper is the proof of a descriptive characterization of 2-dimensional jump systems. Further results in the direction of characterizing higher-dimensional jump systems follow. In addition, a few other directions are handled yielding results that inspire further pursuit.

1 Jump Systems-Introduction

1.1 Introduction

A jump system is a set of lattice points in any number of dimensions that satisfy a certain existence axiom. The idea of jump systems was conceived by Bouchet and Cunningham in order to generalize the sets of bases of a matroid, degree sequences of subgraphs of a graph, and others. While many applications of jump systems lay in the formerly-stated areas, it is possible to naively work with jump systems, ignoring the details of their inspiring topics.

1.2 Two-Step Axiom and Basic Definitions

Definition 1.1. *Let S be a finite set. For $x, y \in \mathbb{Z}^S$ we use the [so-called* taxicab] norm

$$\|x\| = \sum_{i \in S} |x_i|$$

and the corresponding distance

$$d(x, y) = \|x - y\|.$$

Definition 1.2. *For vectors $x, y \in \mathbb{Z}^S$, a* step from x to y *is a vector $s \in \mathbb{Z}^S$ such that $\|s\| = 1$ and $d(x + s, y) = d(x, y) - 1$. $St(x, y)$ denotes the set of all steps from x to y. Note that if s is a step from x to y then $s = \pm e_i$, where e_i*

is i^{th} *standard unit vector. For notational convenience, we will sometimes use* $x \xrightarrow{y} x+s$ *to denote a step from* x *to* $x+s$ *in the direction of* y.

Definition 1.3. ***Jump System*** *Given a collection of points* $J \subseteq \mathbb{Z}^S$*, we say that* J *is a* jump system *if and and only if* J *satisfies Axiom 1.4.*

Axiom 1.4. ***(2-Step Axiom)*** *If* $x, y \in J \subseteq \mathbb{Z}^S$*,* $s \in St(x, y)$*, and* $x + s \notin J$*, then there exists* $t \in St(x + s, y)$ *with* $x + s + t \in J$*.*

1.3 Jump System Operations and Additional Definitions

The following operations allow us to simplify many of the later proofs concerning various properties of jump systems.

Definition 1.5. *Let* J *be a jump system and let* $a \in \mathbb{Z}^S$*. Then the* translation J' of J by a *is defined by* $J' = \{x + a : x \in J\}$*.*

Example 1.6. *Let* $J = \{(1, 1), (1, 3)\}$ *and* $a = (2, 4)$*. Then the translation of* J *by* a *is* $J' = \{(3, 5), (3, 7)\}$*.*

Definition 1.7. *Let* J *be a jump system and let* $N \subseteq S$*. We call* J' *the* reflection of J in N *if and only if* $J' = \{x' : x \in J, x'_j = x_j \ for \ j \notin N, x'_j = -x_j \ for \ j \in N\}$*.*

Example 1.8. *Let* J *be as in Example 1.6, with* $S = \{1, 2\}$ *and* $N = \{1\}$*. Then the reflection of* J *in* N *is* $J' = \{(-1, 1), (-1, 3)\}$*.*

Definition 1.9. *The* sum *of two jump systems* $J_1 \in \mathbb{Z}^S$ *and* $J_2 \in \mathbb{Z}^S$*, denoted* $J_1 + J_2 = J$*, where* $J = J_1 + J_2 = \{x + y : x \in J_1, y \in J_2\}$*.*

Example 1.10. *Let* $J_1 = \{(2, 2), (2, 3)\}$ *and* $J_2 = \{(0, 0), (1, 0), (3, 0)\}$*. Then* $J = J_1 + J_2 = \{(2, 2), (2, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$*.*

Definition 1.11. *Let* J *be a jump system in* $\mathbb{Z}^S$*. Let* $v \in \mathbb{R}^S$*. Then if* $v^T x = v_1 x_1 + \ldots + v_{|S|} x_{|S|} = \sum_{i \in S} v_i x_i$ *remains bounded for all* $x \in J$ *and* $\omega_v = max\{v^T x : x \in J\}$ *we call* $\mathfrak{f}_v = \{x : x \in J, v^T x = \omega_v\}$ *a* face *of J. For notational convenience we define the set* $V = \{v : v \in \{-1, 0, 1\}^S, v \neq 0)\}$*.*

Since the jump systems addressed in this paper are finite, the reader should note that all faces are in fact nonempty sets.

Example 1.12. *Let* J *be as in Example 1.10 and* $v = (1, 0)$*. Then* $\omega_v = 5$ *and* $\mathfrak{f}_v = \{(5, 2), (5, 3)\}$*.*

The geometry of two and three-dimensional jump systems and their ensuing faces yield great insight into the general properties of faces. Later it will be shown that the faces $\mathfrak{f}_v$ with $v \in V$ are sufficient for working with jump systems.

Theorem 1.13. *[Bouchet, Cunningham] Let* J *be a jump system in* $\mathbb{Z}^S$*. Then:*

1. *The translation of* J *is a jump system.*

2. *The reflection of J is a jump system.*

3. *The sum of two jump systems is a jump system.*

4. *If $v \in V$, then $\mathfrak{f}_v$ is a jump system.*

Since a jump system is a collection of points in n-dimensional space, we need a convention for determining the location of a point with respect to the extremes of the jump system.

Definition 1.14. *Let J be a jump system in $\mathbb{Z}^S$. Then we associate a collection of points in $\mathbb{Z}^S$ called a* polytope *with J. The polytope, denoted P_J, is the set*

$$P = \{x : x \in \mathbb{Z}^S, v^T x \leq \omega_v \forall v \in \mathbb{R}^S\}$$

Definition 1.15. *Let $a_i \leq b_i$ for all $i \in \{1, \ldots, |S|\}$ Then the set of points $\{x : a_i \leq x_i \leq b_i$ for all $i\}$is called a* box*. A box B can be denoted by $B = \prod_{i=1}^{|S|} [a_i, b_i]$.*

Note that while it is possible for $a_i \in (\mathbb{R} \cup -\{\infty\})$ and $b_i \in (\mathbb{R} \cup \{\infty\})$, we will always refer to boxes with finite dimensions and integral boundaries unless stated otherwise.

2 2-Dimensional Characterization and Additional Properties

Theorem 2.1. *Let J be a jump system and let $v \in V$. We define the set S_1 such that $S_1 = \{i \mid v_i \neq 0\}$. Let $S_2 \subset S_1$ with $|S_2| = |S_1| - 2$. If $a, b \in \mathfrak{f}_v$ and $a_i = b_i$ for all $i \in S_2$, then all the points between a and a', and b and b', where* $a'_i = \begin{cases} b_i & \text{if } i \in S_1, \\ a_i & \text{otherwise.} \end{cases}$ *and* $b'_i = \begin{cases} a_i & \text{if } i \in S_1, \\ b_i & \text{otherwise.} \end{cases}$
are also in $\mathfrak{f}_v$.

Proof. By reflection, translation, and coordinate-swapping, we may assume the following:
1)$v = (1, \ldots, 1, 0, \ldots, 0)$, where the first m entries of v are ones, and the remaining $|S| - m$ entries are zero.
2)$S_1 = \{1, \ldots, m\}$, $S_2 = \{3, \ldots, m\}$.
3)$a = (0, 0, 0, \ldots, 0)$.
4)$b = (b_1, -b_1, 0, \ldots, 0, b_{m+1}, \ldots, b_n)$ where $b_1 > 0$.

