GEOMETRIC ANALYSIS AND THE MOUNTAIN PASS THEOREM

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Abstract. These notes are designed to serve as a template of a LaTeX article. In the process we will describe some notions of Geometric Analysis pertaining to the Mountain Pass Theorem. Little attempt was made to be a publishable set of notes, but instead to provide examples of commonly used commands, environments, and symbols in LaTeX.

Introduction

The subject of Geometric Analysis is motivated by viewing analytical problems via an understanding the Geometric properties of the functionals associated with these problems. As an illustration of this type of analysis consider the solution for the following linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{0.1}$$

For instance, each individual equation in (0.1) can be viewed geometrically as an affine hyperplane, thus the analytical problem of finding solutions is now shifted to a geometric problem of intersections of hypersurfaces.

As an illustration, consider the following:

$$2x_1 + 3x_2 + 4x_3 = 0 \tag{0.2}$$

$$4x_1 + 6x_2 + 8x_3 = 1 \tag{0.3}$$

Then, subtracting (0.3) from (0.2) we have

$$2x_1 + 3x_2 + 4x_3 = 1$$

Which contradicts the initial set of equations, hence there cannot be a solution. Alternatively, each equation (0.2) and (0.3) gives an affine hyperplane with normal $\langle 2, 3, 4\rangle$, therefore they are parallel. Since the distance to the origin (given by b_i) is different, they cannot intersect, hence a solution does not exist.

This simple idea of viewing the interplay between Analysis and Geometry is a very influential and classical area of Mathematics, some famous problems are as follows.

1. Variational Calculus
   "Find a solution of $f(x) = 0$ by finding Critical Points of the antiderivative $F(x)$"
2. Yamabe Problem - Uniformization Theorem for Surfaces (M^2, g)
   "Can we find a metric conformal to g with constant scalar curvature?" That is, solve
   $$\Delta u - K(x) - e^{2u} = 0.$$
3. Minimal Surface Theory and Bubbles " Given a simple wire frame in $\mathbb{R}^3$, can you find a smooth surface with the given frame as its boundary with least area?" That is, minimize the *energy* associated to the frame and show that the minimizer corresponds to a (classical) smooth surface.

In these notes we will focus on ideas from Variational Calculus, in particular, Critical Point Theory and discuss the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [1]. The approach here follows the presentation in the book of Jabri [3].

1. Geometric Analysis

Variational and topological methods have proved to be powerful tools in the solution of concrete nonlinear boundary value problems appearing in many disciplines where classical methods have failed. The ideas to be presented here use the analysis inspired in the geometry of a mountain pass.

The abstract process in modern critical point theory has its roots in the Dirichlet principle which he postulated at Göttingen. Given an open bounded set Ω in the plane and a continuous function $h : \partial\Omega \to \mathbb{R}$, the boundary value problem

$$\begin{cases} -\Delta u &= 0 \quad , \text{ in } \Omega \\ u &= h \quad , \text{ in } \partial\Omega \end{cases} \tag{1.1}$$

admits a *smooth* solution u that minimizes the functional

$$\Phi(u) = \int_\Omega \sum_{i=1}^{2} (D_i h(x))^2 \, dx \tag{1.2}$$

in the set of smooth functions defined on Ω that are equal to h on $\partial\Omega$. The *Euler equation* corresponding to (1.2) is the equation (1.1). Euler established the principle above by showing that any smooth minimizer of (1.2), such that $u = h$ on $\partial\Omega$, is a solution of (1.1).

Weierstrass pointed out in 1870 that the existence of the minimum is not assured in spite of the fact that the functional Φ may be bounded from below. The subtle difference between minimum and *infimum*, not yet perceived in those these early times, was made. He proved that the functional

$$\Psi(u) = \int_{-1}^{1} (xu'(x))^2 dx \tag{1.3}$$

possesses an infimum but does not admit any minimum in the set

$$\mathfrak{C} = \left\{u \in C^1[-1,1]; u(-1) = 0, u(1) = 1\right\}.$$

Indeed, if we consider the sequence

$$u_n = \frac{1}{2} + \frac{\arctan(\frac{x}{n})}{2\arctan(\frac{1}{n})},$$

then, $u_n \in \mathfrak{C}$ and $\Psi(u_n) \to 0$. If some u was a minimum, then $xu'(x) = 0$ on $[-1,1]$. Therefore, $u =$ constant, in contradiction with $u(-1) = 0$ and $u(1) = 1$.

