High School Math Reform and Combinatorics

Anna Grossman
Advised by Dr. Brian Miceli
Senior Project
Fall 2007

1 Introduction

As a math major planning to continue to get my Masters in the Art of Teaching, I decided to pick a project that combined my two worlds of math and education. Education reform in high schools is a hot topic all across the world, and the mathematics classroom is definitely not exempt from that discussion. In fact, there are debates constantly about how to implement the research on math education reform in the classrooms. Based on the research available, one of the focuses is taking students beyond the basic algebra-geometry-algebra-calculus track and providing them authentic, challenging, and thought-provoking questions. I combined this knowledge with my strong interest and appreciation for combinatorics, and thus I chose to focus on how combinatorics fits effectively into the current math education reform. Therefore, this paper will explore current math education reform and how combinatorics fits into the reform as well as provide example combinatorial problem sets which can be used in the classroom.

2 Current State of High School Math and Suggested Reform

Traditionally, math is an endless memorization task for students over facts and procedures, which they most often quickly forget. Essentially, how the common math classrooms have been run is as follows: the teacher shows students examples of how to solve certain types of problems, and then students practice the method with class work and homework. This method has been called "mindless mimicry mathematics" where students are not necessarily understanding, but rather they are just copying the technique (Battista 1999). Scientific studies have shown that traditional methods for high school math instruction are ineffective and do not foster the growth of math reasoning or problem solving skills. Poor education in high school math classrooms has many consequences. Students are not moving forward to undergraduate study and the workforce prepared for the higher level math that exists in both environments. Additionally, math anxiety is widespread across many generations; people readily admit they are not good at math or just do not understand it. If people had better high school math experiences, maybe they would not be so quick to jump on the "I hate math" bandwagon. Notice, however, that although so many people cannot read, admitting this is not nearly as acceptable. This demonstrates the stigma that this country suffers from that deems math boring and hard. Still more disheartening is the fact that even the "bright" students in today's classrooms are struggling to grasp the the deeper understanding of math as they are guided to focus their learning on the formulas alone not the justification behind the formulas (Battista 1999). Battista provides an example of such a student. A female eighth grade student about to finish a geometry class, meaning that she is ahead of the classic math schedule, is shown a box and rectangular packages formed from two cubes. She is given a picture showing how many packages can fit along the length, width and height of the box and in which direction, and she is then asked to find how many of the packages it takes to completely fill the rectangular box. The student immediately reverts to the formula for volume, $V = lwh$, and simply multiplies together what she was given. She fails to take into account, however, which direction the boxes are positioned to see if they would actually fill the box. In fact her answer is incorrect. So, we see that she knows formulas and procedures yet her knowledge and understanding of what they represent and what they calculate is quite superficial; she knew how to calculate volume, but she does not truly know *what* it is. These findings in advanced students are quite frustrating as these students have the capability to understand math but they are not being given a quality opportunity with the current mathematics instruction. Another good example of this superficial knowledge in all ability levels of students is dividing fractions. If asked to divide $\frac{5}{2}$ by $\frac{1}{4}$, many students can revert to their "invert and multiply rule" and tell you that the answer is $(\frac{5}{2})(\frac{4}{1}) = 10$ (Battista, 1999). Conceptually, though, most students would not know how to represent this, in a picture for example, or figure it out without their go-to rule. The learning of symbolic manipulations should never become disconnected from the underlying mathematical reasoning (Battista, 1999).

Far too often, citizens, lawmakers, and those who make education decisions are swayed by arguments supporting traditional math curricula, and they adopt policies that do not take into account recommendations from professionals, scholars and scientists on the issue of mathematics teaching (Battista 1999). However, in 1989, the National Council of Teachers of Mathematics (NCTM) released a set of recommendations and standards for school mathematics; since then, the main focus of the movement to reform math education has been on implementing these standards and the recommendations of other education and math professionals. The NCTM standards set many goals encompassing not just the material that should be taught, but *how* that material should be taught. One of the goals set forth is that high school math classrooms should produce students who are mathematically literate in the current technological world, the world where math is constantly growing and being applied to more diverse fields. The standards also call for new content like probability, statistics, and discrete math to be included in the curriculum as well as new approaches to the existing curriculum (Ross 1995). The hope is that students will acquire the ability to be mathematical problem solvers, and in order to do this, NCTM envisions that students will have the opportunity to solve complex and authentic problems. These problems provide the chance for students to draw their own conclusions using demonstrations, drawings and real life objects in addition to formal math and logic arguments (Battista 1999). Currently, when students are learning through the routine and repetition model employed by most math classrooms, they learn just those routines. They forfeit the ability to understand the math that surrounds them. Additionally, these students struggle when they reach undergraduate math classes since they have forgotten most of the math they have learned between elementary school and college because it was so disconnected from their thought processes and intuitions. Essentially, the current high school math system produces students that memorize and forget, thus explaining the lack of interest in continuing mathematics in students' undergraduate years and beyond (Battista 1999). Therefore, the need to implement the reform suggested by NCTM and other professionals to the high school math curriculum is clear.

