A brief Incursion into Knot Theory

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Mathematics Department

Major Seminar, Fall 2008

Outline

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A Fundamental Problem

Given two objects, how to tell them apart?

- When are two surfaces homeomorphic?

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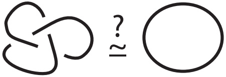
Given two objects, how to tell them apart?

- When are two surfaces homeomorphic?

- When are two knots equivalent?



Injectivity via Geometric and Topological Methods

- Foliation Theory

A Fundamental Problem
Main Research Area

Injectivity via Geometric and Topological Methods

- Foliation Theory
- Spectral Theory

Injectivity via Geometric and Topological Methods

- Foliation Theory
- Spectral Theory
- Variational Calculus

Injectivity via Geometric and Topological Methods

- Foliation Theory
- Spectral Theory
- Variational Calculus
- Global Embedding of Submanifolds

Sample Topics

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Project Problem (Foliation Theory)

Discuss Fibrations versus Foliations, e.g., characterize the foliations of the Euclidean Plane, that is, $\mathbb{R}^2$.

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Project Problem (Spectral Theory)

Discuss the eigenvalues of the Laplacian on a bounded region of $\mathbb{R}^n$, *e.g., Rayleigh and Min-Max Methods.*

Sample Topics

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Project Problem (Variational Calculus)

Discuss any result in the Geometric Analysis report, e.g., understand the MPT and its proof.

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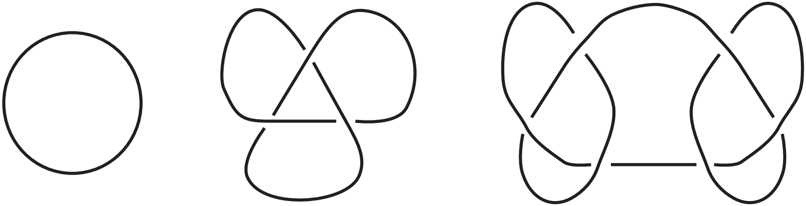
Project Problem (Geometry)

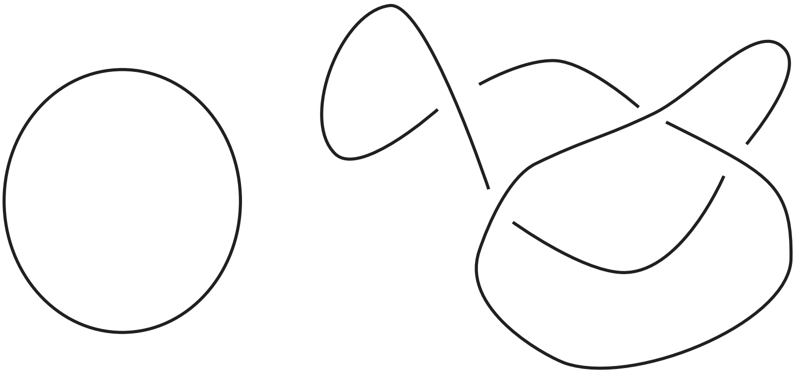
Learn about Classification of Surfaces via the Euler Characteristic and/or understand the Gauss-Bonnet formula

$$\int_M K \, dA = 2\pi\chi(M)$$

Are the knots below the same?



How about these?



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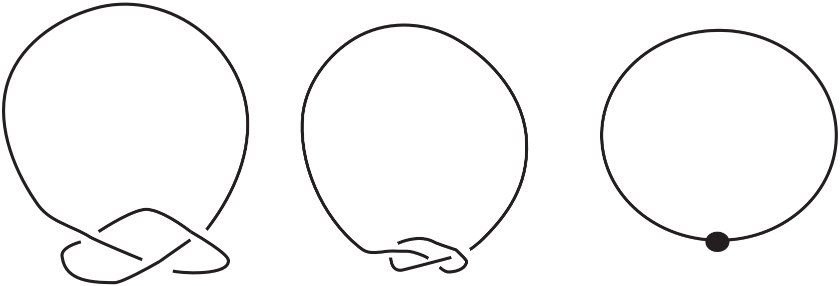
  There is a homeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that

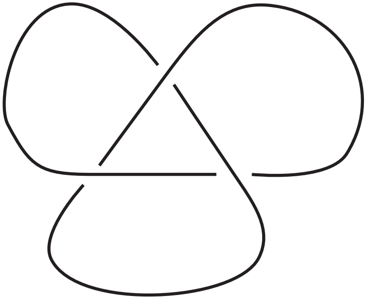
$$\varphi(K_1) = K_2$$

The map φ itself must be "nice" $\longleftrightarrow$ **Isotopic to the Identity**

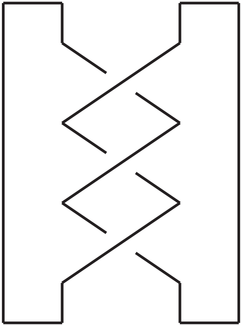
- No situations such as:



Two Knots with **similar** Diagram must be the same.



