Tessellations: The Link Between Math and Art

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Contents

1 Motivation

My interests in both art and mathematics serve as the motivation for this project. Although math and art seem unrelated, there are connections between the two disciplines. Historically, math and art have been associated with one another. The Greeks are famous for using mathematic ratios in the construction of their temples such as the Parthenon [14]. During the Renaissance, mathematicians and artists worked together to accurately represent the three-dimensional world on a flat piece of paper. With the understanding of perspective, projective geometry was developed. The mathematical study of projective geometry provided tools for artistic representation of the world on a flat surface [10]. Symmetry is an important element for artists to incorporate into their artwork. Symmetry creates balance in a piece of art, and the lack of symmetry can destroy the sense of balance. The study of symmetries in geometry is a connection between math and art.

Tessellations are the specific connection between math and art that I chose to study. I was inspired by the artwork of M.C. Escher, particularly his tessellation of the hyperbolic plane *Circle Limit III.* I studied geometries, tessellations and created my own tessellations based on my research.

2 Geometry

I began my research by looking at Euclidean Geometry. The practical applications of geometry in sciences and engineering led to the study of Euclidean geometry [8, page 7]. Around the year 300 B.C., Euclid published a book entitled *Elements* in which he listed five postulates describing geometry. The modern reformulation of Euclid's postulates are:

- Any two points can be joined by a straight line.
- Any straight line segment can be extended indefinitely in a straight line.
- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- All right angles are congruent.
- **Parallel Postulate:** Given a line and a point not on that line, there exists only one parallel line to the original line going through that point.

2.1 The Question of the Fifth Postulate

It was unknown for a long time whether Euclid's Fifth Postulate was required in addition to the other four postulates to describe Euclidean Geometry. Many mathematicians did not believe this statement was needed as a single postulate. Numerous attempts were made to prove the Parallel Postulate from Euclid's other four postulates, but all were unsuccessful. During the late 17th century, Saccheri failed to prove Euclid's Fifth Postulate from the other four, and stumbled upon elements of hyperbolic geometry [8, page 17]. The establishment of the hyperbolic plane proved the necessity of the Fifth Postulate in Euclidean geometry and "freed mathematicians from the assumption that geometry was limited to Euclidean Space" [1, pages 450-451].

2.2 The Erlanger Program

Felix Klein developed a method to distinguish different types of geometries by looking at groups of transformations and their invariants [10]. Klein's method for studying geometries is called the *Erlanger Program.* I have used the Erlanger Program to define and discuss the geometries in this paper. Euclidean, hyperbolic and spherical geometries are different examples of geometries in the Erlanger Program.

A *geometry* is a pair (S, G) consisting of a nonempty set S and a *transformation group* G, where G is a collection of transformations with composition as the group operation. G is the group of transformations $T : S \to S$ such that:

1. G is closed under compositions of transformations,
2. G is associative,
3. G contains the identity,
4. transformations are invertible and their inverses are in G.

The transformations I am interested in are the *isometries* (or rigid motions) of the plane. *Isometries* are distance preserving bijections from the plane to itself [1, page 133]

3 Euclidean Geometry

The first example of geometry is on the Euclidean plane $\mathbb{R} \times \mathbb{R}$, with the Euclidean metric. The Euclidean metric is defined as the distance between two points p, q such that $p = (x_1, y_1)$ and $q = (x_2, y_2)$. The distance between p and q is given by

$$\mathrm{d}(p,q)= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A *geodesic* is the shortest path between two points. In the Euclidean plane, a geodesic is a straight line. The following theorems describe the transformation group of the Euclidean plane.

Theorem 3.1. *Every isometry of the Euclidean plane is a reflection, rotation, translation, glide reflection, or the identity map [1, page 140].*

Theorem 3.2. *An isometry of the Euclidean plane is a composition of three or fewer reflections [19, page 78].*

The isometries of the Euclidean plane can be distinguished by the number of points that are fixed. The identity map fixes all points. Reflections fix infinitely many points along the line of reflection. Rotations fix the one point about which the rotation is made. Translations and glide reflections have no fixed points.

An important transformation that is not an isometry of the Euclidean plane is a *circle inversion*. The circle inversion preserves the circle of inversion and maps the points in the interior of the circle to points on the exterior, and points on the exterior are mapped to points in the interior [15, page 22]. Circle inversions will provide a geometric interpretation of isometries in the hyperbolic plane.

4 Non-Euclidean Geometry

There are other geometries in addition to Euclidean geometry. They are called non-Euclidean geometries and consider ideas that are not necessarily true in Euclidean geometry. The examples of non-Euclidean geometries considered in this paper are hyperbolic and elliptic geometries.

4.1 Hyperbolic Geometry

Hyperbolic geometry was discovered by mathematicians in the nineteenth century while searching for a proof that the Parallel Postulate follows from the other four Euclidean axioms. In the search for a proof, properties were discovered to obey Euclid's first four postulates while violating the fifth. A reformulation of the Fifth Postulate in the hyperbolic plane is:

Theorem 4.1. *Through a given point x not on a given line $\mathcal{L}$ pass an infinite number of lines not intersecting $\mathcal{L}$ [19, page 371].*

There are several models of the hyperbolic plane. I used the Poincaré disc model as the model of the hyperbolic plane in my research. In the Poincaré disc, the entire hyperbolic plane is represented inside the unit disc of the complex plane [9, page 166]. We will call the Poincaré disc $\mathcal{D}$ and define it as follows

$$\mathcal{D} = \{z : |z| < 1\}$$

[8, page 78]. The interior points of the boundary disc are the *points* of the hyperbolic plane [5, page 23]. The boundary points of the circle are called *ideal points* or *points at infinity*. Ideal points are not contained in the hyperbolic plane.