One successful new math program that takes into account many of NCTM's standards and goals as well as other suggested reform demonstrates that it is possible to implement these new ideas in a positive manner: The Interactive Math Program (IMP). It has a very innovative style and completely changes the way that high school math is approached. IMP is a comprehensive four year math program of problem based mathematics that adds important topics such as statistics and probability to the existing material of algebra, geometry, and trigonometry. IMP presents different topics throughout the four years using a thematic approach so that many different types of material are taught simultaneously allowing the focus to be much broader than just on one type of math as traditional curriculum dictates. Studies show that after going through the IMP problem based curriculum, students were more confident in math, more likely to view math as meeting the needs of society, and more likely to see math in everyday activities (The Research

Base of IMP). IMP is successful due to the attention it gives to how students learn math most effectively and how much the current curriculum needs to be revamped. One main focus of the program is that curriculum should allow students to see math as a useful tool to facilitate learning. Topics and ideas should be introduced concretely so that the math is immediately accessible to the students instead of starting with abstract ideas and showing their concrete applications. As applied to combinatorics, this means that combinatorial problems should start with tangible, hands-on problems rather than starting with combinatorial coefficients that have little or no context. "Studying mathematics in the contexts of problems motivates students to think mathematically and to make connections between skills and different mathematical topics" (The Research Base of IMP, p11). Essentially, putting math into real life problems that build on general experiences not only puts math in a context students understand but also motivates students. IMP is a perfect example of how capturing students' attention and connecting math to their world can really improve their math experience.

3 Combinatorics and Its Place Within Math Reform

As stated before, curriculum in the majority of math classrooms today is dominated by the traditional algebra-geometry-pre-calculus-calculus sequence. Incorporating combinatorics into the high school curriculum provides variation from this sequence as well as opportunities to fulfill the math reform goals and standards. Kapur (1970) is quoted as supporting combinatorics in a high school classroom for many reasons. The independence of combinatorics from calculus facilitates the tailoring of suitable problems for different grades, and usually, very challenging problems can be discussed with students so that they discover the need for more sophisticated mathematics to be created. Kapur also asserts that combinatorics has many applications in many different fields, and it can be used to train students in "enumeration, making conjectures, generalization, and systematic thinking" (Batanera et all 1997, p181).

Combinatorics, in a basic sense, is the math involved in counting. Essentially, combinatorics focuses on the numbers of different combinations or groupings of numbers. As far as combinatorics in high school math curriculum, the main goal is to understand how to enumerate the possible permutations or combinations of a finite set. Permutations are the groupings of distinct objects in which order is important. One way to introduce permutations to students is to suppose that there are n people that show up at a movie theater, but inside the theater there is only room for r people to line up between the ticket counter and the door (Nom). The number of different lines that can be formed of r people is then found by calculating how many choices for each spot there are:

$$n(n-1)(n-2)...(n-r+1).$$

Thus the number of permutations of n distinct objects taken r at a time can be written as

$$nPr = \frac{n!}{(n-r)!}.$$

Combinations, on the other hand, are the ways to choose a subset of objects from a main set. In combinations, order is not important, just the number of ways you can choose the objects. Going back to the example of a line at a movie theater, when n people show up at the door, you pick k people that can fit inside the door (Nom); this is $nCk = \binom{n}{k}$. The formula is given by

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}.$$

According to studies, combinatorial thinking is part of Piaget's formal operational stage of cognitive development (Janackova, 2006). Combinations are actually operations on operations so they are, in fact, very characteristic of the formal operational stage. This stage of cognitive development normally is reached during adolescence, but may not be reached until adulthood if it is reached at all. Piaget and Fischbein describe the different approaches students have to combinatorial problems and how these approaches demonstrate their different levels of cognitive development. During Stage I, students use random listing that lacks any form of systematic strategy, and during Stage II, students will start to use trial and error while starting to discover practical procedures to organize the elements (Batanero, 1997). Once at the formal operational stage, students can discover systematic procedures of combinatorial constructions. It has been shown though that combinatorial thinking may not come automatically; in fact, Fischbein (1970) found that even at the formal operational level, combinatorial problem solving capacity may only be reached through instruction (Batanero, 1997). Since this is not a type of thinking that comes naturally, the teacher holds a very important role in guiding students to achieve this type of thinking. Though this might be challenging, Fischbein and Gazit (1988) found that with proper instruction, even ten year olds can learn some combinatorial ideas (Batanero, 1997). Such a study implies that if middle school students can grasp some of these topics, high school students can grasp them as well, and they should be challenged to do so.