Major contributions looking for absolute minimizers of functionals bounded from below were also made by pioneers like Lagrange, Legendre, Jacobi, Hamilton, Poincaré, etc. This was revisited by Birkhoff in 1917 who succeeded to obtain a *minimax principle* were critical points u are such that $\Phi(u) = \inf_{A\in\mathcal{A}} \sup_{x\in A} \Phi(x)$ and $\mathcal{A}$ is a family of particular sets. The remaining ingredient for the modern theory of minimax theorems was a notion of *compactness* introduced in the 1960s by Palais, Smale, and Rothe as we will see in the sequel.

2. First Steps Towards the Mountains

We begin by defining what is meant by the Palais-Smale condition, denoted by (PS) which is the analogue for compactness in Variational Calculus.

Definition 2.1. Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional. We say that Φ satisfies the *Palais-Smale condition*, denoted by (PS), if any sequence (u_n) in X such that

$$\Phi(u_n) \text{ is bounded and } \Phi'(u_n) \to 0, \tag{PS}$$

admits a convergent subsequence.

Any sequence satisfying (PS) is called a Palais-Smale sequence.

Definition 2.2. Let X and Φ be as in the former definition, and $c \in \mathbb{R}$. The functional Φ is said to satisfy the (local) Palais-Smale condition at the level c, denoted by $(\text{PS})_c$, if any sequence (u_n) in X such that

$$\Phi(u_n) \to c \tag{2.1}$$

$$\Phi'(u_n) \to 0, \tag{2.2}$$

admits a convergent subsequence.

Remark 2.3. The condition $(\text{PS})_c$ is a compactness condition on the functional Φ, in the sense that the set $\mathbb{K}_c$ of critical points of Φ at the level c,

$$\mathbb{K}_c = \{u \in X; \Phi(u) = c, \Phi'(u) = 0\},$$

is compact.

We should point out the following classical results about the size of critical values for smooth maps.

Theorem 2.1 (Morse Theorem). *If $\Phi \to \mathbb{R}^N$ is of class C^N on the open set U of $\mathbb{R}^N$, then the set of critical values of Φ has measure zero.*

Theorem 2.2 (Sard Theorem). *If $\Phi : U \subset \mathbb{R}^N \to \mathbb{R}^M$ is of class C^r on the open set U of $\mathbb{R}^N$, the the set of critical values of Φ has measure zero provided $r \geq N - M + 1$.*

2.1. **The Finite Dimensional Mountain Pass Theorem.** Ideas for the Mountain Pass Theorem (MTP) can be found as early as Calculus I, for instance, consider the following well-know resutl.

Theorem 2.3 (Rolle). *Let $f \in C^1([x_1, x_2], \mathbb{R})$. If $f(x_1) = f(x_2)$, then there exists $x_3 \in (x_1, x_2)$ such that $f'(x_3) = 0$*

This type of results, when generalized to higher dimensions, provide the landscape in Fig. 2.1.