4.1.1 Definitions and Theorems

In the disc model of the hyperbolic plane, the *distance* between two points z_1 and z_2 is given by

$$d(z_1, z_2) = \ln(z_1, z_2, q_2, q_1),$$

where z_1 and z_2 are points on the geodesic γ. The ideal endpoints of γ are q_1 and q_2 [8, page 95]. *Geodesics* of the hyperbolic plane are the Euclidean circles and lines (*clines*) forming right angles with the boundary of the unit disc. Two geodesics *intersect* if they share a common point in $\mathcal{D}$. *Parallel* geodesics meet at a point at infinity, and *ultra-parallel* geodesics never intersect in $\mathcal{D}$ or at the circle at infinity [15, page 22].

As in Euclidean geometry, hyperbolic isometries are described by the number of points they fix and can be expressed as the composition of non-Euclidean reflections [15, page 22]. The isometries of the hyperbolic plane are:

- *Circle Inversion* (non-Euclidean reflection), which has infinitely many fixed points along the arc of inversion (a geodesic in $\mathcal{D}$)
- *Hyperbolic Translation* (non-Euclidean translation), which fixes no points in the hyperbolic plane, but two on the boundary
- *Parabolic Translation* (non-Euclidean translation), which fixes no points in the hyperbolic plane, but one at the boundary
- *Elliptic Transformation* (non-Euclidean rotation), which fixes one point in the interior of the disc [12]

The reflections in $\mathcal{D}$ are given by circle inversions over geodesics. The geodesic is a Euclidean circle $\mathcal{C}$ and will be preserved. Therefore, any point of $\mathcal{D}$ on $\mathcal{C}$ is fixed. Any point of $\mathcal{D}$ in the interior of $\mathcal{C}$ will map to the exterior and any point on the exterior of $\mathcal{C}$ will map to the interior of $\mathcal{C}$ [15, page 22]. Figure 1 is an illustration of the Circle Inversion in the Poincaré disc model of the hyperbolic plane.

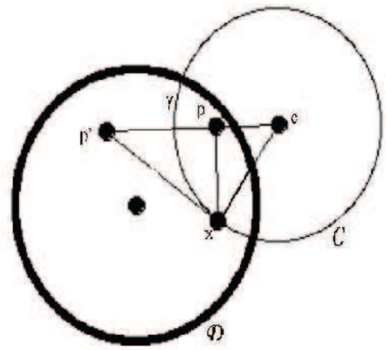

Figure 1: Circle Inversion

The hyperbolic translation is the composition of two circle inversions across ultra-parallel geodesics [12]. Through these ultra-parallel geodesics exists a unique Euclidean circle $\mathcal{C}$ that is perpendicular to both geodesics. The two points on the boundary where $\mathcal{C}$ intersects the boundary are fixed points under hyperbolic transformation. Notice these are not fixed points in $\mathcal{D}$. Under the inversions, points move along $\mathcal{C}$ and are translated in $\mathcal{D}$ [15, page 23]. Figure 2 illustrates a hyperbolic translation.

A parabolic translation is formed by two circle inversions across parallel geodesics [12]. The point of intersection of the geodesics on the boundary is b, which is fixed under a parabolic transformation. A *horocycle* is a circle that is tangent to the boundary of $\mathcal{D}$ at a single point [8, page 86]. Any horocycle that is tangent to $\mathcal{D}$ at b is preserved and the points transformed by the isometry travel along the horocycle. Therefore, the parabolic transformation acts as a translation in $\mathcal{D}$ and is shown in Figure 3 [12].

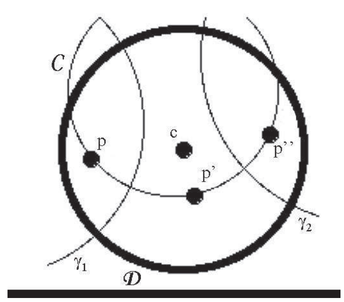

Figure 2: Hyperbolic Translation

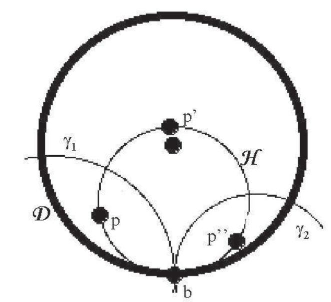

Figure 3: Parabolic Translation

An elliptic transformation is given by inversions across two geodesics that intersect in $\mathcal{D}$ and is analogous to an Euclidean rotation. Points in $\mathcal{D}$ move along hyperbolic circles centered at the point of intersection of the two geodesics [15, page 23].

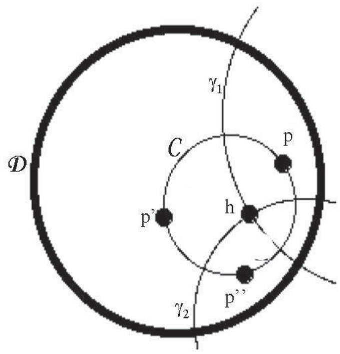

Figure 4: Elliptic Transformation

4.2 Elliptic Geometry

Another example of a non-Euclidean geometry is the geometry on the surface of the sphere. Spherical geometry is the two-dimensional case of a more general geometry called *elliptic geometry.* I will only consider the geometry on the surface of a two-dimensional sphere.