To prove that all the points between a and a' and between b and b' are in $\mathfrak{f}_v$, it is enough to show that the point $(1, -1, 0, \ldots, 0) \in \mathfrak{f}_v$ because if we prove that, we can recursively let $a = (1, -1, 0, \ldots, 0)$ and prove that $(2, -2, 0, \ldots, 0) \in \mathfrak{f}_v$, and so on, all the way to $(b_1, -b_1, 0, \ldots, 0)$. Take a step $a \xrightarrow{b} (1, 0, \ldots, 0)$. We see that $(1, 0, \ldots, 0) \notin J$ because $v^T(1, 0, \ldots, 0) > v^T a$. By Axiom 1.4, we can take a second step that will get us back into J. The only possible second step is

$(1,0,\ldots,0) \xrightarrow{b} (1,-1,0,\ldots,0)$, because a step in the last $|S|-m$ coordinates will take us out of $\mathfrak{f}_v$. Therefore, $(1,-1,0,\ldots,0) \in J$. Also, by an analogous argument, we can prove that all the points between b and b' are in $\mathfrak{f}_v$. $\square$

Example 2.2. *Let $v = (1,1,1,1,0,0)$, $a = (3,4,5,6,7,11)$, and $b = (0,4,5,9,10,2)$ with $a, b \in \mathfrak{f}_v$. Then $S_1 = \{1,2,3,4\}$ and $S_2 = \{2,3\}$. Now, $a' = (0,4,5,9,7,11)$ and $b' = (3,4,5,6,10,2)$. The above lemma states that the points between a and a', $(3,4,5,6,7,11)$, $(2,4,5,7,7,11)$, $(1,4,5,8,7,11)$, $(0,4,5,9,7,11)$, and the points between b and b', $(0,4,5,9,10,2)$, $(1,4,5,8,10,2)$, $(2,4,5,7,10,2)$, $(3,4,5,6,10,2)$, are all in $\mathfrak{f}_v$.*

Corollary 2.3. *If $J \subseteq \mathbb{Z}^2$ is a jump system and $a, b \in \mathfrak{f}_v$ where the support of v is 2, then all the points between a and b are also in the jump system.*

Proof. $S_1 = \{1,2\}$ and $S_2 = \{\}$. We see that theorem 2.1 applies here, with $a' = b$. Therefore all points between a and b are in the jump system. $\square$

Theorem 2.4. *[Lovász]*

Let J be a jump system and let $v, w \in V$. Define the function $abs(a) = (|a_1|,\ldots,|a_n|)$. If the dot product of $abs(v)$ and $abs(w)$ equals zero, then $(\mathfrak{f}_v)_w \cap \mathfrak{f}_{(v+w)} \neq \emptyset$.

Lemma 2.5. *Let $J \subseteq \mathbb{Z}^2$ be a jump system and $a \in P_J \setminus J$ such that $v^T a = \omega_v$ for some $v \in V$. Then there exist points $x, y \in \mathfrak{f}_v$ such that a lies on the line segment connecting x and y.*

Proof. We will prove this in two cases. The first case will deal with v whose support is one, and the second case will deal with v whose support is two. By reflection, translation, and coordinate flipping, we will assume that $a = (0,0)$ and it lies on $\mathfrak{f}_{(1,0)}$ in case one and $\mathfrak{f}_{(1,1)}$ in case two.

Case 1: $(1,0)^T a = \omega_{(1,0)}$.

Assume there are no points in J that are above a. By theorem 2.4, we know that $\mathfrak{f}_{(1,0)} \cap \mathfrak{f}_{(1,1)} \neq \emptyset$, so there must have been some point below a that is in $\mathfrak{f}_{(1,1)}$. But that would imply that a is outside $\mathfrak{f}_{(1,1)}$ and therefore not in the polytope of J. This is a contradiction.

Case 2: $(1,1)^T a = \omega_{(1,1)}$.

Assume there are no points in J that are of the form $(-\alpha, \alpha)$, $\alpha > 0$. By theorem 2.4, we know that $\mathfrak{f}_{(0,1)} \cap \mathfrak{f}_{(1,1)} \neq \emptyset$, so there must be a point of the form $(\alpha, -\alpha)$ that is in $\mathfrak{f}_{(1,1)}$ and $\mathfrak{f}_{(0,1)}$. But that would imply that a is outside $\mathfrak{f}_{(0,1)}$ and therefore not in the polytope of J. This is a contradiction.

$\square$

Theorem 2.6. *Let $J \subseteq \mathbb{Z}^2$. Then J is a jump system iff*
1) *For all $x', x'' \notin J$ with $d(x', x'') = 1$ that are in the polytope of J, there are no points $x \in J$ that are on the line that passes through x' and x''*
and
2) *each face of J is a jump system.*

Proof. Throughout the proof, the word *gap* will signify a point z for which $z \in P_J \setminus J$. Also, the expression y *is on a face* will mean $v^T y = \omega_v$ for some $v \in V$ regardless of whether $y \in J$ or $y \notin J$.
($\Rightarrow$) Assume that J is a jump system.

Because our conclusions are invariant under reflection and translation, we may, without loss of generality, assume that $x = (0,0) \in J$, $x' = (1,0) \notin J$, $x'' = (2,0) \notin J$ with x, x', x'' in the polytope of J. This implies that there are no points in J of the form $(k,0)$ where $k > 0$.

There are two cases we have to consider. The first case is when x' is on a face and the second is when x' is not. If x' is on a face, we see that x and x'' are on that same face. This observation holds because any face $\mathfrak{f}_w$ that contains x' and neither x nor x'' places x and x'' on opposite sides of itself. Then we would have either $x \notin P_J$ or $x'' \notin P_J$, contradicting our hypothesis.

Case (1) x, x', x'' are on a face $\mathfrak{f}_v$ for some $v \in V$.
We may assume that $v = (0,1)$, (i.e. x lies on the north face of J). Since we know there are no points to the right of x, x will be on the $(1,0)$ face of $\mathfrak{f}_v$. Therefore, $x \in \mathfrak{f}_{(1,1)}$ of J by 2.4.

Thus our assumption that x' is in the polytope of J is false, because $(1,1)^T x' = 1$, which is greater than $\omega_{(1,1)} = 0$, and therefore outside the polytope.

Case(2)$v^T x' < \omega_v$ for all $v \in V$.
Since we know that there are no points in J of the form $(k,0)$, with $k > 0$, and $(1,0)$ is in the polytope, there must be some gap $y = (m,0), m > 1$ on some face of J. By corollary 2.3 and lemma 2.5, we know that a "diagonal" face of a 2-dimensional jump system cannot contain any gaps in it, therefore $(m,0)$ must be on the $(1,0)$ face. By the previous case, we know that if $(m,0) \notin J$ is on a face, then either $(m,1)$ or $(m,-1)$ are in J. If we take a step $(m,\pm 1) \xrightarrow{(0,0)} (m,0)$, we will not have a second step toward $(0,0)$ that will land on a point $\in J$. Therefore J is not a jump system, and we have a contradiction.

($\Leftarrow$)Assume (1) and (2)
We will show that the two-step axiom holds for any arbitrary pair of points, $x, y \in J$. Because we can reflect and translate the jump system, we can, without loss of generality, assume $x = (0,0)$ and $y = (p,q)$ where $p, q \geq 0$. To avoid triviality, we assume that $d(x,y) > 2$ and that x and y are not on the same face. There are two cases to consider. The first case is where p or q equal zero (because of symmetry, we will only prove this case for $q = 0$) and the second is where $p, q > 0$.

Case (1)$q = 0$.

By (1) we know that $(1,0)$ or $(2,0)$ is in J. Thus, the two-step axiom holds. By symmetry, this case also applies to $y = (0,q)$.

Case (2)$p \geq 1$. $q \geq 1$.
Without loss of generality, we can assume that the first step is $x \xrightarrow{y} (1,0)$ and we assume that $(1,0) \notin J$. We will show that either
i)$(1,1) \in J$ or
ii)$(2,0) \in J$.

If $(1,1) \in J$, then it is a step from $(1,0)$ to y and the 2-step axiom is satisfied.

So we will assume that $(1,1) \notin J$. Clearly, $(1,1)$ is in the polytope of J. Now we will show that $(1,0)$ is in the polytope. Assume for the sake of contradiction that $(1,0)$ is not in the polytope. Then, $(0,0)$ must have been on a face of J. Since $v^T(1,0) > v^T(0,0) = 0$ for some v, we know that v_1=1. So the possible faces that $(0,0)$ could be on are: $\mathfrak{f}_{(1,0)}$, $\mathfrak{f}_{(1,1)}$, or $\mathfrak{f}_{(1,-1)}$. If x is on $\mathfrak{f}_{(1,0)}$ or $\mathfrak{f}_{(1,1)}$, then we have a contradiction because $v^T y > 0$ and thus y is not in J.

If $(0,0) \in \mathfrak{f}_{(1,-1)}$, then by Lovász, we know that there must be a common point on $\mathfrak{f}_{(1,-1)}$ and $\mathfrak{f}_{(1,0)}$. Since, $x \notin \mathfrak{f}_v$, there must be a point $(\alpha,\alpha) \in J$ where $\alpha > 0$. But now, Corollary 2.3 implies that $(1,1) \in J$, which is a contradiction. Therefore the point $(1,0)$ is in the polytope. Thus, by (1), we know there are no points of the form $(1,q)$ (because $(1,0)$ and $(1,1)$ are in the polytope, but not in J). So $y = (p,q)$, where $p > 1$.