Batanera, Navarro-Pelayo, and Godino conducted research on the effect of instruction on basic combinatorial reasoning ability in 14-15 year old students. Essentially, the research was to support the claim above that with instruction, students can master combinatorial ideas. Quoted in the study is Dubois (1984) who determined that simple combinatorial questions can be classified into three models. The first is *selections* which represent the concept of sampling; there are m objects from which n objects are drawn. The different possibilities for the selection model depend on whether or not replacement is allowed and whether or not the order of the objects is important. The next model is the *distribution* or *mapping model*: n objects are distributed into m cells. In this model, the different possibilities and structures depend on the following: whether the objects are identical or not, whether the containers are identical or not, how many objects can be placed in each cell, and whether the order the objects are placed into the containers is important or not. The last model defined is the *partitions* model in which sets are partitioned into subsets: splitting a set of n objects into m subsets. According to Dubois, however, there is a bijective correspondence between the models of partition and distribution even though this might not be evident to students. While these models may use the same combinatorial operation, it cannot be assumed that the different types of questions have the same level of difficulty.

The study included a thirteen item questionnaire that was developed to allow a balanced, representative sample of problems from all three models. There were 720 students between the ages of 14 and 15, and 352 students had received combinatorial instruction while 348 had not. The questionnaire showed no difference in performance based on sex or order of the questions, so the difference in results could legitimately be attributed to whether or not the students received instruction. The experiment showed that there was a significant difference after instruction which greatly increased students' combinatorial capacity for solving problems. The average number of errors was 10.59 for students without instruction and 7.01 for students with instruction. Many students struggled with seeing that two problems with different combinatorial models were the same even when the solution to both answers used the same combinatorial operations. The results of the study conclude that when organizing teaching over basic combinatorial models, the emphasis should be placed upon translating combinatorial problems to the different models and the recursive reasoning and systematic listing, rather than just algorithmic aspects and definitions. "Understanding a concept (e.g. combinations) cannot be reduced to simply being able to reproduce its definition. Concepts emerge from the system of practices carried out to solve problem situations" (Batanera et all 1997, p196). This experiment serves as a great example of the fact that students can be guided through proper instruction to understand combinatorial problems. Additionally, it reinforces the notion that mathematics instruction needs to focus deeper on the *why* and *how* instead of just regurgitating formulas.

Additionally, combinatorics offers an opportunity to enhance the high school math curriculum while implementing the new standards and suggested reforms. There are many combinatorial problems that are examples of "real life" situations, so students have an easier time conceptualizing the problems and coming up with ways to approach them. With all the different approaches, combinatorial questions also provide challenging experiences for learners of all academic levels. Bharath Sriraman conducted a study in 2004 when faced with the discovery by studies that mathematically gifted students have already mastered about sixty percent of the content in high school mathematics (Sriraman 2004). Sriraman thought the answer to how educators can keep students interested and exercising their mathematical capabilities was to have more emphasis on enumerative and discrete math due to its independence from calculus. In the study he conducted, students had to maintain a journal where they worked on combinatorial and number theory problems with clues to guide them. His pedagogical goal was to "pose problem situations that had an underlying structure or principle which could be potentially discovered by the gifted students, but simple enough for all students to have the opportunity to create representations, think abstractly, and create generalizations at varying levels" (Sriraman 2004, p35). In his first experiment, he assigned eight problems over three months and half of them used a combinatorial structure titled the Steiner Triple Model. The five gifted students in the class were able to uncover the sequence and began to understand the Steiner Triple Model. They devised an efficient strategy for counting triples and then were given the opportunity to apply their findings to more general questions; for example, in an arrangement of n objects in triples, how many ways can each pair of objects appear once and only once? The second experiment spanned three months and included five sequentially assigned questions representing diverse situations all based on the Pigeonhole Principle. Four out of nine gifted students discovered the Pigeonhole Principle by looking at the structural similarities of the assigned questions.