Trefoil



Braided Trefoil

Knot Theory - Diagrams

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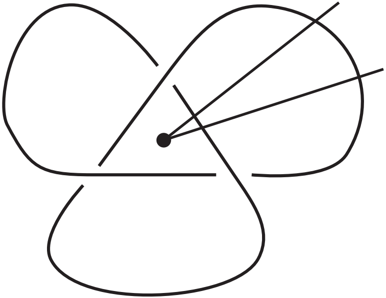

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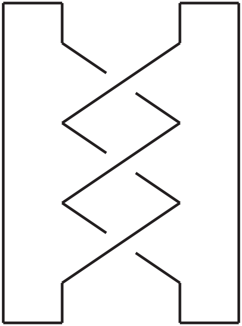

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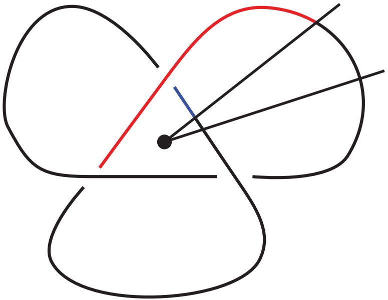

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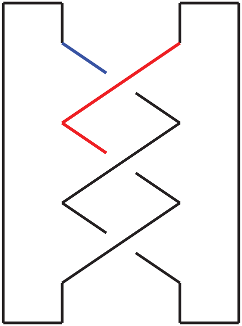

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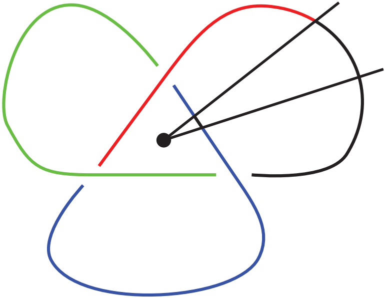

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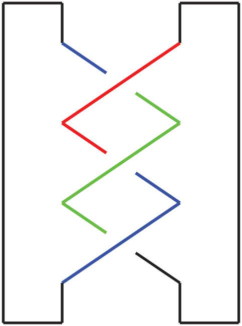

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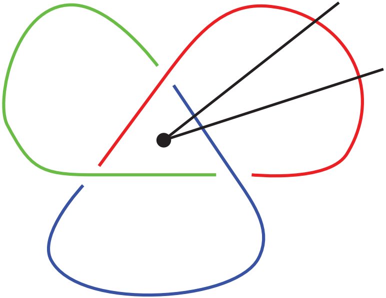

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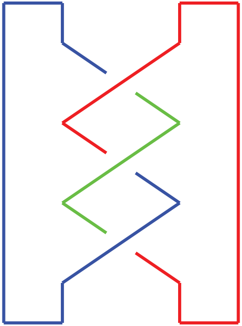
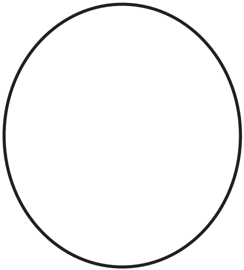
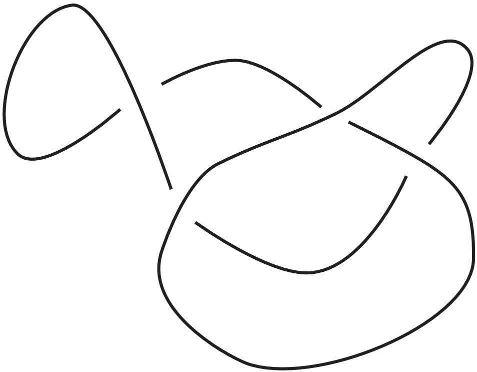


Braided Trefoil

Even if not exactly the diagram same, they could be the same...