Now we will show that $(2,0)$ is in the polytope. Assume for the sake of contradiction that $(2,0)$ is not in the polytope. Then $v^T(1,0) = \omega_v$ for some $v \in V$. Since $v^T(2,0) > v^T(1,0) = 0$ for some v, we know that v_1=1. But $(1,0)$ cannot be on $\mathfrak{f}_{(1,1)}$ or $\mathfrak{f}_{(1,-1)}$ because of corollary 2.3 and lemma 2.5. And it can not be on $\mathfrak{f}_{(1,0)}$ because $(1,0)^T y > (1,0)^T(1,0) = 1$. Thus, we have shown that the point $(2,0)$ is in the polytope. By (1) we know that $(2,0) \in J$, since the line passing through $(1,0)$ and $(2,0)$ also passes through $(0,0)$. There fore $(1,0) \xrightarrow{y} (2,0)$ satisfies the 2-step axiom. ☐

3 Additional Properties of Jump Systems

Theorem 3.1. *Let $J \subseteq \mathbb{Z}^S$ with associated polytope P_J. Let $x \in J$, $x + e_n \in P_J \setminus J$, $x + 2e_n \in P_J \setminus J$ for some $n \in S$. If $k \geq 1$, $\sum_{i \neq n} |k_i| \leq k-1$, $k_i \in \mathbb{Z}$, then*

$$x + ke_n + \sum_{i \neq n} k_i e_i \notin J.$$

Proof. Let $y \in J$ such that $y = x + ke_n + \sum_{i \neq n} k_i e_i$ with $d(x,y)$ minimal. Through reflection we may assume that $k_i \geq 0$ for all i. Step $y \xrightarrow{x} y - e_i$ for some e_i. Then $y - e_i$ violates minimal choice of y, so $y - e_i \notin J$. By Axiom 1.4 there exists a second step from y to x contained in J. There are three possible choices for this second step:

1. $y - e_i \xrightarrow{x} y - 2e_i$, which violates the minimal choice of y.
2. $y - e_i \xrightarrow{x} y - e_i - e_j$ for some $j \neq i$ and $j \neq n$, which also violates the minimality of y.
3. $y - e_i \xrightarrow{x} y - e_i - e_n$, which again violates our minimal choice of y.

Thus all three possibilities from Axiom 1.4 result in contradictions, and we have that $y \notin J$. ☐

We define some notation to make the following theorem less cumbersome. Consider $\mathbb{Z}^3$ and let $v \in V$ where the support of v is 3. Let $x \in \mathbb{Z}^3$, and define the following set $\mathcal{M}_v(x) = \{y : y \in J,\ y_i \geq x_i \text{ for } v_i = 1,\ y_i \leq \text{ for } v_i = -1\}$.

Theorem 3.2. *Let $J \subseteq \mathbb{Z}^3$ be a jump system. If $x \in P_J \setminus J$ and $v \in V$ with support equal to three, then there exists a point $z \in J$ such that $z \in \mathcal{M}_v(x)$.*

Proof. We prove the theorem for $v = (1,1,1)$, which is sufficient due to the symmetry of the arguments.

Consider $\mathfrak{f}_{e_1}$. Then $a_1 \geq x_1$ for all $a \in \mathfrak{f}_{e_1}$, or else $x \notin P_J$. Maximize a_3 over $\mathfrak{f}_{e_1}$, yielding points in the set $(\mathfrak{f}_{e_1})_{e_3}$. By [Lovász] we know that $(\mathfrak{f}_{e_1})_{e_3} \subseteq \mathfrak{f}_{(1,0,1)}$. Maximize a_2 over $(\mathfrak{f}_{e_1})_{e_3}$, yielding $((\mathfrak{f}_{e_1})_{e_3})_{e_2})$, which is contained in $\mathfrak{f}_{(1,1,1)}$ by [Lovász]. Thus for all points $a \in ((\mathfrak{f}_{e_1})_{e_3})_{e_2})$, we have that $a_1 \geq x_1$, $a_1 + a_3 \geq x_1 + x_3$, and $a_1 + a_2 + a_3 \geq x_1 + x_2 + x_3$.

Define the set $A = \{y \in J,\ y_1 \geq x_1,\ y_1 + y_3 \geq x_1 + x_3,\ y_1 + y_2 + y_3 \geq x_1 + x_2 + x_3\}$. It is clear that $((\mathfrak{f}_{e_1})_{e_3})_{e_2}) \subseteq A$, so A is nonempty.

Consider $\mathfrak{f}_{e_3}$. Then $b_3 \geq x_3$ for all $b \in \mathfrak{f}_{e_3}$, or else $x \notin P_J$. Maximize b_1 over $\mathfrak{f}_{e_3}$, yielding points in the set $(\mathfrak{f}_{e_3})_{e_1}$. By [Lovász] we know that $(\mathfrak{f}_{e_3})_{e_1} \subseteq \mathfrak{f}_{(1,0,1)}$. Maximize b_2 over $(\mathfrak{f}_{e_3})_{e_1}$, yielding $((\mathfrak{f}_{e_3})_{e_1})_{e_2})$, which is contained in $\mathfrak{f}_{(1,1,1)}$ by [Lovász]. Thus for all points $b \in ((\mathfrak{f}_{e_3})_{e_1})_{e_2})$, we have that $b_3 \geq x_3$, $b_1 + b_3 \geq x_1 + x_3$, and $b_1 + b_2 + b_3 \geq x_1 + x_2 + x_3$.

Define the set $B = \{y \in J,\ y_3 \geq x_3,\ y_1 + y_3 \geq x_1 + x_3,\ y_1 + y_2 + y_3 \geq x_1 + x_2 + x_3\}$. It is clear that $((\mathfrak{f}_{e_3})_{e_1})_{e_2}) \subseteq B$, so B is nonempty.

Choose $a \in A$ and $b \in B$ such that a_3 and b_1 are maximal. Define the set $\mathcal{M}_{13} = \{y \in J,\ y_1 \geq x_1,\ y_3 \geq x_3,\ y_1 + y_2 + y_3 \geq x_1 + x_2 + x_3\}$. Then $\mathcal{M}_{13}$ is nonempty, or else $a_3 < x_3$ and $b_1 > x_1$.

Assume $\mathcal{M}_{13}$ is empty, and take the step $a \xrightarrow{b} a + e_3$. We know that $a + e_3 \notin J$, because $(1,1,1)^T(a + e_3) > (1,1,1)^T a = \omega_{(1,1,1)}$. Then by Axiom 1.4 one of $a + 2e_3 \in J$, $a + e_3 - e_1 \in J$, or $a + e_3 - e_2 \in J$ must hold. However, each of these possibilities violates the maximal choice of a, and therefore $\mathcal{M}_{13}$ must be nonempty.

Through analogous arguments, the sets $\mathcal{M}_{12} = \{y \in J, y_1 \geq x_1, y_2 \geq x_2, y_1 + y_2 + y_3 \geq x_1 + x_2 + x_3\}$ and $\mathcal{M}_{23} = \{y \in J, y_2 \geq x_2, y_3 \geq x_3, y_1 + y_2 + y_3 \geq x_1 + x_2 + x_3\}$ are nonempty.

Choose $f \in \mathcal{M}_{13}$ with f_2 maximal, $g \in \mathcal{M}_{23}$ with g_1 maximal, and $h \in \mathcal{M}_{12}$ with h_3 maximal. If $f_2 \geq x_2$, $g_1 \geq x_1$, or $h_3 \geq x_3$, then we are done, so assume otherwise. Thus $f_2 < x_2$, $g_1 < x_1$, and $h_3 < x_3$. Since $f_2 < x_2$, we may assume without loss of generality that $f_1 > x_1$ in order to preserve $f_1 + f_2 + f_3 \geq x_1 + x_2 + x_3$.

Examine the step $f \xrightarrow{g} f + e_2$. Since $f + e_2$ violates the maximal choice of f, we know that $f + e_2 \notin J$. Axiom 1.4 thereby states that one of $f + 2e_2$, $f + e_2 + e_1$, or $f + e_2 + e_3$ is in J. However, each of these points contradicts the maximal choice of f. Therefore there exists a point $z \in J$ such that $z_1 \geq x_1$, $z_2 \geq x_2$, and $z_3 \geq x_3$. □

Theorem 3.3. *Let $J \subseteq \mathbb{Z}^3$ be a jump system with associated polytope P_J. Let $x \in P_J \setminus J$, $x + e_3 \in P_J \setminus J$, and $x + 2e_3 \in J$. Then the eight points in $\{x \pm e_1 \pm e_2; x \pm e_1 + e_3; x \pm e_2 + e_3\}$ are contained in J.*

Proof. Since x is contained in the polytope P_J, by Theorem 3.2 we know that there exists a point $b \in f_{(1,1,-1)}$ such that $b_1 \geq x_1$, $b_2 \geq x_2$, and $b_3 \leq x_3$. Step $x + 2e_3 \xrightarrow{b} x + e_3 \notin J$. Then by Axiom 1.4 there exists a second step from $x + e_3$ to b. Since $x + e_3 \xrightarrow{b} x \notin J$ by our hypothesis, it must be the case that either $x + e_1 + e_3 \in J$ or $x + e_2 + e_3 \in J$. Whichever occurs, we will show that $x + e_2 + e_3 \in J$.

