A brief Incursion into Knot Theory

Eduardo Balreira

Trinity University
Mathematics Department

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Outline

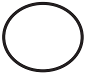
A Fundamental Problem

Given two objects, how to tell them apart?

- When are two surfaces homeomorphic?

 $\overset{?}{\simeq}$ 

- When are two knots equivalent?

 $\overset{?}{\simeq}$ 

A Fundamental Problem
Main Research Area

Injectivity via Geometric and Topological Methods

- Foliation Theory
- Spectral Theory
- Variational Calculus
- Global Embedding of Submanifolds

Sample Topics

Project Problem (Foliation Theory)

Discuss Fibrations versus Foliations, e.g., characterize the foliations of the Euclidean Plane, that is, $\mathbb{R}^2$.

Project Problem (Spectral Theory)

Discuss the eigenvalues of the Laplacian on a bounded region of $\mathbb{R}^n$, *e.g., Rayleigh and Min-Max Methods.*

Sample Topics

Project Problem (Variational Calculus)

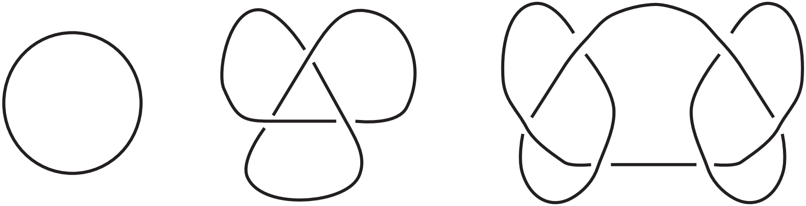
Discuss any result in the Geometric Analysis report, e.g., understand the MPT and its proof.

Project Problem (Geometry)

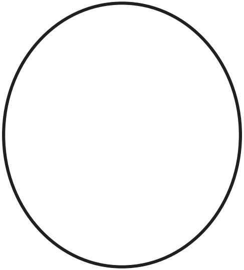
Learn about Classification of Surfaces via the Euler Characteristic and/or understand the Gauss-Bonnet formula

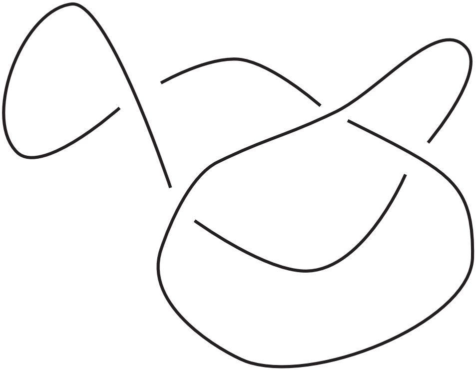
$$\int_M K \, dA = 2\pi\chi(M)$$

Are the knots below the same?



How about these?





A knot is an injective map $h : S^1 \to \mathbb{R}^3$

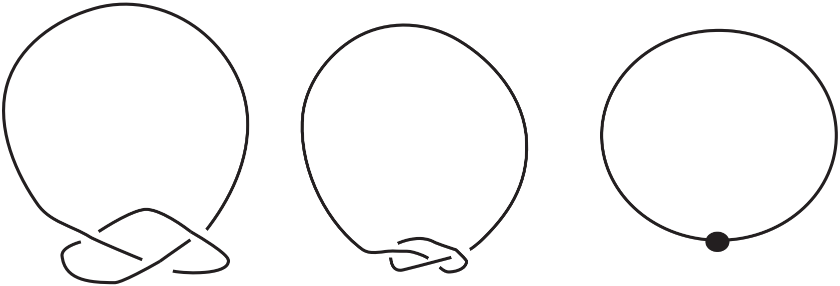
- Picture in the plane (or slide) - Diagram with crossing
- Tame Knots
  - Finite Number of arcs
  - Only two strands at a crossing
  - "nice"
- Invariant Property: $K_1 \sim K_2$ if,

  There is a homeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that

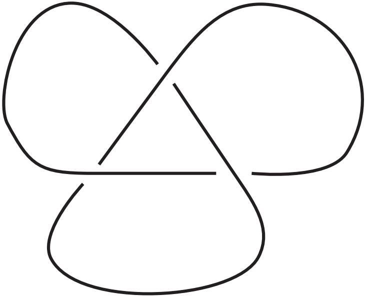
$$\varphi(K_1) = K_2$$

The map φ itself must be "nice" $\longleftrightarrow$ **Isotopic to the Identity**

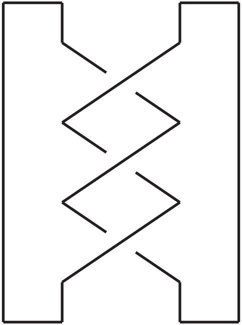
- No situations such as:



Two Knots with **similar** Diagram must be the same.