Sriraman found that choosing problems which involve self discovery of a combinatorial problem or structure along with the use of journals ultimately led to positive results. This innovative technique of approaching math exploration really takes into account the research on making math meaningful and understandable. The study led to "the positive outcomes of gifted students abstracting structural similarities, conceptually linking related problems, utilizing their creativity pursuing general solutions and creating theoretical generalizations" (Sriraman 2004, p37). In addition to just the gifted students though, the problems were easy to understand and grasp so they were accessible to the weaker students as well. These students were able to model the problems through trial and error which, by itself, is a beneficial strategy of problem solving and systematic thinking. Thus, students of all ability levels gained valuable advancements in their mathematical education through combinatorial problems which fostered students' independent thinking and encouraged them to work at their personal level. Additionally, since combinatorial problems have many different approaches, these questions provided students the opportunity to share ideas as they worked through the different contexts and representations. Essentially, Sriraman's study is a thorough example of how beneficial combinatorial problems are in the high school math classroom (Sriraman 2004).

4 Framework for Combinatorial Problem Sets

I have developed some combinatorial problem sets that could be used in a high school math classroom. These problem sets are meant to enhance the very basic combinatorial ideas that students are currently introduced to in the high school curriculum by presenting *real life* problems that can be approached with combinatorics. The examples are structured so that a teacher would present them after the basic combination/permutation lesson so that students would already have a basic knowledge of when to do which operation and how to calculate each one. Students will hopefully reach some generalizations and deeper understanding of the concept after working through these two problem sets as they are geared to provide authentic applications of their knowledge. The questions in the problem set are unique as they demonstrate that there are many different ways to look at and approach the same questions; these questions do not just represent a straightforward application of a given formula. They are really geared to helping students see and understand what they are doing. The chronology of the problem set mirrors how a teacher would work through the set in the classroom to guide her students.

The first problem set I structured is one that focuses on a deck of cards and five card poker hands. Using different poker scenarios, this problem set demonstrates that there are different methods to get to the same answer, i.e. not everyone has to be able to think the same way. These questions also lend themselves to many different difficulty levels as a result of this characteristic. Classic combinatorial poker problems get very difficult very quickly, so this problem set helps students understand the inherent complexities of math. The second problem set uses two strings of problems. The first one focuses on the number of paths to get from point$A$ to point$B$ and the second one looks at the number of partial scores that can possibly lead to a final score. This problem set shows how combinatorial problems are very often isomorphic to each other, and it challenges students to think on a new plane of trying to make generalizations about the different questions to see that they are asking the same thing. Math as a whole is isomorphic as well, so this is a great skill to teach students.

5 Problem Set 1 - Cards

This problem, set focused on cards and poker hands, is a great way to expand on combinatorics for students. A deck of playing cards is very familiar to students, so it provides an example of where math comes into play in the *real world*. This problem set also allows students to look at the questions in more ways than one, and thus, more students will hopefully be able to visualize and understand the problems and concepts that they might not have otherwise grasped. If students are having issues with the concepts behind the questions and the math involved, the teacher could scale down the problems and work with a 10 card deck and physically show the students how the ideas connect. Finally, this problem set demonstrates how quickly the problems get complicated and complex, so these basic and intermediate questions are a doorway to higher level combinatorial thinking.

Question 1: How many different five card hands are possible?

The first method for approaching this question is simply to notice that this is a combination since order does not matter. So there are 52 cards total and we want to know how many ways there are to choose 5 of them. So, we have

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{(5!)(47!)} = 2,598,960$$

possible five card hands. For students who do not immediately make a connection that this is a combination, the teacher can guide them through with a step by step method. Notice that there are 52 choices for the first card in the hand, then once a card is dealt to the hand, there are only 51 choices left. Similarly there are then 50, 49, and 48 choices for the next three cards respectively: $52 * 51 * 50 * 49 * 48$. However, it is necessary to account for the fact that order is not a defining factor in a hand so we must divide the product by 5! to get rid of the implication that order matters. Thus the total number of five card hands is

$$\frac{52 * 51 * 50 * 49 * 48}{5!} = 2,598,960.$$

As the students will hopefully then see, these answers are exactly the same, how they want to work the problem is completely dependent on how they mentally visualize the problem.

Question 2: How many different ways are there to get a pair, two cards with the same numerical value, on the first two cards dealt?