Reidemeister Moves

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Type I

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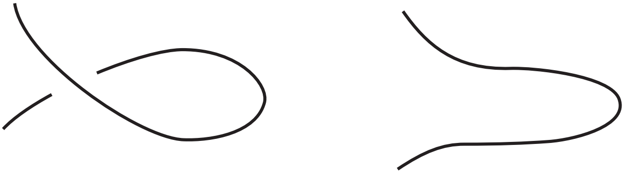

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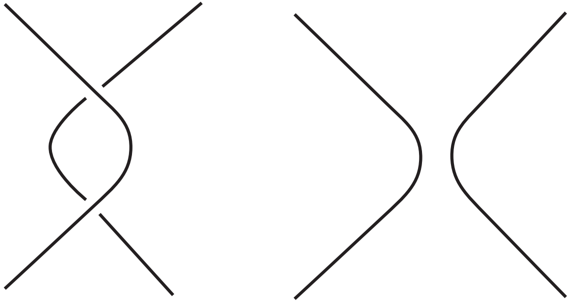
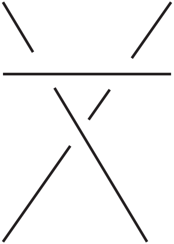
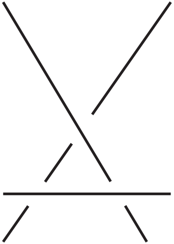
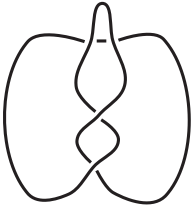


Type III

Slide a strand across a crossing

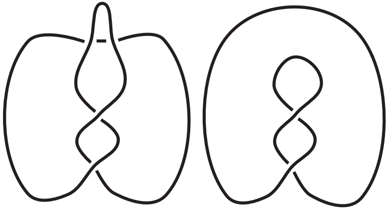






Unknotting a Knot

Using Reidemeister moves



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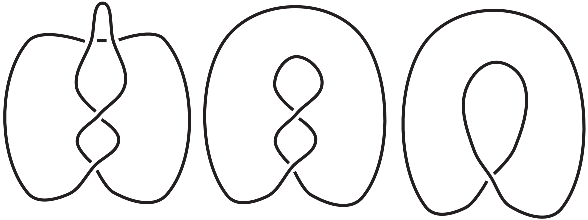

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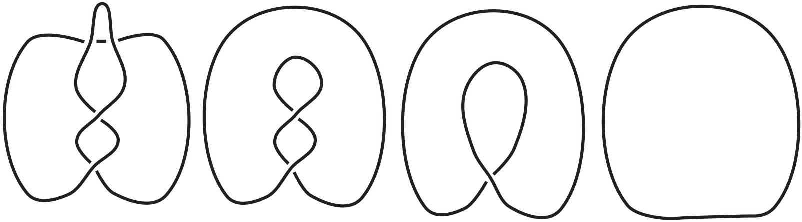

Theorem (Reidemeister, 1932)

Two Knot diagrams of the same knot can be deformed into each other by a finite number of moves of Type I, II, and III.

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Project Problem

Discuss the result above, its proof, and its significance in other areas. For instance, Physics - the structure of the atom and Chemistry - the structure of the DNA replication.

Invariants

- Colorability

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- Polynomial Invariants

Sample Problems in Knot Theory

Project Problem (Colorability)

Understand the definition and application as well as generalizations for mod p *colorability. This project has a Topological and Number Theoretical flavor.*

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Project Problem (Unknotting number)

Discuss classical result and classification of knots via crossing and unknotting numbers, e.g., minimal diagrams and prime knots.

More Sample Problems in Knot Theory

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Project Problem (Genus of a knot)

Learn surface theory in order to understand Seifert surfaces. Show that the genus of a knot, $\frac{-\chi(M)+1}{2}$ *is a knot invariant.*

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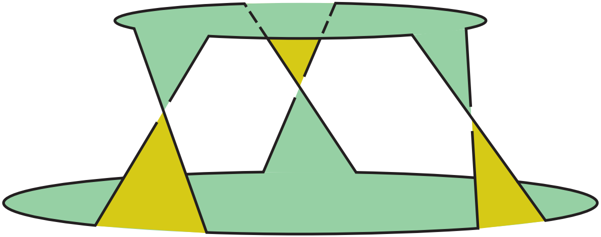

Even More Sample Problems in Knot Theory

Project Problem (Knot Group)

Learn to compute the knot group and discuss group presentations and possibly learn about representation theory.

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Learn to compute the knot group and discuss group presentations and possibly learn about representation theory.

$$\pi_1(\mathbb{R}^3 - T) = \langle a, b | \; aba = bab \rangle = \langle x, y | \; x^2 = y^3 \rangle$$

Can we make sense of the expressions above?