Great circles are circles on the surface of the sphere centered at the center of the sphere. Thus, great circles are the largest circles that can be drawn on the sphere. Great circles are the *lines* on the sphere and any two lines in the sphere will intersect. Thus, the Parallel Postulate fails in elliptical geometry [4, page 14].

Mathematicians had difficulties axiomatizing spherical geometry from Euclid's postulates. As problems arose with the First and Second Postulates. Consider the North and South poles on the globe. Infinitely many lines of longitude can be drawn through these two points. So, there are an infinite number of lines on the sphere that pass through two distinct points.

Klein discovered how to "rid spherical geometry of its one blemish," the lack of a unique line through two points. Klein redefined the concept of *point* in elliptic geometry to be the pair of antipodal points identified to each other [4, page 13]. We have the following theorem:

Theorem 4.2. *Two distinct points determine a unique line in elliptic geometry [8, page 118].*

In the mid-1850s, Riemann noticed the great circles on a sphere with the antipodal point identification can be extended into lines of infinite length. This was the final step needed in the axiomazation of spherical geometry [4, page 11].

4.2.1 Definitions and Theorems

On the surface of a sphere, the distance between two points p and q is given by

$$d(p, q) = r\theta$$

where r is the radius of the sphere, c is the center of the sphere, and θ is the measure of the angle $\angle pcq$ [14]. A *geodesic* in elliptic geometry is a great circle connecting a pair of antipodal points which have been identified together [4, page 13].

The isometries of the sphere are *rotations* about a diameter of the sphere and *reflections* about a geodesic [14]. Since every line on the sphere intersects, two reflections will result in a rotation [8, page 296]. So, we can see that a rotation on the sphere is a composition of at least two reflections.

5 Similarities and Differences of Euclidean, Hyperbolic and Elliptic Geometries

There are similarities between Euclidean, hyperbolic and elliptic geometries. In the study of isometries, we see Euclidean analogs of the transformations in both the hyperbolic and elliptic plane. Isometries can be expressed as the composition of reflections in all three planes.

Many similarities exist between Euclidean and hyperbolic geometries and have been axiomatized in Absolute geometry. *Absolute geometry* is given by a set of axioms that does not assume the Parallel Postulate (or any of its equivalent statements). The theorems of Absolute geometry are true for both hyperbolic and Euclidean geometries. Several interesting results of Absolute geometry are included below.

Theorem 5.1. *The following are equivalent conditions in Absolute Geometry that hold on the absolute plane:*

- *Angles of one triangle sum to 180 degrees,*
- *Angles of every triangle sum to 180 degrees,*
- *For every triangle ABC with a right angle, there is a point D such that $ABCD$ is a rectangle,*
- *There is a rectangle in the absolute plane,*
- *For every point p and every line l not containing p, p lies on a unique line parallel to l,*
- *There is a point p and a line l not containing p such that p lies on a unique line parallel to l,*
- *There exists triangles that are similar but not congruent in the absolute plane. [1, page 520].*

The statements of this theorem correspond directly to Euclidean geometry. The converse of these statements also holds in the absolute plane, but correspond to hyperbolic geometry.

Theorem 5.2. *The following are equivalent conditions in the absolute plane:*

- *The sum of the angles of every triangle is less than 180 degrees,*
- *The sum of the angles of one triangle is less than 180 degrees,*
- *There is a triangle, ABC, with a right angle such that no point D forms a rectangle $ABCD$,*
- *There are no rectangles in the absolute plane,*
- *There is a point p and a line l not containing p such that p lies on more than one line parallel to l,*
- *For every point p and every line l not containing p, p lies on more than one line parallel to l,*
- *Any two similar triangles are congruent in the absolute plane. [1, pages 526-527].*

A similarity between hyperbolic and the geometry on the surface of the sphere is the lack of a unique parallel line to a given line. Many differences exist between the three geometries, and one of the biggest is the properties of a triangle in the respective planes.

Theorem 5.3. *The sum of the angles of a triangle in the Euclidean plane is always equal to 180 degrees [1, page 9].*

Theorem 5.4. *The sum of the measures of the angles of any triangle in the hyperbolic plane is less than 180 degrees [19, page 381].*

Theorem 5.5. *The sum of the measures of the angle of any triangle is greater than 180 degrees on the surface of the sphere [19, page 396].*

The difference in the sums of the angles of a triangle will provide different and interesting tilings in the different planes.

6 Tessellations

A *tessellation*, or *tiling*, is the covering of the plane by closed shapes, called *tiles*, without gaps or overlaps [17, page 157]. Tessellations have many real-world examples and are a physical link between mathematics and art. Simple examples of tessellations are tiled floors, brickwork, and textiles. Artists are interested in tilings because of their symmetry and easily replicated patterns. Mathematicians are interested in learning how tiles can cover the plane, other surfaces and spaces. They want to know if and how a tile can cover the plane, how the tiles are surrounded by other tiles, and if a patch of tiling can be extended to cover the entire plane.

Tessellations of the Euclidean plane are generally formed with copies of a single tile (often a polygon) and the transformation of copies of the tile by isometries creates a pattern over the plane [15, page 24]. A *monohedral tiling* is the combination of copies of a single shape, called a *prototile*, under a symmetry group [17, page 158]. The simplest shapes for monohedral tilings are convex polygons. Not all convex polygons can tile the plane. Only certain pentagons can tile the plane, and no polygon of seven or more sides can tile the Euclidean plane [17, page 158]. A *regular tiling* of the plane is created by using congruent copies of a regular polygon with p-sides to create the tiling [11]. A regular tiling is denoted by $\{p, q\}$ for the pattern of q-number of p-gons at a single vertex [6, page 3].