Although students' first inclination might be that the answer to this question is $\binom{52}{2}$, it is actually a little more difficult that that. First, you must decide which number value you are going to make a pair with. Since there are 13 values, or ranks, in a deck $(A, 2, 3, \cdots, 10, J, Q, K)$, choosing one is given by

$$\binom{13}{1} = \frac{13!}{1!(13-1)!} = \frac{13!}{12!} = 13.$$

Once the number has been chosen, there are four of each rank (one of each suit) and you must choose which two of those will be in the pair. This is represented as

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{(2!)(2!)} = 6.$$

Then, we multiply to find out how many times these both happen at the same time, and we are left with the number of possible ways to get a two of a kind on the first two cards dealt is

$$\binom{13}{1} * \binom{4}{2} = 13 * 6 = 78.$$

The thought process of working through this problem without using combinations is as follows. You can be dealt any of the 52 cards for your first card. Once that card is dealt, the next card must match it in rank in order to produce a pair, so there are 3 possible cards to be dealt next. Again we must compensate for the order of the hand not being important, so we must divide out by 2!. Therefore, the number of ways to get a pair on your first two cards of the hand is

$$\frac{(52)(3)}{2!} = 78.$$

Again it is clear that it does not matter which method is used because both produce the same, correct answer.

Question 3: How many different ways are there to get a flush (all cards the same suit) in a five card poker hand?

In order to solve this question using combinations, it is necessary to again use two different combinations. First, we must take into account that there are four different suits with which a flush can be accomplished: hearts, diamonds, spades, and clubs. Choosing one suit is represented by

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{3!} = 4.$$

After choosing a suit, there are 13 cards in the suit and we need five of those to complete our flush, so we have

$$\binom{13}{5} = \frac{13!}{5!(13-5)!} = \frac{13!}{(5!)(8!)} = 1,287.$$

Thus to complete the calculation, we need to multiply the two conditions giving us that the number of possible ways to get a flush is

$$\binom{4}{1} * \binom{13}{5} = 4 * 1,287 = 5,148.$$

This problem can also be looked at on a card-by-card basis since not all students are comfortable enough with the concept to resort to formulas. There are 52 possible cards to receive on the first card dealt. Once that card is dealt, however, the suit has been chosen for the flush. Therefore, there are now 12 options for the second card, 11 for the third, 10 for the fourth and 9 for the last card. So we have, $52 * 12 * 11 * 10 * 9$. We are not done though as it is not important which order the cards were dealt; the same hand results no matter what. Therefore, we must divide the product by 5! to account for the different orderings all producing the same hand. So, we end up with

$$\frac{52 * 12 * 11 * 10 * 9}{5!} = 5,148$$

ways to get a flush. At this point the teacher should reemphasize that the answer is again the same as the first method.

If teachers have extra time or a fast moving class, this problem set could be expanded upon by combining it with a probability lesson by looking at the likelihood that someone will actually get a flush or a pair. Essentially though, this problem set is intended to concretely demonstrate how combinatorics comes into play. Poker hands will catch high school students' attention and interest because it is something that is a part of their lives. This is a really important aspect when trying to keep students engaged in math while trying to challenge them to expand their math reasoning skills. This problem set also shows the transition from very basic combinatorial problems to more intermediate problems. For example, if you present some other types of hands students will be able to notice that they are not so clear cut or easily defined. Students, therefore, are able to see the mathematical progression and the complexities that quickly arise.

6 Problem Set 2 - Isomorphic Paths and Scores Problems

This problem set is really two different problem sets combined. The purpose of combining these two problem sets is to demonstrate the isomorphic nature of combinatorial problems and how the realization that certain problems are isomorphic can lead to a deeper understanding of the concepts. Isomorphic problems are defined by Siegler (1977) as "problems that are formally identical but differ in their surface structure" (Janackova et all, 2006, p130). This helps break students away from only comprehending problems on the surface and attaching certain words and situations to certain formulas. As stated above in the framework, this problem set consists of two different sets of problems. The first one focuses on the possible paths to get from point A to point B while the second one focuses on the different possible partial scores that can lead to given final score. As different restrictions are placed on each set of problems, the students work to discover the answer. At the end of the two problem sets, and in response to the careful board work of the teacher, students will see that although the questions were presented very differently, the problem sets actually contained the same three questions mathematically.

6.1 Path Questions

In this set of questions, there will be a "town" marked out on the first quadrant of a cartesian plane, and students must determine the number of different possible paths with different circumstances or restrictions to get from a starting point A (normally at point $(0, 0)$ in the bottom left corner of the "town") to point B (normally at point (m, n) in the top right corner of the "town"). One of the conditions that will hold throughout the problems is that the only acceptable moves throughout the grid are north (up) and east (right). This set of problems is beneficial for a high school classroom as it offers more visual students an opportunity to actually draw out the possible paths. This is an option that is not normally available in standard math questions. It is important for students of all ability levels and all learning types to be given opportunities to succeed and apply what they are good at to math problems.