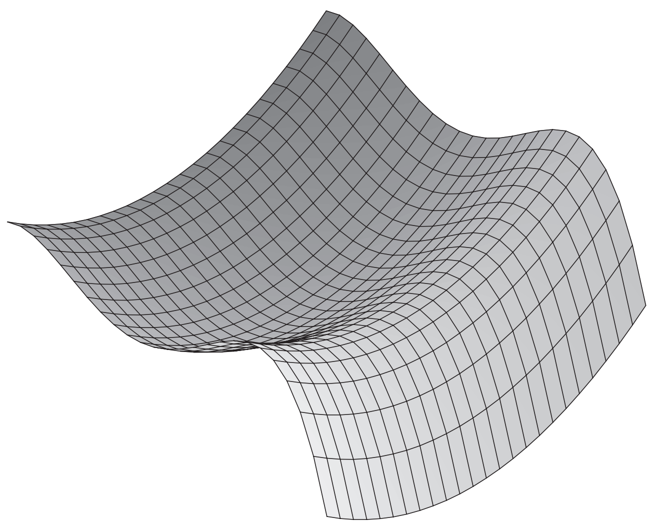

Figure 2.1. The MPT landscape.

Theorem 2.4 (Finite Dimensional MPT, Courant). *Suppose that $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ is coercive and possesses two distinct strict relative minima x_1 and x_2. Then φ possesses a third critical point x_3 distinct from x_1 and x_2, characterized by*

$$\varphi(x_3) = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} \varphi(x)$$

where $\Gamma = \{\Sigma \subset \mathbb{R}^n; \Sigma$ is compact and connected and $x_1, x_2 \in \Sigma\}$.
Moreover, x_3 is not a relative minimizer, that is, in every neighborhood of x_3 there exists a point x such that $\varphi(x) < \varphi(x_3)$.

For a proof, see [5].

2.2. **The Topological Mountain Pass Theorem.** It turns out that the MPT is a topological result, in the sense that no differential structure on X is needed or used. First some definitions.

Definition 2.4. A topological space is *compactly connected* if for each x_1, x_2 in X there exists a compact connected set containing both x_1 and x_2.

Definition 2.5. Let $f : X \to \mathbb{R}$ be a functional. A point$x \in X$ is called a *mountain pass point* if for every neighborhood $\mathcal{N}$ of x, the set defined by

$$\mathcal{N} \cap \{y \in X; f(y) < f(x)\}$$

is disconnected.

Definition 2.6. A functional $f : X \to \mathbb{R}$ is said to be *increasing at infinity* if for all $x \in X$, there is a compact subset $K \subset X$ such that

$$f(z) > f(x), \text{ for all } z \notin K$$

The main result is as follows.

Theorem 2.5 (Katriel, [4]). *Let X be a topological space locally connected, compactly connected, and admitting a continuous functional φ that is increasing at infinity. Let $S \subset X$ be a set that separates x_1 and x_2 and suppose that*

$$\max\{\varphi(x_1), \varphi(x_2)\} < \inf_{x \in S} \varphi(x) = p.$$

Then, there is a third point x_3 which is either a local minimum or a mountain pass point of φ with $\varphi(x_3) = c \geq p > \max\{\varphi(x_1), \varphi(x_2)\}$. Moreover, the value of c is characterized by the minimax argument

$$c = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} \varphi(x)$$

where $\Gamma = \{\Sigma \subset X; \Sigma$ is compact and connected and $x_1, x_2 \in \Sigma\}$.

the proof is, of course, topological in nature and needs the following

Lemma 2.6. *Let X be a compactly connected, locally connected, and locally compact, and let U be and open connected subset of X. Then, U is compactly connected.*

3. The Mountain Pass Theorem

The mountain pass theorem of Ambrosetti and Rabinowitz is a result of great intuitive appeal as well as practical importance in the determination of critical points of functionals, particularly those which occur in the theory of ordinary differential equations. The celebrated result in [1] is as follows.

Theorem 3.1 (Ambrosetti-Rabinowitz, [1]). *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional satisfying (PS). Suppose that there is $e \in X$, $\|e\| > r > 0$ and*

$$\alpha = \max\{\gamma(0), \gamma(e)\} < \inf_{u \in S(0,\rho)} \Phi(u) = \beta.$$

Then, Φ possess a critical value $c \geq \beta$ characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \Phi(u)$$

where

$$\Gamma = \{\gamma \in C([0,1]; X); \gamma(0) = 0, \gamma(1) = e\}.$$

4. Applications

As a very simple application of the MPT, we report a proof of a *global homeomorphism* theorem. A priori, we must assume that the map is locally invertible, but that does not ensure global invertibility. For instance, think of the real valued function arctan in the real line.