Assume that $x+e_1+e_3 \in J$ and consider the step $x+e_1+e_3 \xrightarrow{b} x + e_1 \notin J$. Then there must be a second step from $x+e_1$ to $x+2e_1$, $x+e_1+e_2$, or $x+e_1-e_3$, although Theorem 3.1 dictates that $x + e_1 - e_3 \notin J$.

Assume that $x + e_1 + e_2 \notin J$, implying that $x + 2e_1 \in J$. We will show that this results in a contradiction, and $x + e_1 \xrightarrow{b} x + e_1 + e_2 \in J$.

Then by Theorem 3.2 we know that there is a point $c \in \mathfrak{f}_{(-1,1,-1)}$ such that $c_1 \leq x_1$, $c_2 \geq x_2$, and $c_3 \leq x_3$. Consider the step $x + 2e_3 \xrightarrow{c} x + e_3 \notin J$. By Axiom 1.4, there is a second step $x+e_3 \xrightarrow{c} x - e_1 + e_3$ or $x+e_3 \xrightarrow{c} x + e_2 + e_3$, because x is ineligible due to our hypothesis.

If $x-e_1+e_3 \in J$, then $x-e_1+e_3 \xrightarrow{x+2e_1} x - e_1 \notin J$, and $x \notin J$ by hypothesis. The only other possible step from $x - e_1$ to $x + 2e_1$ is $x - e_1 \xrightarrow{x+2e_1} x \notin J$, which violates Axiom 1.4, so we cannot have $x - e_1 + e_3 \in J$.

If $x + e_2 + e_3 \in J$, then $x + e_2 + e_3 \xrightarrow{x+2e_1} x + e_2 \notin J$. There are two possible second steps from $x + e_2$ to $x + 2e_1$, namely x and $x + e_1 + e_2$. However, $x \notin J$ by hypothesis, so we must have $x + e_1 + e_2 \in J$, contradicting our earlier assumption. Therefore $x + e_1 + e_2 \in J$.

Through a symmetric argument we conclude that $x - e_1 - e_2 \in J$. Then taking $x - e_1 - e_2 \xrightarrow{x+e_1+e_2} x - e_1 \notin J$, it must be the case that $x - e_1 + e_2 \in J$, because we disregard x by our hypothesis. Also, by taking $x - e_1 - e_2 \xrightarrow{x+e_1+e_2} x - e_2 \notin J$, it must be the case that $x + e_1 - e_2 \in J$.

Consider the step $x + e_1 + e_2 \xrightarrow{x+2e_3} x + e_1 \notin J$. Then since $x \notin J$, we must have $x + e_1 + e_3 \in J$, because our hypothesis disallows $x \in J$. Similarly, $x + e_1 + e_2 \xrightarrow{x+2e_3} x + e_2 \notin J$. Then because $x \notin J$, we arrive at $x + e_2 + e_3 \in J$. Analogously, we can step $x - e_1 - e_2 \xrightarrow{x+2e_3} x - e_1 \notin J$ and $x - e_1 - e_2 \xrightarrow{x+2e_3} x - e_2 \notin J$, forcing $x - e_1 + e_3 \in J$ and $x - e_2 + e_3 \in J$, respectively. □

Definition 3.4. *Let J be a jump system. Then the* reduction *J' of J, written as $J' = J_{(x_1,\dots,x_i+x_j,\dots,x_{|S|})}$ means that for all $x = (1, \dots, |S|) \in J$, $x' = (x_1, \dots, x_{i-1}, x_i + x_j, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{|S|}) \in J'$.*

Theorem 3.5. *[Ponomarenko] The reduction operation preserves the jump system property.*

Definition 3.6. *Let $J \subseteq \mathbb{Z}^S$ be a jump system and $i \in S$. Then a* projection *J' of J, written as $J' = J_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{|S|})}$ means that for all $x = (x_1, \dots, x_{|S|}) \in J$, $x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{|S|}) \in J'$.*

Theorem 3.7. *The projection operation preserves the jump system property.*

Example 3.8. *example of reduction*

Definition 3.9. *Let J be a jump system. Then the* strong reduction *J' of J, written as $J' = J_{(x_1,\dots,x_i+\alpha x_j,\dots,x_{|S|})}$ where $\alpha \in \{-1, 0, 1\}$, means that for all $x \in J$, $x' = (x_1, \dots, x_{i-1}, x_i + \alpha x_j, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{|S|}) \in J'$.*

Corollary 3.10. *The strong reduction operation preserves the jump system property.*

Proof. Case 1: $\alpha = 1$ or $\alpha = -1$

This case is exactly the reduction operation (with reflection having been used for $\alpha = -1$), so by Theorem 3.5, we know that J' is a jump system.
Case 2: $\alpha = 0$. This case is exactly the projection operations Thus by Theorem 3.7, we know that J' is a jump system. □

Theorem 3.11. *If J is a collection of points and there exists a $v \in \{-1, 0, 1\}^S$ such that $v^T x$ is a constant for all $x \in J$ then J is a jump system if and only if every strong reduction is also a jump system.*

Proof. By Theorem **??**, we immmediately get one direction of the proof. For the other direction, we will show that if J is not a jump system, then there exists a strong-reduction that is also not a jump system. Since J is not a jump system, there exist points $a, b \in J$ and $s \notin J$ such that after taking a step $a \xrightarrow{b} s$, there are no steps from s to b. By reflection, translation, and coordinate swapping, we can assume the following:
1)$v = (1, \dots, 1, 0, \dots 0)$. v contains n elements with the first m of them being ones and the rest zeroes.
2)$a = (0, 0, \dots, 0)$.
3)$v^T x = 0$ for all $x \in J$, that is $\sum_{i=1}^{m} x_m = 0$ (Direct consequence of above)
4)$b = (b_1, b_2, \dots, b_n)$.

There are two cases that we have to consider. The first one is when s is such that $s_i = \bar{b}_i$ for some $i \leq m$ and zero everywhere else. The second case is when $s_i = \bar{b}_i$ for some $i > m$ and zero everywhere else. For the first case, without loss of generality, we will assume that $i = 1$ and for the second case, we will assume that $i = m + 1$.

Case I: $s = (\bar{b}_1, 0, \dots, 0)$.

We may assume that $\bar{b}_1 = -\bar{b}_2$ and $m > 2$. (since there must exist a b_i with a sign opposite b_1 in order for $v^T b = 0$ and we just let $i = 2$). We will break up this case into the following four subcases:
1)$(2\bar{b}_1, 2\bar{b}_2, 0, \dots, 0) \in J$.
2)$(\bar{b}_1, 2\bar{b}_2, 0, \dots, \bar{b}_i, \dots, 0) \in J$ where $\bar{b}_1 = \bar{b}_i$ and $i \leq m$.
3)$(\bar{b}_1, \bar{b}_2, 0, \dots, \bar{b}_i, \dots, 0) \in J$ where $i > m$.
4)Subcases 1-3 are false.

Subcase 1: $(2\bar{b}_1, 2\bar{b}_2, 0, \dots, 0) \in J$.

Let $J' = J_{(x_1 - x_2, x_3, \dots, x_n)}$. We know that $a' = (0, \dots, 0)$ and $b' = (4\bar{b}_1, 0 \dots, 0)$ are in J'. To prove J' is not a jump system, we will first prove that $e = (\bar{b}_1, 0, \dots, 0) \notin J'$ and then prove that $2e = (2\bar{b}_1, 0, \dots, 0) \notin J'$. Thus, after $a' \xrightarrow{b'} e$, the only next valid step towards b', mainly $2e$, is not in J', which is a violation of the 2-step axiom.

Assume $(\bar{b}_1, 0, \dots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_2 = \bar{b}_1$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_m = 0$. So we have the following two equations: $x_1 - x_2 = \bar{b}_1$ and $x_1 + x_2 = 0$. Therefore $x_1 = \bar{b}_1/2$, which is absurd, and so we have a contradiction and hence $(\bar{b}_1, 0, \dots, 0) \notin J'$.