Question 1: In a town that measures 2 units North/South and 3 units East/West, how many possible ways are there to get from $A(0, 0)$ to $B(3, 2)$ remembering that you may only move North and East?

More visual learners might want to first draw the possible paths and then list them out letting N denote a move North and E denote moves East. Each path in the list must contain $2Ns$ and $3Es$ After drawing the paths, this should be the list generated by the different paths (Please note that when listing the paths on the board, it will be more visually appealing and more organized to write the possible paths vertically):

- $NNEEE$; $NENEE$; $NEENE$; $NEEEN$; $ENEEN$; $EENEN$; $EEENN$; $EENNE$; $ENENE$; $ENNEE$

As students will be able to see, there are exactly 10 possible paths. When generating lists with students about possible paths, it is very important to encourage organized and strategic listing. Based on the research, this is an area that students struggle in, but when following the lead of the teacher, they should be able to develop organized and thorough listing techniques. Another method to approaching this problem is to notice that it is essentially a combination question. There are five total moves necessary to get from A to B, and, in those five moves, there are exactly two moves that need to be made going North. Thus, this can be represented as the number of different ways to choose 2 Ns out of a set of 5 moves:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{(2!)(3!)} = 10.$$

This is perfect time to remind students that this is the exact same as choosing $3Es$ out of 5 total moves:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{(3!)(2!)} = 10.$$

The concept to reinforce here is by choosing the $2Ns$ you are also, by default, choosing the 3 moves that must be Es. This is the identity that

$$\binom{n}{k} = \binom{n}{n-k}.$$

(This identity will resurface in the next question as well as in the first two questions of the second part of the problem set, so it is important for teachers to make sure their students grasp this concept.) Just as in problem set one, students will see that both methods led them to the correct answer of ten possible paths.

Question 2: In a town that measures 7 units North/South and 8 units East/West, how many possible way are there to get from $A(0,0)$ to $B(8,7)$ again remembering that you may only move North and East?

For this question teachers should not immediately discourage students from listing out the options, as they will figure out on their own that it is neither a productive use of their time nor is there an effective way to list over 6,000 items by hand. With that in mind, teachers should refer to **Problem 1** and have students recall what they needed to do to figure out how to set up the combinations. Even math that is uncomfortable for students can be made easier when it has been seen before so this should help guide the students to work on this more complex question. In the previous problem, we focused on the number of total moves and then whether we wanted to choose the moves North or East. So, in this problem, there 15 moves total needed to get from A to B: $7Ns$ and $8Es$. Therefore, for this question, the number of possible paths is given by either

$$\binom{15}{7} = \frac{15!}{7!(15-7)!}$$

or

$$\binom{15}{8} = \frac{15!}{8!(15-8)!}$$

both of which will give us

$$\frac{15!}{(7!)(8!)} = 6,435$$

possible paths.

Question 3: In a town that measures 4 units North/South and 4 units East/West, how many possible ways are there to get from $A(0,0)$ to $B(4,4)$ only moving North and East if now you may not go above, but you may touch, the diagonal that connects A and B?

The only way to approach this problem without higher level combinatorics that utilizes the Catalan Numbers, is to simply list out the possibilities. It would be beneficial for teachers to give students an ample amount of time to generate the lists on their own or in groups so they can be sure to be thorough in their efforts. In order to get students thinking about the restrictions, teachers should plot A and B on a coordinate plane at $(0,0)$ and $(4,4)$ respectively and then draw the straight line that connects the two points. This is a challenging exercise, but it is very beneficial to encourage students to develop organization in their listing system. If students seem to be getting stuck, the teacher might want to go ahead and give them the number of possible paths so that students can work towards that goal and have a way to know that there effort is complete. This information should not be offered right away though as teachers do not want to keep anyone from having the satisfaction of coming up with the answer on their own. The number of possibilities is given by the fourth Catalan Number,

$$C_4 = \frac{1}{5} * \binom{8}{4} = 14.$$

The goal, however, is for the students to generate the following list of possible paths.