Theorem 6.1. *The only three regular tessellations of the Euclidean plane are those with the square, equilateral triangle and the regular hexagon [19, page 190].*

Proof. Let p be the number of sides of a polygon and q be the number of polygons meeting at a single vertex. We can express the measure of an angle in a polygon with two equations and we will set those equations equal to each other and solve.

$$\begin{aligned}
\frac{\pi(p-2)}{p} &= \frac{2\pi}{q} \\
pq\pi - 2q\pi &= 2p\pi \\
pq - 2q - 2p &= 0 \\
pq - 2q - 2p + 4 &= 4 \\
(p-2)(q-2) &= 4
\end{aligned}$$

Notice that the only possible ways to factor $(p-2)(q-2) = 4$ are as follows:
$p = 4$ and $q = 4$, which is a square, $p = 6$ and $q = 3$, which is a hexagon and $p = 3$, $q = 6$, which is a triangle. The sum of the six angles at the vertex is 360°, so each angle is 60°. The triangle must be equilateral as there are six other polygons meeting at a vertex. A similar argument holds for the hexagon and the square. □

A *symmetry* is a geometric transformation of a tile and generates patterns [17, page 161]. Symmetry is found by the patterns created by reflections, translations, rotations and glide reflections in the plane [20].

Symmetry groups are the sets of all symmetries of a particular tiling that maps the pattern onto itself. Symmetry groups provide classification means for patterns and can generate periodic tilings

[17, page 161]. *Periodic tilings* have the property that under transformations of the symmetry group, the original design is mapped onto itself [18, page 439].

Triangles with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ can create periodic tilings of the Euclidean plane, hyperbolic plane and the surface of the sphere. The selection of p, q, r will determine which plane the triangle will tile. The symmetry group of the tiling is (p, q, r), which corresponds to rotations about the corresponding angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ [3, page 20].

In Euclidean geometry, the patterns satisfy the equality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

In spherical geometry, patterns of tilings of triangles satisfy the following inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Hyperbolic triangle tessellations are of the form

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

[3, page 20].

There are only three triangles that can tile the Euclidean plane. They are denoted by the symmetry groups: $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$. We can see geometrically that these are the only triangles that tile the Euclidean plane from the three regular polygons that tile the plane. The $(3, 3, 3)$ symmetry group represents an equilateral triangle that has angle measures $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$. We have from Theorem 6.1 that the equilateral triangle will regularly tile the Euclidean plane. The $(4, 4, 2)$ triangle is made by cutting a square along its diagonal. It is easy to see that two such triangles can be put together with their hypotenuses adjacent and will form a square. Hence, they will then tile the plane (again following from Theorem 6.1). The last triangle is $(6, 3, 2)$ with angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$. This triangle can be formed by bisecting one of the angles of an equilateral triangle. Diagrams of triangular tilings in the Euclidean plane are shown below.

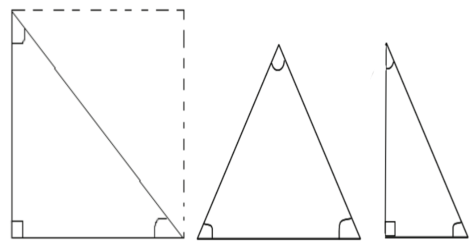

Figure 5: Triangle Tilings in the Euclidean Plane

The only triangle tilings in spherical geometry are: $(p, 2, 2)$, $(2, q, 2)$ $(3, 3, 2)$, $(4, 3, 2)$, $(3, 4, 2)$, $(5, 3, 2)$ and $(3, 5, 2)$. Regular polygons that tile the surface of the sphere are of the form $\{p, q\}$ such that $(p - 2)(q - 2) < 4$ [6]. Bisecting the polygon will give a tiling by triangles. The triangle will have angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$. The solutions to $(p - 2)(q - 2) < 4$ will be the possible tilings on the surface of the sphere. When $q = 2$, any value of p will satisfy the inequality (similarly if $p = 2$, any value of q will satisfy the inequality). The other symmetry groups follow as the remaining solutions.

We prove the hyperbolic tiling in a similar manner. The solutions to $(p - 2)(q - 2) > 4$ are the tilings of the hyperbolic plane. It is easy to see there are infinitely many solutions, and thus, there are infinitely many tilings of the hyperbolic plane [6].

A *semi-regular* tiling is made with two or more regular polygons to tile the plane [22]. A more complicated tiling than a regular tiling is called a dual tiling. A *dual tessellation* is made by taking the center of each polygon as a vertex and joining the incenters of adjacent polygons to create a new polygon to tile the plane [21]. The *incenter* of a triangle is the point at which the angle bisectors intersect inside the triangle [1, page 100].

Theorem 6.2 (Existence of Derived Tilings)**.** *If the regular tiling $\{p, q\}$ exists, then the following regular and semi-regular tilings also exist:*

1. *the dual regular tiling $\{q, p\}$,*
2. *$\{p, 2q, 2q\}$ and $\{2p, 2p, q\}$*
3. *$\{4, 2p, 2q\}$*
4. *$\{p, q, p, q\}$*

5. $\{4, p, 4, q\}$
6. $\{3^2, p, 3, q\}$
7. $\{3, p, 3, p, 3, \frac{q}{2}\}$, *for* $q \geq 6$ *and* q *even. [11]*

The proof of Statement 1 will follow directly after the following lemma. The proofs of the remaining statements of Theorem 6.2 will be omitted, but can be found in the reading "Constructing Semi-Regular Tilings" [11].