Assume $(2\bar{b}_1, 0, \dots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_2 = 2\bar{b}_1$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_m = 0$. So we have the following two equations: $x_1 - x_2 = 2\bar{b}_1$ and $x_1 + x_2 = 0$. Therefore $x_1 = \bar{b}_1$ and $x_2 = -\bar{b}_1 = \bar{b}_2$. But this would mean that $(\bar{b}_1, \bar{b}_2, 0, \dots, 0) \in J$ and that would be a step from s to b which contradicts our initial assumption, that (a, b, s) violates the 2-step axiom. Thus, J' is not a jump system and subcase 1 is proved.

Subcase 2: $(\bar{b}_1, 2\bar{b}_2, 0, \dots, \bar{b}_i, \dots, 0) \in J$ where $\bar{b}_1 = \bar{b}_i$ and $i \leq m$.

Let $J' = J_{(x_1, x_i - x_2, x_3, \dots, x_n)}$. We know that $a' = (0, \dots, 0)$ and $b' = (\bar{b}_1, 3\bar{b}_1, 0 \dots, 0)$ are in J'. To prove that J' is not a jump system, we will first prove that $e = (\bar{b}_1, 0, \dots, 0) \notin J'$ and then prove that $f = (\bar{b}_1, \bar{b}_1, 0, \dots, 0) \notin J'$. Thus after $a' \xrightarrow{b'} e$, the only next valid step toward b', mainly f, is not in J', which is a violation of the 2-step axiom.

Assume $(\bar{b}_1, 0, \dots, 0) \in J'$. Then, for some $x \in J$, we have $x_1 = \bar{b}_1, x_i - x_2 = 0$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0$. So we have the following two equations: $x_i - x_2 = 0$ and $x_i + x_2 = -\bar{b}_1$. That implies $x_i = -\bar{b}_1/2$, which is absurd, and we have a contradiction.

Assume $(\bar{b}_1, \bar{b}_1, 0, \dots, 0) \in J'$. Then, for some $x \in J$, $x_1 = \bar{b}_1, x_i - x_2 = \bar{b}_1; x_3, \dots, x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0$. So, we have the following two equations: $x_i - x_2 = \bar{b}_1$ and $x_i + x_2 = -\bar{b}_1$. That implies $x_i = 0, x_2 = -\bar{b}_1 = \bar{b}_2$. But, this would mean that $x = (\bar{b}_1, \bar{b}_2, 0, \dots, 0) \in J$ and that would be a step from s to b, which contradicts our initial assumtion that (a, b, s) violates the 2-step axiom. Thus J' is not a jump system, and subcase 2 is proved.

Subcase 3: $(\bar{b}_1, \bar{b}_2, 0, \dots, \bar{b}_i, \dots, 0) \in J$ where $i > m$.

Let $J' = J_{(x_1 - x_2, x_3, \dots, x_n)}$. We know that $a' = (0, \dots, 0)$ and $b' = (2\bar{b}_1, 0 \dots, \bar{b}_i, \dots, 0)$ are in J'. To prove that J' is not a jump system, we will first prove that $e = (\bar{b}_1, 0, \dots, 0) \notin J'$ and then prove that

both $2e = (2\bar{b}_1, 0, \ldots, 0)$ and $f = (\bar{b}_1, 0, \ldots, \bar{b}_i, \ldots, 0)$ are not in J'. Thus, after $a' \xrightarrow{b'} e$, the only next valid steps toward b', mainly $2e$ and f, are not in J', which is a violation of the 2-step axiom.
Assume $(\bar{b}_1, 0, \ldots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_2 = \bar{b}_1$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_m = 0$. So we have the following two equations: $x_1 - x_2 = \bar{b}_1$ and $x_1 + x_2 = 0$. Therefore $x_1 = \bar{b}_1/2$, which is absurd, and so we have a contradiction and hence $(\bar{b}_1, 0, \ldots, 0) \notin J'$.
Assume $(2\bar{b}_1, 0, \ldots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_2 = 2\bar{b}_1$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_m = 0$. So we have the following two equations: $x_1 - x_2 = 2\bar{b}_1$ and $x_1 + x_2 = 0$. Therefore $x_1 = \bar{b}_1$ and $x_2 = -\bar{b}_1 = \bar{b}_2$. But this would mean that $(\bar{b}_1, \bar{b}_2, 0, \ldots, 0) \in J$ and that would be a step from s to b which contradicts our initial assumption.
Assume $(\bar{b}_1, 0, \ldots, \bar{b}_i, \ldots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_2 = \bar{b}_1$ and $x_3 = \cdots = x_n = 0$. Also, we know that $x_1 + x_2 + \cdots + x_m = 0$. So we have the following two equations: $x_1 - x_2 = \bar{b}_1$ and $x_1 + x_2 = 0$. Therefore $x_1 = \bar{b}_1/2$, which is absurd, and so we have a contradiction. Thus J' is not a jump system because (a', b', e) violates the 2-step axiom, and subcase 3 is proved.

Subcase 4: Subcases 1-3 are false.
Let $J' = J_{(x_1, x_3, x_4, \ldots, x_n)}$. We know that $a' = (0, \ldots, 0)$ and $b' = (b_1, b_3, b_4, \ldots, b_n)$ are in J'. Let $s' = (\bar{b}_1, 0, \ldots, 0)$ be a step from a' to b'. We see that the only way s' can be in J', is if $(\bar{b}_1, \bar{b}_2, 0, \ldots, 0)$ were in J. But, if that were the case, then the latter would be a step from s to b, which would contradict our initial assumption. Now, we will show what happens if there exists a step in J' from s' to b'. Assume $(\bar{b}_1, 0, \ldots, \bar{b}_i, \ldots, 0)$ where $i \leq m$ and $\bar{b}_1 = -\bar{b}_i$ is in J'. Then we see that $(\bar{b}_1, 0, 0, \ldots, \bar{b}_i, \ldots, 0)$ must have been in J, which is a step from s to b, and thus a contradiction. Now, assume $(2\bar{b}_1, 0, \ldots, 0) \in J'$. Then we see that $(2\bar{b}_1, 2\bar{b}_2, 0, \ldots, 0)$ must have been in J. But that is subcase 1, which is false by assumption, and thus we have a contradiction. Now, assume $(\bar{b}_1, 0, \ldots, \bar{b}_i, \ldots, 0)$ where $i \leq m$ and $\bar{b}_1 = \bar{b}_i$ is in J'. Then we see that $(\bar{b}_1, 2\bar{b}_2, 0, \ldots, \bar{b}_i, \ldots, 0)$ must have been in J. But that is subcase 2, which is false by assumption, and thus we have a contradiction. Now assume $(\bar{b}_1, 0, \ldots, \bar{b}_i, \ldots, 0)$ where $i > m$ is in J'. Then we see that $(\bar{b}_1, \bar{b}_2, 0, \ldots, \bar{b}_i, \ldots, 0)$ must have been in J. But that is subcase 3, which is false by asumption, and thus we have a contradiction. Thus J' is not a jump system because (a', b', s') violates the 2-step axiom. And so, subcase 4 and case I are proved.

Case II: $s = (0, \ldots, 0, \bar{b}_{m+1}, 0, \ldots, 0)$.
We shall break this case up into two subcases:

1)$(-\bar{b}_i, 0, \dots, 0, \bar{b}_i, 0, \dots, 0, \bar{b}_{m+1}, 0, \dots, 0) \in J$.
2)Subcase 1 is false.

Subcase 1: $(-\bar{b}_i, 0, \dots, 0, \bar{b}_i, 0, \dots, 0, \bar{b}_{m+1}, 0, \dots, 0) \in J$.
Let $J' = J_{(x_1 - x_i, x_2, x_3, \dots, x_n)}$. We see that $a' = (0, \dots, 0)$ and $b' = (-2\bar{b}_i, 0, \dots, \bar{b}_{m+1}, \dots, 0)$ are in J'. We will prove that J' is not a jump system by first proving $e = (0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \notin J'$ and then proving that $f = (-\bar{b}_i, 0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \notin J'$. Thus, after $a' \xrightarrow{b'} e$, the ony valid step from e to b', mainly f, is not in J', which is a violation of the 2-step axiom.
Assume $(0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_i = 0$ and $x_2 = \dots = x_{i-1} = 0, x_{i+1} = \dots = x_m = 0, x_{m+1} = \bar{b}_{m+1}$. Also, we know that $x_1 + \dots + x_i + \dots + x_m = 0$. So we have the following two equations: $x_1 - x_i = 0$ and $x_1 + x_i = 0$. Thus, $x_1 = x_i = 0$ and therefore $(0, \dots, 0, \bar{b}_{m+1}, 0 \dots, 0)$ must have been in J. But that is a contradiction.
Assume $(-\bar{b}_i, 0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \in J'$. Then for some $x \in J$, we have $x_1 - x_i = -\bar{b}_i$ and $x_2 = \dots = x_{i-1} = 0, x_{i+1} = \dots = x_m = 0, x_{m+1} = \bar{b}_{m+1}$. Also, we know that $x_1 + \dots + x_i + \dots + x_m = 0$. So we have the following two equations: $x_1 - x_i = -\bar{b}_i$ and $x_1 + x_i = 0$. Thus, $x_1 = -\bar{b}_i/2$, which is absurd and we have a contradiction. And the subcase is proved.