- $EEEENNNN$; $EEENENNN$; $EENEENNN$; $ENEEENNN$; $ENEENENN$; $ENENEENN$; $ENENENEN$; $EENNENEN$; $EENNEENN$; $EENENENN$; $EENENNEN$; $EEENNENN$; $EEENNNEN$; $ENEENNEN$

While generating this list either as a class or in groups, it is important to emphasize which paths are not possible; to do this, teachers should encourage students to look at the graph and the diagonal line as they are working. Hopefully as students start to see which paths cross the diagonal, and thus violate the necessary conditions, they will create the generalization that there may never more Ns than Es. If there were, the path would then cross the diagonal. To enhance the lesson, or if a teacher is short on time, she may scale the problem down so it is looking at the paths from $(0, 0)$ to $(3, 3)$ instead, and work this in addition to the problem above or in place of depending on the situation. For this adjusted problem, the total number of possible paths is the third Catalan Number,

$$C_3 = \frac{1}{4} * \binom{6}{3} = 5$$

, and the list of possible paths is as follows:

- $EEENNN$; $EENENN$; $ENEENN$; $ENENEN$; $EENNEN$

6.2 Score Questions

In this set of questions, two teams, the Eagles and the Ninjas, are playing soccer, and students must determine the number of possible partial scores that could have led to the given final score under different restrictions. Scores at the end of sports games or competitions are certainly something that students have encountered before. These problems can be generalized to a certain sport if a particular school is strong in a given sport or if there are large number of athletes in a given class. This way, the material is even more authentic for the students. Additionally, scores can be written down and listed as they are thought about. Since the goal of this second part of the problem set is to draw connections about the isomorphic nature of the problems, it is imperative that the teachers guide students to be able to make those generalizations. This includes setting up lists in similar ways and following similar procedures. If teachers completely stray from what has already been completed, students will be at a disadvantage for discovering the isomorphic identities of the questions. It is important for teachers to understand, however, that the discovery that these problems are isomorphic might not come immediately to students; in fact, it might require some further exploration and reflection on the material. This problem set, though, does provide a good start to developing an understanding of a very important mathematical concept while simultaneously delving deeper into combinatorial problems.

Question 4: How many different partial scores could there have been leading to the final score of Eagles 3 - Ninjas 2?

For this question, like Question 1 of the problem set, some students will be more comfortable listing out the different possible scores. With the intention of leading students towards understanding the overlap of these questions, teachers should encourage their students to denote each set of partial scores as a list of who scored and in what order. For example, if the Eagles scored first, then the Ninjas, then the Eagles, Ninjas, and Eagles again, students should list that set of partial scores as $ENENE$. So the each item or partial score would be a sequence with Es representing Eagles' points and Ns representing Ninjas' points. The list will look as follows:

- $NNEEE$; $NENEE$; $NEENE$; $NEEEN$; $ENEEN$; $EENEN$; $EEENN$; $EENNE$; $ENENE$; $ENNEE$

The list contains 10 items representing 10 possible partial scores. For students who would rather approach the question using combinations, it must first be recognized that there were 5 points that were scored in all in the game. Of those five points, the Eagles scored 3 and the Ninjas scored 2. Thus, the combination is set up as how many different ways are there for the Ninjas to score their two points, and analogously, how many different ways are there for the Eagles to score their 3 points. So, students would hopefully develop the combination $\binom{5}{2}$ or the combination $\binom{5}{3}$ which are both equivalent. Therefore, the number of partial scores that could lead to this final score is

$$\binom{5}{2} = \binom{5}{3} = \frac{5!}{(3!)(2!)} = 10.$$

Question 5: In a high scoring game, the Ninjas keep the game close but the Eagles are still able to win 8 - 7. How many different partial scores could have led to this particular outcome?

As in Question 2 of the problem set, there are way too many of these to list out; however, teachers should not immediately discourage the method. Either giving students time to start generating the list or presenting the combinations approach first should quickly deter students from trying to list out all the different possibilities. Again, teachers should use the previous question as a guide to help students conceptualize how to turn this question into a combinatorial application. Since the Eagles scored 8 points and the Ninjas scored 7 points, there were 15 points scored all together. Therefore, this can be thought of as the number of different ways the Eagles could have scored their 8 points which would be represented as

$$\binom{15}{8} = \frac{15!}{8!(15-8)!}.$$

Or students can think of the number of different ways the Ninjas could have scored their 7 points:

$$\binom{15}{7} = \frac{15!}{7!(15-7)!}.$$

Both expressions are equivalent, so the total number of possible partial scores is given by

$$\frac{15!}{(7!)(8!)} = 6,435.$$

Question 6: This final game ended up tied: Eagles 4 - Ninjas 4. If the Ninjas were never ahead during the game, though they could have been tied, how many possible partial scores are there leading to this outcome?