Lemma 6.3 (The Incenter Process)**.** *Given a (possibly non-regular) q-gon, such that:*

(i) there is a tiling of the plane using only this q-gon in which adjacent tiles are reflections of each other across their shared edge;

(ii) the q-gon has an incenter.

Then the incenters of adjacent tiles may be joined to produce a semi-regular tiling

$$\{p_1, p_2, \dots, p_q\}$$

where p_i is the number of tiles at the i-th vertex of the q-gon.

Proof. The first statement of the lemma gives us that all the angles at a vertex are the reflections of each other. Then the angle between two adjoining edges is $\frac{2\pi}{p_i}$, for $1 \leq i \leq q$. Statement two of the lemma gives us that the q-gon has an incenter. A p_i-gon is formed by connecting the incenters of the adjacent polygons at the i-th vertex. By definition of incenter, the distance from the incenters of the polygons to the vertex is equidistant. Connecting the incenters about a given vertex will create a polygon with sides of the same length. Then we have the p_i-gon is regular. Repeating the process of connecting the incenters of all the q-gons gives us a sequence of polygons $\{p_1, p_2, \dots, p_q\}$, which is a semi-regular tiling of the plane [11]. □

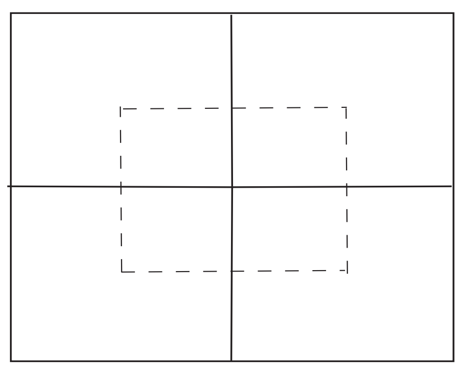

Figure 6: The Incenter Process

Proof of Theorem 6.2. Let the regular tiling $\{p, q\}$ exist. Since a regular p-gon has an incenter and we have q-tiles at every p vertex, the regular tiling $\{p, q\}$ satisfies Lemma 6.3. We join the incenters of the p-gon to form a semi-regular tiling of a q-gon with p number of tiles at each vertex Therefore, the dual tiling $\{q, p\}$ exists [11]. □

7 Hyperbolic Tessellations

As we saw in the previous section, there are infinitely many ways for triangles to tile the hyperbolic plane. We study regular tilings of the hyperbolic plane by polygons.

Theorem 7.1 (Hyperbolic Regular Tilings)**.** *There are regular tilings of the hyperbolic plane of vertex type $\{p, q\}$, where p is the number of sides of a polygon and q such polygons meet at a single vertex, such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.*

Proof. Let $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$ be the angles of triangle $\mathcal{T}$. From the Hyperbolic Triangle Lemma it follows that

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

We have a tiling of the plane by a triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$. We will create a polygon by removing sides of the triangle to join multiple triangles together. First, remove the side of triangle $\mathcal{T}$ opposite the angle of measure $\frac{\pi}{q}$ and combine the adjacent triangles along this side. Now we have an isosceles triangles with angles $\frac{2\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{q}$. Next erase the legs of the isosceles triangle to form a polygon with p number of sides and q such polygons meeting at a single vertex. Hence, there is a regular tiling of the hyperbolic plane of the form $\{p, q\}$ [11]. ☐

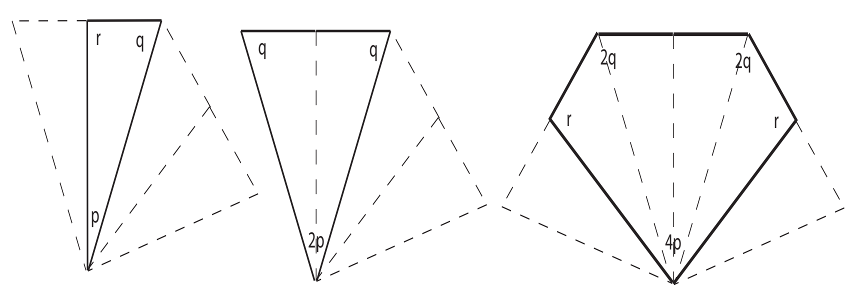

Figure 7: Hyperbolic Regular Tiling

8 The Connection Between Mathematics and Art

"For me it remains an open question whether [this work Circle Limit III] pertains to the realm of mathematics or to that of art."
-M.C. Escher [20]

M.C. Escher was a Dutch graphic artist with a strong interest in math. He studied mathematical topics as a way to realize his artistic visions. Particular topics of interest to Escher were regular divisions of the plane, the 17 symmetry groups and topological spaces. Escher was also friends with famous 20th century mathematicians, Roger Penrose and H.S.M. Coxeter. After correspondence with Coxeter about tilings in the hyperbolic plane, Escher was inspired to create *Circle Limit I.* Escher was interested in "patterns with 'motives' getting smaller and smaller till they reach the limit of infinite smallness." Tilings of the hyperbolic plane in the Poincaré disc model were tools Escher used to create images vanishing to infinity [3, page 19].