Subcase 2: 1)$(-\bar{b}_i, 0, \dots, 0, \bar{b}_i, 0, \dots, 0, \bar{b}_{m+1}, 0, \dots, 0) \notin J$.
Let $J' = J(x_2, x_3, \dots, x_n)$. We see that $a' = (0, \dots, 0)$ and $b' = (b_2, b_3, \dots, b_n)$ are in J'. Let $s' = (0, \dots, \bar{b}_{m+1}, 0, \dots, 0)$. This is a step from a' to b'. We see that the only way s' can be in J' is if $(0, 0, \dots, \bar{b}_{m+1}, 0, \dots, 0)$ were in J. But, by our assumption, we know that that the latter is not in J. Now, we consider all steps from s' to b'. Assume $(0, \dots, 2\bar{b}_{m+1}, 0, \dots, 0) \in J')$. That would imply that $(0, 0, \dots, 2\bar{b}_{m+1}, 0, \dots, 0)$ was in J. But that would be a step from s toward b, which contradicts our initial assumption. Now, assume $(0, \dots, \bar{b}_{m+1}, 0, \dots, \bar{b}_i, 0, \dots, 0) \in J')$. That would imply that $(0, 0, \dots, \bar{b}_{m+1}, 0, \dots, \bar{b}_i, 0, \dots, 0)$ was in J. But that would also be a step from s to b and thus a contradiction. Now, assume $(0, \dots, \bar{b}_i, 0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \in J'$. But that would imply that $(-\bar{b}_i, 0, \dots, \bar{b}_i, 0, \dots, \bar{b}_{m+1}, 0, \dots, 0) \in J$. But this contradicts our assumption. There are no other possible steps from s' to b', and thus, case II and the theorem are proved.

□

Lemma 3.12. *Let $v \in V$, $\alpha \in \mathbb{Z}^S$ such that $v^T \alpha = b$, and $\beta \in \mathbb{Z}^S$ such that $v^T \beta > b$. Then there exists $i \in S$ such that $v_i \beta_i > v_i \alpha_i$.*

Proof. Assume the lemma is not true. Then $\beta_i > \alpha_i \Rightarrow v_i \in \{-1, 0\}$, and $\beta_i < \alpha_i \Rightarrow v_i \in \{0, 1\}$. These facts imply that $v_i(\beta_i - \alpha_i) \leq 0$ for all

$i \in \{1, \ldots, |S|\}$. Hence, $v^T\beta - v^T\alpha = v^T(\beta - \alpha) = \sum_i v_i(\beta_i - \alpha_i) \leq 0$, which contradicts the hypothesis that $v^T\beta > v^T\alpha$. $\square$

Lemma 3.13. *Let $v \in V$, $\alpha \in \mathbb{Z}^S$ such that $v^T\alpha = b$, and $\beta \in \mathbb{Z}^S$ such that $v^T\beta > b$. Then there exists $\delta \in \mathbb{Z}^S$ such that $\delta \in St(\alpha, \beta)$ and $v^T\delta = b + 1$.*

Proof. $v^T\alpha = b$ and $v^T\beta > b \Rightarrow v^T\beta - v^T\alpha > 0$. This inequality implies that $v_1(\beta_1 - \alpha_1) + \ldots + v_{|S|}(\beta_{|S|} - \alpha_{|S|}) > 0$. By Lemma 3.12 there exists $i \in \{1, \ldots, |S|\}$ with $v_i(\beta_i - \alpha_i) > 0$.

Take a step $\alpha \xrightarrow{\beta} \delta$ such that $\delta = \alpha \pm e_i$. Then: $d(\alpha, \delta) = 1$, $d(\delta, \beta) = d(\alpha, \beta) - 1$, and $v^T\delta = v^T(\alpha \pm e_i) = v^T\alpha + 1 = b + 1$. $\square$

Definition 3.14. *Let $J \subseteq \mathbb{Z}^S$ be a jump system and $v \in V$. Then a set of points $\{x : v^Tx = b \ for \ b \in \mathbb{Z}, x \notin J\}$, denoted $R(v, b)$, is called a* rift. *We say that J admits $R(v, b)$.*

Theorem 3.15. *Let J be a jump system in $\mathbb{Z}^S$, $v \in V$, and $\alpha \in J$ such that $v^T\alpha = b-1$. If J admits both $R(v, b)$ and $R(v, b+1)$, then $v^T\beta > b+1 \Rightarrow \beta \notin J$.*

Proof. Let $J \subseteq \mathbb{Z}^S$ be a jump system, $v \in V$, and $\beta \in J$ such that $v^T\beta > b + 1$. Let $\alpha \in J$ such that $v^T\alpha = b - 1$ and $d(\alpha, \beta)$ is minimized. Let J admit $R(v, b)$ and $R(v, b + 1)$. Apply Lemma 3.13. Thus there exists δ such that $d(\alpha, \delta) = 1$, $d(\delta, \beta) < d(\alpha, \beta)$, and $v^T\delta = b$. Take a step $\alpha \xrightarrow{\beta} \delta$. $\delta \notin J$ by our hypothesis, so there must be a step $\delta \xrightarrow{\beta} \delta'$ such that $\delta' \in J$. Also, $v^T\delta' \neq b - 1$, or else the minimal choice of α would be violated. Thus by Lemma 3.13 we take a step $\delta \xrightarrow{\beta} \delta'$ such that $d(\delta, \delta') = 1$, $d(\delta', \beta) < d(\delta, \beta)$, and $v^T\delta' = b + 1$ or b, both of which imply that $\delta' \notin J$. Therefore the pair α, β violates Axiom 1.4, and β cannot exist. $\square$

Theorem 3.16. *Let $x \in P_J$. If $x + e_i \notin P_J$ or $x - e_i \notin P_J$ for some $i \in \{1, \ldots, |S|\}$, then $v^Tx = \omega_v$ for some $v \in V$. Furthermore, if $x \in J$, then $x \in \mathfrak{f}_v$.*

Proof. Let $x \in P_J$ and $x + e_i \notin P_J$ for some $i \in \{1, \ldots, |S|\}$. Then $x \in P_J \Rightarrow$ for all $v \in V$, $v^Tx \leq \omega_v$. $x \pm e_i \notin P_J \Rightarrow$ there exists $v' \in V$ such that ${v'}^T(x \pm e_i) > \omega_{v'}$. In particular we have that $v_i \in \{-1, 1\}$, or else ${v'}^T(x + e_i) = v'_1x_1 + \ldots + v'_i(x_i + e_i) + \ldots + v'_{|S|}x_{|S|} = v'_1x_1 + \ldots + v'_ix_i + \ldots + v'_{|S|}x_{|S|}$, which is a contradiction of our hypothesis.

Thus ${v'}^T(x + e_i) > \omega_{v'} \geq {v'}^Tx$ and ${v'}^T(x + e_i) = {v'}^Tx + 1$. So ${v'}^Tx + 1 > \omega_{v'} \geq {v'}^Tx$, and since $\omega'_v \in \mathbb{Z}$, we have that ${v'}^Tx = \omega_{v'}$. $\square$

4 Prime Jump Systems

Definition 4.1. *A jump system $J \in \mathbb{Z}^S$ is called a* prime *jump system iff*

1. $|J| \geq 2$

2. *There does not exist jump systems J_a and J_b such that both $|J_a| \geq 2$, $|J_b| \geq 2$, and $J_a + J_b = J$.*

Theorem 4.2. *Given a jump system $J \subseteq \mathbb{Z}^S$ where $|J| = 3$, we have that $J = \{a, b, c\}$. Then there exists $J_x \subseteq \mathbb{Z}^S, J_y \subseteq \mathbb{Z}^S$ such that $|J_x| = 2$, $|J_y| = 2$ and $J_x + J_y = J$ where c is the point of J obtainable by two different combinations of points in J_x, J_y iff*

1. *$c_k = (\frac{a_k+b_k}{2})$ for all coordinates,*
2. *$a_k = b_k = c_k$ fails for at most two coordinates, and*
3. *$\sum_{i \in S} |a_i - b_i| = 0, 2$ or 4.*

Proof. ($\Rightarrow$) Let $J \subseteq \mathbb{Z}^S$ be a jump system such that $|J| = 3$ and $J = J_x + J_y$ where $|J_x| = 2$, $|J_y| = 2$. Thus $J = \{a, b, c\}$, $J_x = \{(x, x')\}$, and $J_y = \{(y, y'\}$, and we can assume that c is the point obtained through summation in two different ways. With this assumption we have

$$x_i + y_i = a_i \quad x'_i + y'_i = b_i \quad x_i + y'_i = c_i \quad x'_i + y_i = c_i \text{ for all } i,$$

which implies $2c_i = a_i + b_i \Rightarrow c_i = \left(\frac{a_i+b_i}{2}\right)$. This statement proves the first condition of the theorem.