The best way to go about this problem is by listing the possible sequences of scores, just like students did in Question 4. As lists are being generated, either as a class or in groups, special attention should be paid to and conversations had about which scores are not possible. Noticing that the Ninjas may never be ahead and employing the listing technique utilized in Question 3, students will hopefully make the connection that their list can never have more Ns than Es since the Ninjas may never have more points than the Eagles. From this understanding, the list of partial scores will look as follows:

- $EEEENNNN$; $EEENENNN$; $EENEENNN$; $ENEEENNN$; $ENEENENN$; $ENENEENN$; $ENENENEN$; $EENNENEN$; $EENNEENN$; $EENENENN$; $EENENNEN$; $EEENNENN$; $EEENNNEN$; $ENEENNEN$

As in Question 3, this list might take a while to fully develop, but the process of coming up with the list is very important so students should not be rushed. Additionally, it is important to notice that while some students did not understand the list that was the solution to Question 3, they might understand this much better simply because they are picturing a score board rather than a map. This is great as more students are being reached but the same mathematical and combinatorial concepts are being applied. The overall result then is more students understand more math, which is a very positive result of this problem set. Similarly to the first part of the problem set, this question can be modified to a 3-3 tie in order to save time of enhance the lesson. In that case, the number of possible partial scores is 5, and the list is represented below.

- $EEENNN$; $EENENN$; $ENEENN$; $ENENEN$; $EENNEN$

Although this problem set may seem to have gotten tedious, the sequence of questions provides a thorough guide to how many combinatorial questions are essentially asking the same thing but are just hidden under different covers. Teachers can implement this problem set in a classroom in a couple different ways. If a school is on a block schedule and thus the math class has approximately 90 minutes each day, then a teacher could try and work through this Isomorphic Problem set in just one day; otherwise, clearly the problem set can be broken up into 2 sections and worked on consecutive days. It is important though that the problem set is worked in a condensed period of time in order to reach the most students that will be able to make the connections that the problems are the same. If the problem set is too spread out, the short term memory of high school students will win and thus the underlying theme will be lost. After completing the problems, teachers could assign students to look through the work they have done to compare and contrast the two different parts of the problem set. Once students are given the opportunity to look back and reflect, they will hopefully see, even if it is simply from the fact that they used the same operations, that two seemingly different problems are actually the same. The goal of this problem set, is for students to reach that coveted "ah-hah" moment that the combinatorial and mathematical reasoning and operations are the exact same on these different problems. This lesson is extended as students are informed that this phenomenom does not only occur in combinatorics, but rather isomorphic problems exist throughout many aspects of mathematics. Indeed, a large portion of understanding complex math questions is figuring out the different ways to look at the question so that a workable, understandable question might surface. Once that realization happens, then many seemingly impossible questions are approachable. This is an idea that can be very intriguing to students and should be emphasized to show its importance.

7 Conclusion

This paper and the work of my senior project focused on the current issues in high school mathematical education, the proposed reforms to high school mathematical education, and the power that combinatorics holds to support those proposed reforms. Additionally, the two problem sets I developed can provide high school teachers with a pre-formulated way to expand on combinatorial ideas in their classrooms. The problem sets are effective on many different levels. Most importantly though, they take intermediate combinatorial ideas and put them into a very accessible context for students of all ability levels and learning types. These questions all dealt with situations and backgrounds that are familiar to most high school students thus bringing the normally distant and disconnected subject of math into the realm of their everyday life. Additionally these questions serve as just some examples of how well combinatorics supports the proposed reforms and standards delineated for high school math classrooms in order to make high school math education a more holistic and idea based subject rather than just the regurgitation of formulas and techniques.

Although I do not know yet what level of high school math I will be teaching in approximately a year and a half, I do know that I will be in a high school math classroom and I plan on implementing much, if not all, of what I have learned throughout the process of developing this project. Combinatorics is just an example of all of the exciting and approachable types of mathematics that are available that we should truly be exposing students to at lower levels of math. Combinatorial problems at a high school level provide a great starting off point in order to guide students to explore the many different ways to think about situations and problems. This would encourage students to stay interested in math, hopefully, and continue pursuing the subject. Ultimately, though, the goal is to get students intrigued and excited about math and to break down the "I hate math" wall that so many students have built up around themselves by the time they reach high school. There is so much more to offer students than just formulas and techniques, high school math teachers have the ability to shape their way of thinking so they can become mathematical thinkers and problem solvers.

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