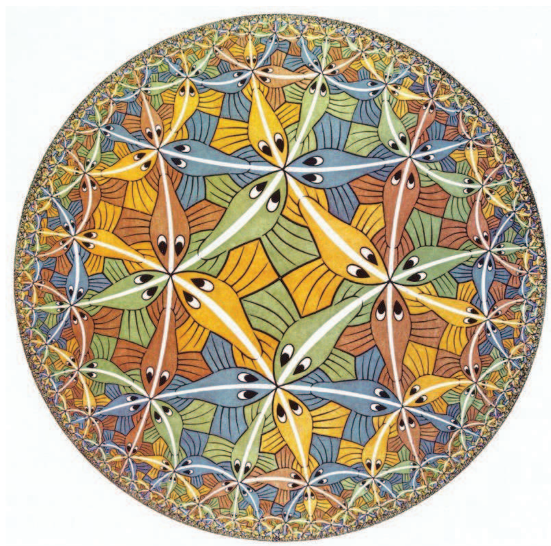

Figure 8: *Circle Limit III*

The tessellations of M.C. Escher inspired me to create my own tessellations. I first created two Euclidean tessellations and then applied my research on hyperbolic tilings to create a tessellation of the hyperbolic plane in the Poincaré disc model. I used Adobe Illustrator to create the Euclidean tessellations and I created the hyperbolic tessellation by hand. After the Euclidean designs were created, I transferred the pattern to a copper plate and made etched prints of the tessellations.

8.1 Designing the Tessellations

Escher's approach to creating Euclidean tessellations was to apply the isometries of the Euclidean plane to the tiles he designed [20]. He created the tiles by distorting polygons known to tile the plane. The distortions of the polygons were best transformed by translations and rotations. Reflections did not usually work because most of Escher's tiles did not preserve a straight line along which to reflect the tile [17, page 162]. I used a process similar to Escher's to change a regular polygon into a more interesting tile to create my tessellations.

The first design I created was based on translational symmetry of a square. I modified the square by adding a line along the length of the base. Then I copied this line and translated it to the top of the square. Next, the top and bottom lines of the original square were removed. To make the design more interesting, I drew additional lines inside the tile before I made copies of the tile. Copies of the tile were slid up, one on top of each other, and over the width of one tile to create two columns of tiles.

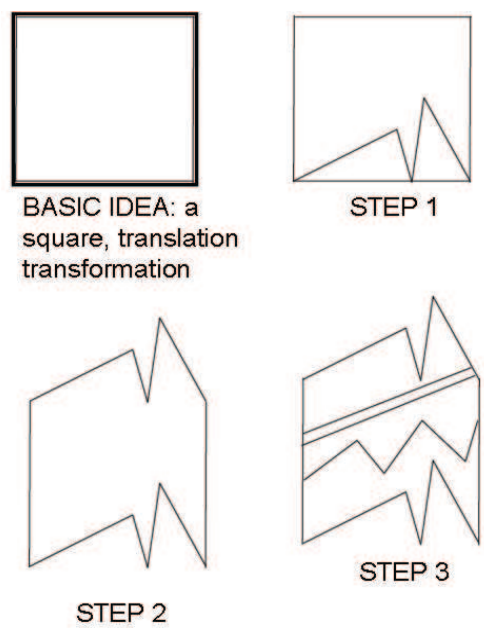

Figure 9: Tessellation Process: Using a Square

The second Euclidean tessellation I created was based on an equilateral triangle with rotational symmetry. Rotations of $60°$ were used to create the tessellation. The first step was to draw a curved line. Then the line was copied, rotated $60°$ and joined to the top of the first line. A straight line was drawn between the bottoms of the two curved lines to serve as a guideline and to enclose the shape in a triangular fashion. The guideline was then bisected and a curved line was drawn along half of the guideline, inside the triangular region. This curved line was copied and reflected across the guideline to connect the remaining length along the guideline. The guideline was then removed and the tile was complete.

Once the tile was completed, I chose to draw two designs inside the tile. One was of a dove and the other a toucan. I alternated the birds and rotated the tiles by $60°$ (or multiples of $60°$). This was repeated until a full unit of $360°$ at the vertex had been created. The tiling was extended by placing the tiles next to each other in a similar manner. Such a tiling can be extended indefinitely to tile the entire Euclidean plane.

I applied my research on hyperbolic geometry and my interest in Escher's *Circle Limit III* to create my hyperbolic tessellation. The $(6, 4, 2)$ tiling of triangles in the Poincaré disc served as my model of the plane, as it did for Escher's *Circle Limit I*.

To create my tessellation, I first grouped two triangles (one black, one white) sharing a hypotenuse to make a kite shape. All the triangles were paired up and colored in an alternating pattern of blue

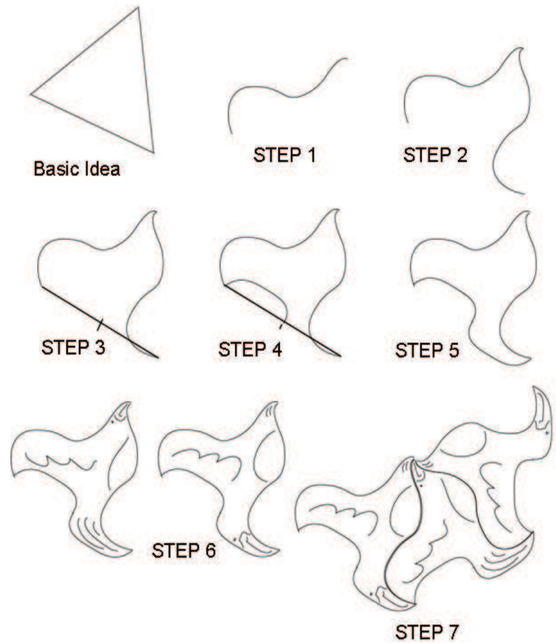

Figure 10: Tessellation Process: Using the Triangle

and white kites. Once all the triangles were paired and recolored into kites, I made alterations to the white kites. The alterations consisted of the removal of a smaller kite-shape from the top, short edge of the big white kite. This small kite was then attached along the long side of the kite by a rotation at the common vertex. I continued this pattern of removing and adding kite pieces around the disc to every white kite. I made smaller kite cut-outs as the larger white kites shrunk in Euclidean size as they approached the boundary of the disc. To help differentiate the design, the kites I altered were colored orange while the kites I did not change were again colored blue.