Suppose $a_i = b_i = c_i$ fails and assume without loss of generality that $b_i > a_i$. Thus $b_i = a_i + k_i$ for some $k_i \in \mathbb{Z}$. If k_i is odd, then $c_i = \frac{2a_i+k_i}{2} \notin \mathbb{Z}$, so k_i must be even. Furthermore, if $k_i \geq 6$, then $c_i \geq a_i + 3$. Then $a \xrightarrow{c} a + e_i \notin J$, so there must be a second step from $a + e_i$ to c that is in J. We know that $a + 2e_i \notin J$, so $a + e_i \pm e_j \in J$ for some $j \neq i$. However, this is impossible, because $b_i > c_i > a_i$, and a, b, and c are the only points in J. Therefore $k_i = 2$ or 4. This conclusion proves that $|a_i - b_i| = 2$ or 4.

Suppose $a_i = b_i = c_i$ fails for all $i \in \{1, \ldots, k\}$where $3 \leq k \leq |S|$. Through reflection and translation we may assume that $b_i > a_i$ for all i. Thus $c_i \geq a_i + 1$ for all $i \in \{1, \ldots, k\}$, and $d(a, c) \geq k \geq 3$. Step $a \xrightarrow{c} a + e_i \notin J$. Then there must be a second step from $a + e_i$ to c. However, we have that $d(a + e_i, c) \geq k - 1 \geq 2$, so any step $a + e_i \xrightarrow{c} a + e_i + e_j \notin J$, which is a contradiction. Thus there can only be two coordinates i for which $a_i = b_i = c_i$ fails.

If $\sum_{i \in S} |a_i - b_i| > 4$, then we know from the first two arguments of the proof that $|a_j - b_j| = 4$ for some $j \in S$ and $|a_k - b_k| = 2$ or 4 for some $k \neq j$, $k \in S$. Thus $d(a, b) = 6$ or 8, $d(a, c) = 3$ or 4, and Axiom 1.4 is violated. Thus $\sum i \in S|a_i - b_i| = 0$, 2, or 4. ☐

Example 4.3. *The jump system $J = \{(0, 0); (1, 1); (2, 0)\}$ is a prime jump system by Theorem 4.2.*

To develop an intuition for determining whether a particular jump system is prime, the reader is encouraged to observe some simple examples of sums of jump systems. Paying special attention to the way in which each of the summands can "tile" the resulting jump system led to a few interesting examples.

Due to the geometric intuition that can be developed in $\mathbb{Z}$ and $\mathbb{Z}^2$, it is reasonable to believe that infinitely many prime jump systems may be constructed using the following iterative methods, although we have not yet arrived at a proof.

Example 4.4. *Define the jump systems $J_1, \ldots, J_n$ where $J_i \subseteq \mathbb{Z}$ defined as follows:*

- $J_1 = \{1\}$
- $J_n + 1 = J_n \bigcup_{i=1}^{n} \{maxJ_n + i + 1\}$.

Example 4.5. *Define jump system $J_0, \ldots, J_k$, where $J_i \subseteq \mathbb{Z}$, as follows:*

- $J_0 = \{(0,0), (1,0), (-1,0), (0,1), (1,0)\}$
- $J_{2n+1} = J_{2n} + (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor) + (1,0)$
- $J_{2n} = J_{2n-1} + (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor) + (0,1)$

The following example shows that if a jump system J can be expressed as $J = J_a + J_b$, then the pair J_a, J_b is not necessarily unique. Furthermore, the example shows that the number of summands is not necessarily unique for a particular jump system.

Example 4.6. *Let $J = \{0,1,2,3\}$, $J_a = \{0,1\}$, $J_b = \{0,1,2\}$, and $J_c = \{0,2\}$. Then $J_a + J_b = J = J_a + J_c$, but the pairs of summands are distinct. Also notice that $J_a + J_c = J = J_a + J_a + J_a$, while $J_c \neq J_a + J_a$.*

5 Hyperplane Separating a Box and a Jump System

Theorem 5.1. *[Lovász] Let J be a jump system in $\mathbb{Z}^S$ and B be a box such that $(b_i - a_1) \geq 1$for all i. Then there exists a vector $v \in \{-1,0,1\}^S$ such that*

$$d(J,B) = \min_{x \in J} v^T x - \max_{b \in B} v^T b.$$

Corollary 5.2. *[Lovász] Let J be a jump system and B a box with $b_i - a_i \geq 1$ for all i. Then $J \cap B = \emptyset$ if and only if there exists $v \in V$ and corresponding ω_v such that*
$v^T x \leq \omega_v$for all $x \in J$ but $v^T b > \omega_v$for all $b \in B$.

Separating Hyperplane Algorithm Given a box B in $\mathbb{Z}^S$ such that $(b_i - a_i) > 0$for all i and a jump system $J \subseteq \mathbb{Z}^S$, the following algorithm determines whether there exists a hyperplane that separates the box from the jump system J.

1. Choose $v \in V$.

2. Find $\beta \in B$ with $v^T\beta$. For $B = \prod_{i=1}^{|S|}[a_i, b_i]$, take $\beta = (\beta_1, \ldots, \beta_{|S|})$ such that $\beta_i = a_i$ if $v_i = -1$, $\beta_i = b_i$ if $v_i = 1$, and β is any integer in $[a_i, b_i]$ if $v_i = 0$. Calculate $v^T\beta = \omega_B$.

3. Find point $j \in J$ such that $-v^T j$ is maximal over all $x \in J$. Let $\omega_J = \max\{-v^T x : x \in J\}$.

4. Compare ω_J with ω_B.

   - If $\omega_J \geq \omega_B$, then the hyperplane defined by v does not separate.
   - If $\omega_J < \omega_B$, then the hyperplane defined by v separates J and B.

5. Unless all $3^{|S|} - 1$ possible choices for v have been made, choose another $v \in V$ and go to step 2.

The worst-case running time for this algorithm is $O(3^{|S|})$ for the case where J intersets B, because the algorithm checks all possible hyperplanes. Since the hyperplanes are chosen naively, the average running time is $O(\frac{3^{|S|}}{2})$, with the best case being where a separating hyperplane is the first to be tested.

6 Jump Systems and NP-completeness

Theorem 6.1. *Let J_i be jump systems such that $J_i \subseteq \mathbb{Z}^{N_i}$, where $N_i \subseteq S$ for $i \in \{1, \ldots, n\}$. Let J be a jump system in $\mathbb{Z}^S$ such that*

$$J = J_1 + J_2 + \ldots + J_n. \tag{1}$$

Then deciding whether a point

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{|S|}) \in J \tag{2}$$

is an NP-complete problem.

Proof. The membership problem is in NP, because given $\alpha = \beta_1 + \beta_2 + \cdots + \beta_n$ where $\beta_i \in J_i$, one can verify in polynomial time whether each β_i is actually in J_i and also whether their sum equals to alpha. To show that it is NP-hard, we will reduce the perfect matching problem to it. Let $G = (V, E)$ be a graph. Then the set of all degree sequences of all subgraphs $H \subseteq G$ forms a jump system. Express this jump system as a sum of $|E|$ $|V|$-dimensional jump systems, each containing two elements. Each of these jump systems corresponds to a particular edge and consists of the $(0, \ldots, 0)$-element and a $(0, 1)$-vector x. The point $x = (x_1, \ldots, x_{|V|})$ where $x_i = 1$ for i corresponding to the vertices adjacent to the relevant edge and $x_i = 0$ for i corresponding to vertices not adjacent to the edge.

Example 6.2.

$$J_1 = \Big\{(0,0,0,0); (1,1,0,0)\Big\} \quad J_2 = \{(0,0,0,0); (0,1,1,0)\}$$

$$J_3 = \{(0,0,0,0);(0,1,0,1)\} \quad J_4 = \{(0,0,0,0);(0,0,1,1)\}$$

$$J = \sum_{i=1}^{4} J_i$$

If $(1,\ldots,1) \in J$, then the graph G has a perfect matching. If $(1,\ldots,1) \notin J$, then J has no perfect matching.