One More Sample Problem in Knot Theory

Project Problem (Polynomial Invariants)

Understand and learn to compute a polynomial invariant of a knot, e.g., Bracket, Kauffman, Jones, Homfly, and Alexander Polynomials.

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- All rules come from the topology of the knots (i.e., crossing types)
- Some are formal Algebraic Computations.
- Others are Geometric in nature.

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- At least two of the colors are used
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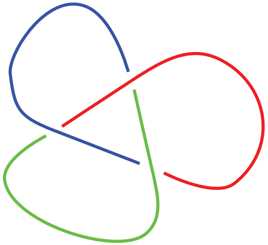

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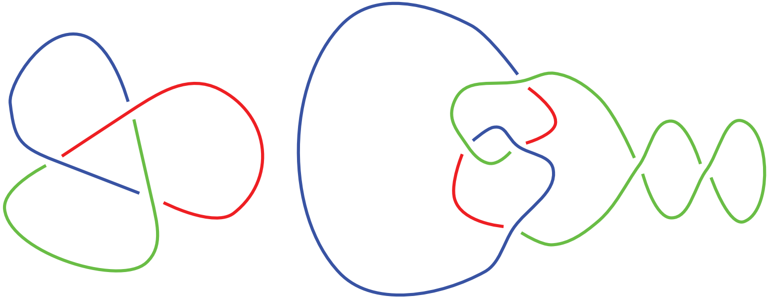
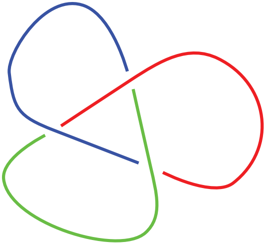
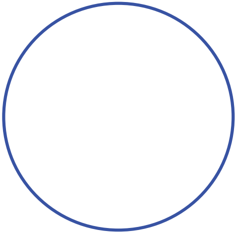


Colorability

The Unknot and Trefoil are different!



Tricolorable



Not tricolorable

An Algebraic Invariant

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- Consider $\varphi\big|_{\mathbb{R}^3 - K_1} : \mathbb{R}^3 - K_1 \to \mathbb{R}^3 - K_2$

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- Same **Fundamental Group**

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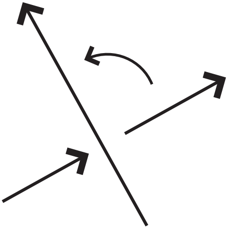
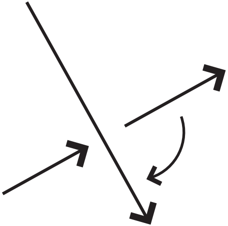
- MATH 4365 - Topology (New Course! Fall 09)
- Some Algebra background would be very nice...



+ Crossing



− Crossing

Computing the Knot group

- Label each arc with a letter.

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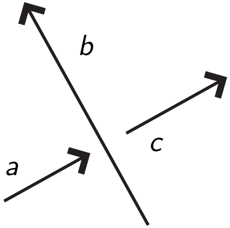
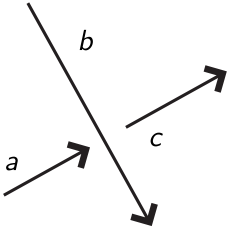
- At a crossing: the emerging arc labeled by the conjugate as follows:

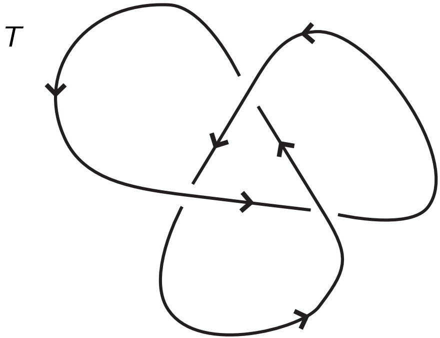


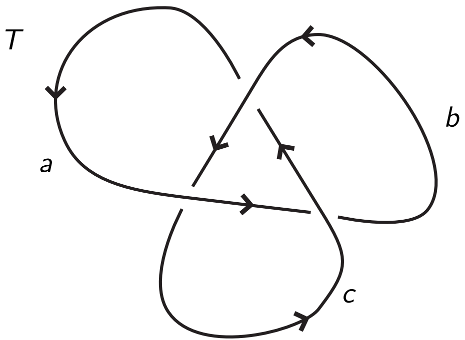
$c = bab^{-1}$



$c = b^{-1}ab$

Computing the Knot group
The Trefoil Group




T
a
b
c

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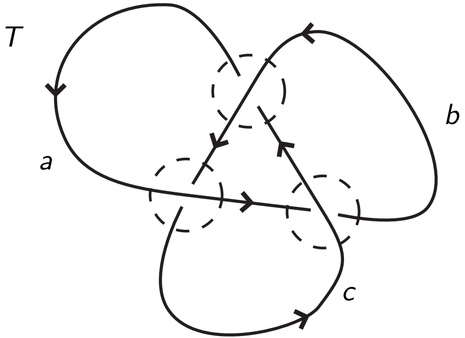
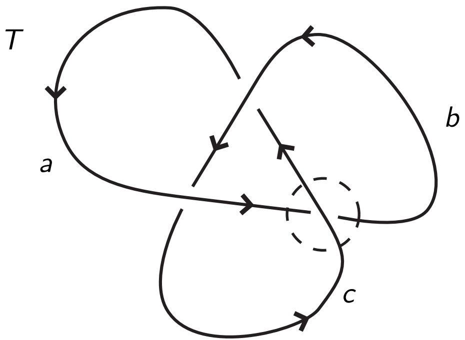
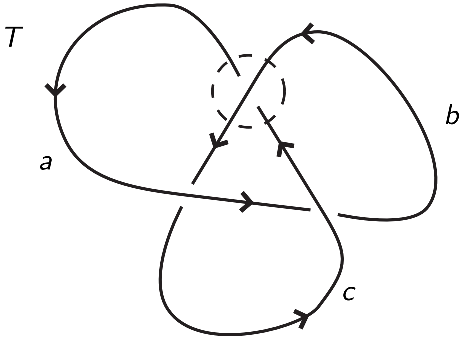




$b = cac^{-1}$



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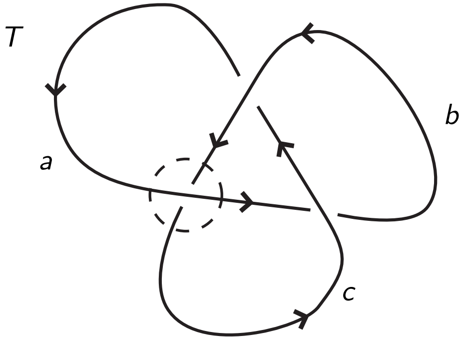

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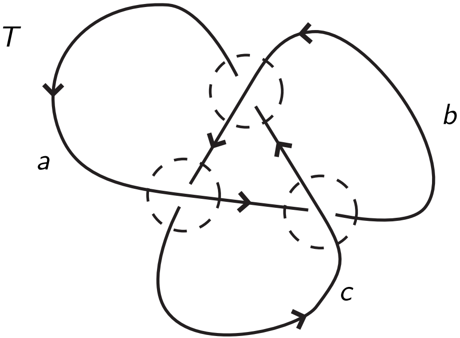

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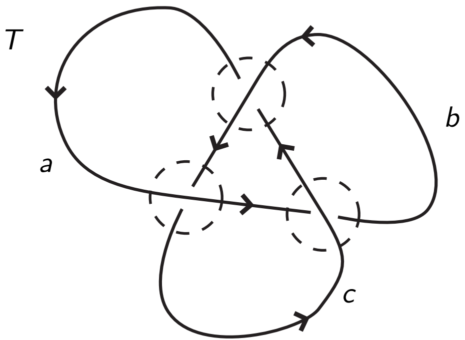

$b = cac^{-1}$

$a = bcb^{-1}$

$c = aba^{-1}$

Thus, $b = aba^{-1}aab^{-1}a^{-1}$ or $bab = aba$

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- $\pi_1(\mathbb{R}^3 - T) = \langle a, b \mid aba = bab \rangle$
- **Fact:** $\pi_1(\mathbb{R}^3 - T)$ is **not** Abelian
- **Fact:** K is the unknot, $\pi_1(\mathbb{R}^3 - K) = \mathbb{Z}$

Computing the Knot group
The Trefoil Group

The computations above show:

- $\pi_1(\mathbb{R}^3 - T) = \langle a, b |\ aba = bab \rangle$
- **Fact:** $\pi_1(\mathbb{R}^3 - T)$ is **not** Abelian
- **Fact:** K is the unknot, $\pi_1(\mathbb{R}^3 - K) = \mathbb{Z}$
    - Unknot $\neq$ Trefoil!