8.2 The Final Tessellations

The three tessellations I designed after my research in geometries and tessellation, which were inspired by the work of M.C. Escher are now shown. *Modified Square* is the title of the tessellation I created with translational symmetry of a square.

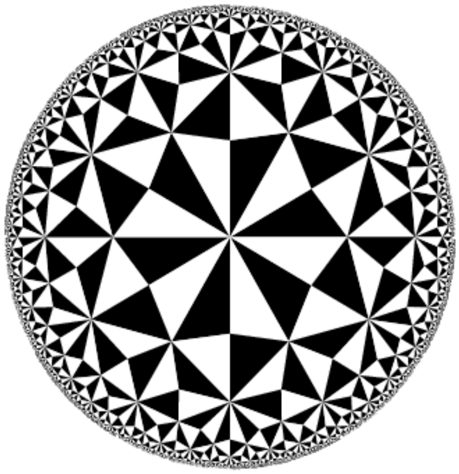

Figure 11: (6,4,2) Tiling of the Hyperbolic Plane

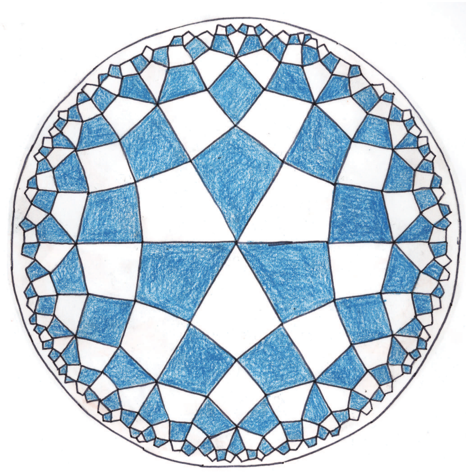

Figure 12: Tiling with Kites

Doves and Toucans is the title of the tessellation I created with rotational symmetry of an equilateral triangle.

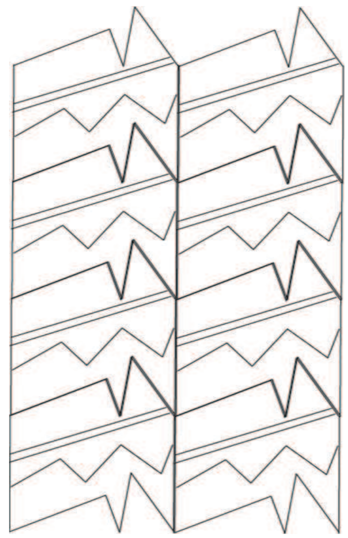

Figure 13: *Modified Square*

My hyperbolic tessellation is titled *Orange and Blue*.

8.3 The Printmaking Process

Escher was a printmaker and I have a strong interest in printmaking, so I decided to use this technique to make prints of my tessellations. The printmaking technique I used to create the Euclidean tessellations is called *etching* and uses a copper plate to hold the design. The copper plate is prepared for printmaking by polishing the front of the plate and coating it with a waxy protective covering called *hardground*. The back of the plate is protected with contact paper. The hardground (and contact paper) protects the copper from acid, which the plate will later be soaked in. The tessellation design is transferred to the plate using carbon paper to draw the lines on the hardground. Then the design is scratched into the hardground using a metal needle. Where the hardground is scratched and removed exposes the copper to acid. The plate is set in an acid bath for 90 minutes. The acid "eats away" the exposed copper, leaving a grooved line. When the lines are sufficiently deep, the acid is rinsed from the plate and the hardground is removed with ammonia. The printing process can now begin.

The etching process turns the lines into small wells that will hold ink. Ink is applied to the plate and pushed into these wells using strips of cheesecloth. The plate is set on a heating plate to warm the metal and the ink. The warmed metal and ink keeps the ink in the wells and makes it easier to remove the excess ink from the surface of the plate. Once the surface of the plate has been wiped clean, the only remaining ink is inside the wells of the lines. The plate is set on the printing press, inked side up, and damp paper is set on top of the plate. Felt blankets are set on top of the paper to protect the paper from the pressure of the press. The moisture from the damp paper helps pull the ink out of the wells and onto the paper. The print is removed from the press and dried flat on a drying rack.