Therefore the perfect matching problem has been reduced to the membership problem, showing that the α-inclusion problem is NP-complete for a jump system J such that J is a sum of jump systems.

□

7 Convex Hulls

Definition 7.1. *Let J be a jump system in $\mathbb{Z}^S$. Then the* convex hull of J, *denoted $Co(J)$, is the set*

$$Co(J) = \bigcup_{v \in \mathbb{R}^S} \{x : x \in \mathfrak{f}_v\}$$

A point $z \in J$ is said to be on the convex hull of J *if $z \in Co(J)$.*

Definition 7.2. *Let J be a jump system in $\mathbb{Z}^S$ and the set V be as in definition 1.11. Then the* orthogonal convex hull of J, *denoted $OCo(J)$, is the set $\bigcup_{v \in V} \mathfrak{f}_v$. A point $z \in J$ is said to be* on the orthogonal convex hull of J *if $z \in OCo(J)$.*

Definition 7.3. *A point $z \in J$ is an* interior point of J *if $z \notin Co(J)$.*

Theorem 7.4. *Let J be a jump system in $\mathbb{Z}^S$. Then $OCo(J) = Co(J)$.*

Proof. Let $x \in OCo(J)$. Then $x \in \mathfrak{f}_v \subseteq Co(J)$ because $v \in V \subseteq \mathbb{R}^S$. Thus we have $OCo(J) \subseteq Co(J)$.

To prove that $Co(J) \subseteq OCo(J)$, we will show that if $x \in Co(J)$, then $x \in \mathfrak{f}_v$ for some $v \in V$, and therefore $x \in OCo(J)$.

Let $x \in \mathfrak{f}_w$ for some $w \in \mathbb{R}^S$. It will be shown that $x \in \mathfrak{f}_v$, where $v_i = \frac{|w_i|}{w_i}$ for all i. Through reflection and coordinate-swapping we may assume that $w_i \geq 0$ for all i and that $w_1 \geq w_2 \geq \ldots \geq w_{|S|}$. Define the set $M = \{i \text{ such that } v_i = 1\}$, where $|M| = m$. Therefore $v = (1,\ldots,1,0,\ldots,0)$ where the first m coordinates are 1, and the remaing $|S| - m$ coordinates are 0.

Assume $x \notin \mathfrak{f}_v$. Then choose $y \in \mathfrak{f}_v$ such that $d(x,y)$ is minimal. Since $v^T y > v^T x$, and $w^T x \geq w^T y$, there must exist $i \in M$ for which $y_i > x_i$ and $j \in M$ for which $x_j > y_j$. If no such i existed, then we would have $v^T x \geq v^T y$, contradicting the assumption that $x \notin \mathfrak{f}_v$. Since i exists, if no such j existed, then we would have $w^T y > w^T x$, contradicting $x \in \mathfrak{f}_w$.

Consider the step $y \xrightarrow{x} y + e_i$. Because $v^T(y + e_i) > v^T y$, we know that $y + e_i \notin J$. Therefore Axiom 1.4 states that there exists a second step $y + e_i \xrightarrow{x}$

$y + e_i \pm e_j \in J$. If $j \notin M$, then $v^T(y + e_i \pm e_j) = v^T(y + e_i) > v^T y$, and then $y+e_i \pm e_j \notin J$. For $j \in M$ we must consider $y+e_i+e_j$ and $y+e_i-e_j$ separately, although both will lead to contradictions.

The point $y+e_i+e_j \notin J$, because $v^T(y+e_i+e_j) > v^T(y+e_i) > v^T y$. Also, $v^T(y + e_i - e_j) = v^T y$, so $y + e_i - e_j \in \mathfrak{f}_v$. However, $d(y + e_i - e_j, x) < d(y, x)$, violating the minimal choice of y.

Therefore $x \in \mathfrak{f}_v$, and $Co(J) \subseteq OCo(J)$, completing the equality. □

Corollary 7.5. *Let J be a jump system with $J \subseteq \mathbb{Z}^2$. Then all faces $\mathfrak{f}_v \in Co(J)$ are contained in lines with slope $m \in \{-1, 0, 1, \infty\}$.*

Proof. This corollary is an immediate consequence of Theorem 7.4 □

Theorem 7.6. *The following algorithm determines the orthogonal convex hull (and hence the convex hull) of a jump system $J \subseteq \mathbb{Z}^2$. The worst-case running time is $O(n)$, while the best-case running time is approximately $O(\sqrt{n})$. Here $n = |J|$.*

Proof. The algorithm is merely an iterative search with each query determined exactly by Axiom 1.4. The worst-case running time is seen when all points of the jump system are contained in the convex hull. The best-case running time is achieved when the values $|a_1 - b_1|$ and $|c_2 - d_2|$ for $a, b, c, d \in J$ are maximized with respect to $|J| = n$.

□

1. Determine $x_1 = max\{y_1 : y \in J\}$. Select a point $x_0 = (x_1, x_2) \in J$. Let $x = x_0$.
2. Check if $x' = (x_1, x_2 + 1) \in J$.
    - $x' \in J \Rightarrow$ Set $x = x'$. Repeat Step 2
    - $x' \notin J \Rightarrow$ Check if $x'' = (x_1, x_2 + 2) \in J$.
        - $x'' \in J \Rightarrow$ Set $x = x''$. Repeat Step 2.
        - $x'' \notin J \Rightarrow$ Go to Step 3.
3. Check if $x''' = (x_1 - 1, x_2 + 1) \in J$.
    - $x''' \in J \Rightarrow$ Set $x = x'''$. Repeat Step 3.
    - $x''' \notin J \Rightarrow$ Go to Step 4.
4. Check if $x' = (x_1 - 1, x_2) \in J$.
    - $x' \in J \Rightarrow$ Set $x = x'$. Repeat Step 4
    - $x' \notin J \Rightarrow$ Check if $x'' = (x_1 - 2, x_2) \in J$.
        - $x'' \in J \Rightarrow$ Set $x = x''$. Repeat Step 4.
        - $x'' \notin J \Rightarrow$ Go to Step 5.
5. Check if $x''' = (x_1 - 1, x_2 - 1) \in J$.

- $x''' \in J \Rightarrow$ Set $x = x'''$. Repeat Step 5.
- $x''' \notin J \Rightarrow$ Go to Step 6.

6. Check if $x' = (x_1, x_2 - 1) \in J$.
   - $x' \in J \Rightarrow$ Set $x = x'$. Repeat Step 6
   - $x' \notin J \Rightarrow$ Check if $x'' = (x_1, x_2 - 2) \in J$.
     - $x'' \in J \Rightarrow$ Set $x = x''$. Repeat Step 6.
     - $x'' \notin J \Rightarrow$ Go to Step 7.
7. Check if $x''' = (x_1 + 1, x_2 - 1) \in J$.
   - $x''' \in J \Rightarrow$ Set $x = x'''$. Repeat Step 7.
   - $x''' \notin J \Rightarrow$ Go to Step 8.
8. Check if $x' = (x_1 + 1, x_2) \in J$.
   - $x' \in J \Rightarrow$ Set $x = x'$. Repeat Step 8
   - $x' \notin J \Rightarrow$ Check if $x'' = (x_1 + 2, x_2) \in J$.
     - $x'' \in J \Rightarrow$ Set $x = x''$. Repeat Step 2.
     - $x'' \notin J \Rightarrow$ Go to Step 9.
9. Check if $x''' = (x_1 + 1, x_2 + 1) \in J$.
   - $x''' \in J \Rightarrow$ Set $x = x'''$. Repeat Step 9.
   - $x''' \notin J \Rightarrow$ Check if $x = x_0$.
     - $x = x_0 \Rightarrow$ Exit: Algorithm completed.
     - $x \neq x_0 \Rightarrow$ Set $x = x'$. Go to Step 10.
10. Check if $x' = (x_1, x_2 + 1) \in J$.
    - $x' \in J \Rightarrow$ Check if $x' = x_0$.
      - $x' = x_0 \Rightarrow$ Exit: Algorithm completed.
      - $x' \neq x_0 \Rightarrow$ Set $x = x'$. Repeat Step 10.
    - $x' \notin J \Rightarrow$ Check if $x'' = (x_1, x_2 + 2) \in J$.
      - $x'' = x_0 \Rightarrow$ Exit: Algorithm completed.
      - $x'' \neq x_0 \Rightarrow$ Set $x = x''$. Repeat Step 10.

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