Trefoil

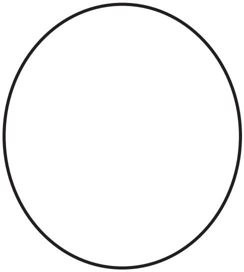


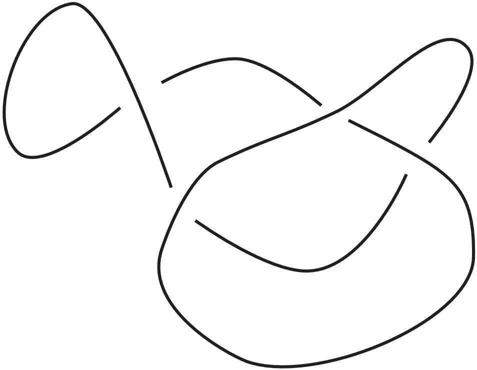
Braided Trefoil

Two Knots with **similar** Diagram must be the same.

Knot Theory - Diagrams

Even if not exactly the diagram same, they could be the same...



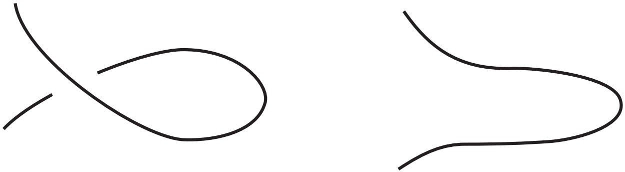


Reidemeister Moves

- Type I: Put or Take out a kink.
- Type II: Slide a strand over/under to creat/remove two crossings.
- Type III: Slide a strand across a crossing.

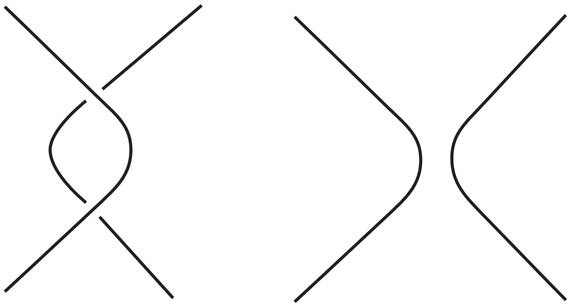
Type I

Put or Take out a kink.



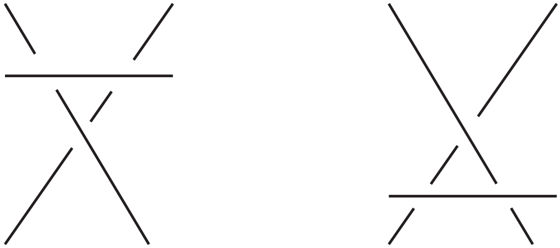
Type II

Slide a strand over/under to creat/remove two crossings.

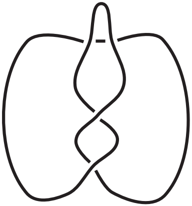


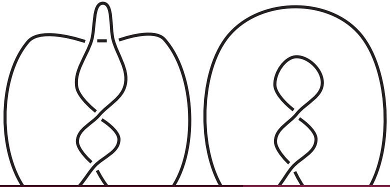
Type III

Slide a strand across a crossing



Unknotting a Knot
Using Reidemeister moves





Theorem (Reidemeister, 1932)

Two Knot diagrams of the same knot can be deformed into each other by a finite number of moves of Type I, II, and III.

Project Problem

Discuss the result above, its proof, and its significance in other areas. For instance, Physics - the structure of the atom and Chemistry - the structure of the DNA replication.

Invariants

- Colorability
- Unknotting number
- Genus of a knot
- Knot Group
- Polynomial Invariants

Sample Problems in Knot Theory

Project Problem (Colorability)

Understand the definition and application as well as generalizations for mod p *colorability. This project has a Topological and Number Theoretical flavor.*

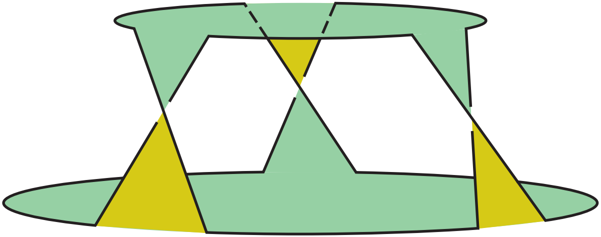
Project Problem (Unknotting number)

Discuss classical result and classification of knots via crossing and unknotting numbers, e.g., minimal diagrams and prime knots.