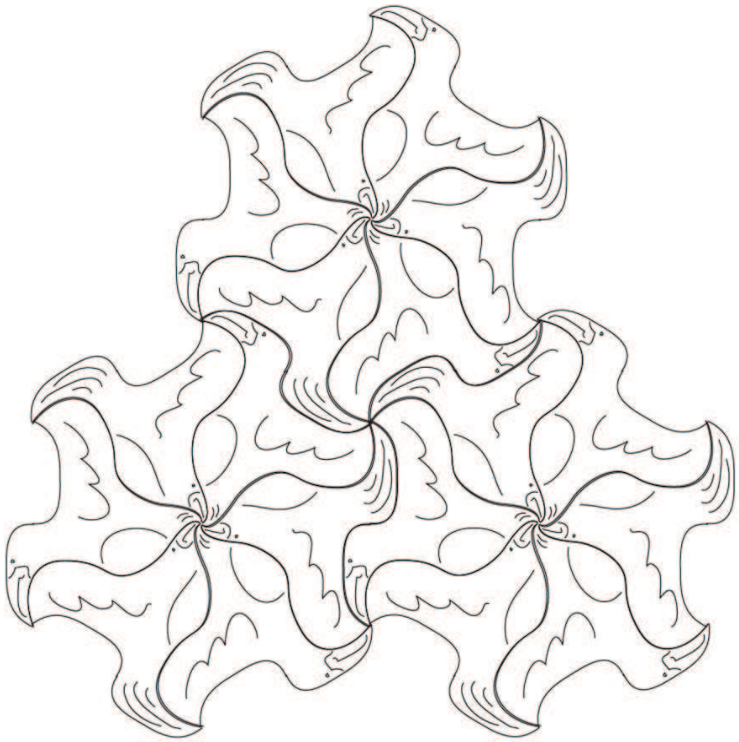

Figure 14: *Doves and Toucans*

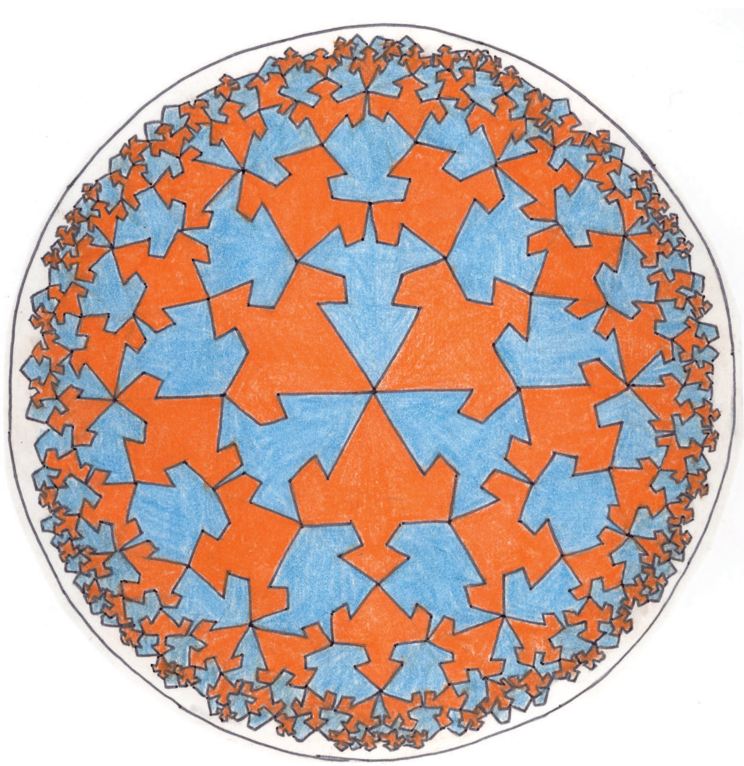

Figure 15: *Orange and Blue*

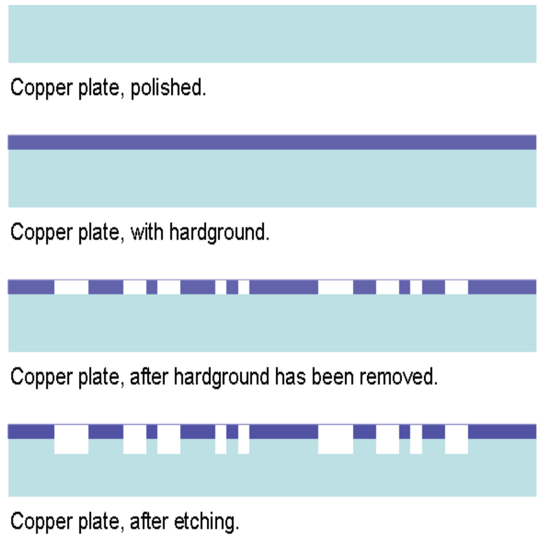

Figure 16: Cross Section of a Copper Plate

9 Conclusions

The mathematical and artistic connection in tessellations is strongly exemplified by Escher's *Circle Limit* series. Coxeter proved the mathematical accuracy of Escher's *Circle Limit III* in the late 1970s. Coxeter showed that Escher was correct within a degree of measurement of the angles the geodesics intersect the boundary. Escher created *Circle Limit III* without computations, calculations or a deep understanding of hyperbolic geometry. However, with a keen artistic eye, he was able to produce what mathematician J. Taylor Hollist claims to be one of the "most beautiful example[s] I have ever seen of the Poincaré circle model for hyperbolic geometry" [13].

Since the late 1950s when Escher began producing his *Circle Limit* prints, mathematicians and computer scientists have continued to study hyperbolic tessellations. Technology has vastly improved from the days when Escher created hyperbolic images by hand. The combination of mathematics, creative thinking and computer technology come together in the study of tessellations and geometry today. There is no algorithm that can determine exactly how tiles can tile the plane or how polyhedra can fill spaces. Therefore, there is much to be learned about tilings, especially in non-Euclidean geometries when the kinds of tilings are infinite in number. "The use of 'visual computers' gives rise to new challenges for mathematicians - at the same time, computer graphics might in the future be the unifying language between art and science" [7, page 3].

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