Theorem 4.1 (Hadamard)**.** *Let X and Y be finite dimensional Euclidean spaces, and let $\Phi : X \to Y$ be a C^1 function such that:*

i. *$\Phi'(x)$ is invertible for all $x \in X$,*
ii. *$\|\Phi(x)\| \to \infty$ as $\|x\| \to \infty$.*

Then Φ is a diffeomorphism of X onto Y.

Proof. By i and the inverse function theorem, Φ is an open mapping, then the range of Φ is open in Y. Using ii and the fact that a bounded closed set in a finite dimensional space is compact, we verify easily that $\Phi(X)$ is closed. Indeed, let $\Phi(x_n)$ be a convergent sequence in Y. Then it is bounded in Y and hence $\{x_n\}$ is bounded in X. Thus, it admits a convergent subsequence converging to some $\overline{x} \in X$ and $\Phi(x_n) \to \Phi(\overline{x})$.

It remains to show that Φ is injective to show it is a global diffeomorphism. By contradiction, suppose that $\Phi(x_1) = \Phi(x_2) = y$ for two points x_1 and x_2 in X, and consider the C^1 function $f : X \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2}\|\Phi(x) - y\|.$$

By ii, $f(x) \to \infty$ as $\|x\| \to \infty$. It is easy to see that x_1 and x_2 are global minima of f by the inverse function theorem $\Phi(x) \neq \Phi(x_i)$ in a neighborhood of x_i (for $i = 1, 2$), which implies that x_1 and x_2 are strict local minima.

Therefore, by the finite dimensional MPT, there exists a third critical point x_3 for f with $f(x_3) > 0$. So $\|\Phi(x_3) - y\| > 0$, and thus $\Phi(x_3 \neq x_3$. But the fact that x_3 is a critical point of f means that

$$\Phi'^{*}(x_3)\,(\Phi(x_3) - y) = 0$$

which contradicts the invertibility of $\Phi'(x_3)$ expressed in i.

□

Next, consider the application to semilinear elliptic Dirichlet problem

$$\begin{cases} -\Delta u(x) = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is supposed to be a Carathéodory function satisfying the growth condition

$$|f(x, s)| \leq a(x) + b|s|^{p-1}, \tag{4.2}$$

where $a(x) \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p \leq \frac{2N}{N-2}$, in $N \geq 3$, and $1 \leq p < \infty$ if $N = 2$.

The energy functional Φ associated to (4.1), defined by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x, u(x))dx,$$

is well-defined on $H_0^1(\Omega)$ and is of class C^1. The critical points of Φ are *weak* solutions of (4.1).

Proposition 4.2. *Suppose f satisfies:*

(1) *$f(x, s) = o(|s|)$ at $s = 0$ uniformly in $x \in \overline{\Omega}$.*
(2) *There are constants $\mu > 2$ and $r > 0$ such that for $|s| \geq r$,*

$$0 < \mu F(x, s) \leq sf(x, s).$$

Then, (4.1) *possesses a nontrivial solution.*

Proof. We show that Φ has the right geometry and then that is satisfies (PS) condition. A simple application of MPT ensures the nontrivial solution. Finally, an application of regularity theory shows that the weak solutions are strong solutions. □

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual Variational Methods in Critical Point Theory and Applications*, J. Funct. Anal. **14** (1973), 349-381.

[2] J. Hadamard, *Sur les Transformations Ponctuelles*, Bull. Soc. Math. France **34** (1906), 7184.

[3] Y. Jabri, The Mountain Pass Theorem, Variants, Generalizations and Some Applications, Cambridge University Press, 2003.

[4] G. Katriel, *Mountain Pass Theorems and Global Homeomorphism Theorems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **11**, number 2, (1994), 189-209.

[5] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant lecture notes in mathematics, no. **6**, 2001.

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