More Sample Problems in Knot Theory

Project Problem (Genus of a knot)

Learn surface theory in order to understand Seifert surfaces. Show that the genus of a knot, $\dfrac{-\chi(M)+1}{2}$ *is a knot invariant.*



Even More Sample Problems in Knot Theory

Project Problem (Knot Group)

Learn to compute the knot group and discuss group presentations and possibly learn about representation theory.

$$\pi_1(\mathbb{R}^3 - T) = \langle a, b |\ aba = bab \rangle = \langle x, y |\ x^2 = y^3 \rangle$$

Can we make sense of the expressions above?

One More Sample Problem in Knot Theory

Project Problem (Polynomial Invariants)

Understand and learn to compute a polynomial invariant of a knot, e.g., Bracket, Kauffman, Jones, Homfly, and Alexander Polynomials.

- All rules come from the topology of the knots (i.e., crossing types)
- Some are formal Algebraic Computations.
- Others are Geometric in nature.

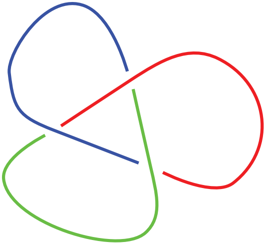
Invariants
Colorability

K is colorable $\leftrightarrow$ each arc has one of 3 colors

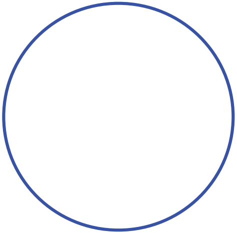
- At least two of the colors are used
- At crossing, either all different or all the same color.

Colorability

The Unknot and Trefoil are different!



Tricolorable



Not tricolorable

An Algebraic Invariant

Topologically, every knot is equivalent to S^1. DONE?

- Knots are different via their embedding in $\mathbb{R}^3$.
- Better question: $K_1 \sim K_2$ if $\exists\, \varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\varphi(K_1) = K_2$$

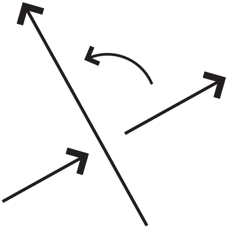
- Consider $\varphi\big|_{\mathbb{R}^3 - K_1} : \mathbb{R}^3 - K_1 \to \mathbb{R}^3 - K_2$
- Thus

$$\pi_1(\mathbb{R}^3 - K_1) \simeq \pi_1(\mathbb{R}^3 - K_2)$$

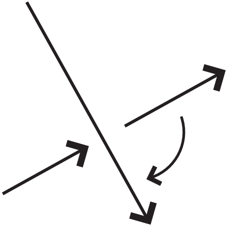
- Same **Fundamental Group**

The Knot Group
An Algebraic Invariant

- $\pi_1(M)$ is the Fundamental Group of M
- For a knot K, $\pi_1(\mathbb{R}^3 - K_1)$ is the **Knot Group** of K
- MATH 4365 - Topology (New Course! Fall 09)
- Some Algebra background would be very nice...



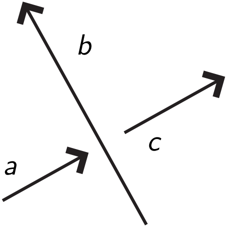
$+$ Crossing



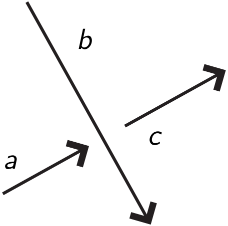
$-$ Crossing

Computing the Knot group

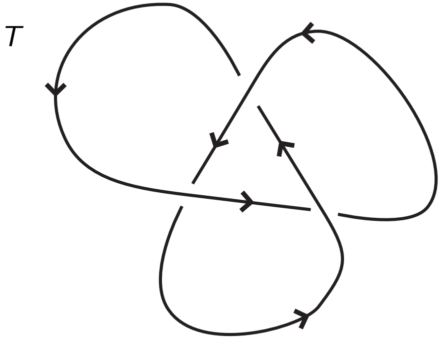
- Label each arc with a letter.
- At a crossing: the emerging arc labeled by the conjugate as follows:



$c = bab^{-1}$



$c = b^{-1}ab$


T


T
b

The computations above show:

- $\pi_1(\mathbb{R}^3 - T) = \langle a, b | \ aba = bab \rangle$
- **Fact:** $\pi_1(\mathbb{R}^3 - T)$ is **not** Abelian
- **Fact:** K is the unknot, $\pi_1(\mathbb{R}^3 - K) = \mathbb{Z}$

  - Unknot $\neq$ Trefoil!