# THE STRUCTURE OF DIFFERENTIAL MANIFOLDS VIA MORSE THEORY

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## 1. Introduction

A fundamental question in any branch of mathematics is the classification problem. In the case of Topology, our question can be translated into the problem of classifying spaces; specifically, classifying differentiable manifolds up to equivalence class. Classically, we consider the equivalence classes of manifolds up to homeomorphism. However, in this paper we will focus on the question of classifying manifolds up to homotopy, although we will address the issue of classification up to homeomorphism as well. For instance, let us examine the torus $T^2$ and the sphere $S^2$. Intuitively, it seems that they are not topologically equivalent. Note that the torus has a *hole* and the sphere does not (see Figure 1). In order to formally prove that they are not topologically equivalent, we need a concrete way to examine them.

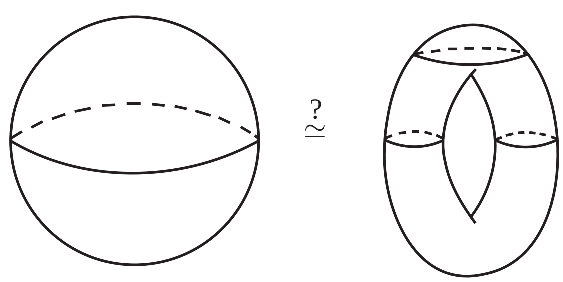

FIGURE 1. $S^2$ and $T^2$

A strikingly simple and effective way to do this is to look at *horizontal slices* of these manifolds and observe what these cross-sections look like. Back to our example, we see that the slices of $S^2$ generally are either circles, empty, or points at the north and south poles of the sphere. However, when we look at $T^2$, we see that the cross-sections generally are either circles, empty, points at the north and south poles like the sphere; but in addition, the slices of the torus look like two disjoint circles around the hole. Intuitively, this is a compelling reason to argue that $S^2$ and $T^2$ are not equivalent. In order to look at these slices more closely, we are going to consider real-valued functions on the manifolds. Our philosophy is that manifolds which fall into different homotopy equivalence classes would have different kinds of function on them. However, before we can study these functions, we must know exactly what a differentiable manifold is and what functions on it we want to study. Hence, we begin the paper with their definitions and some properties.

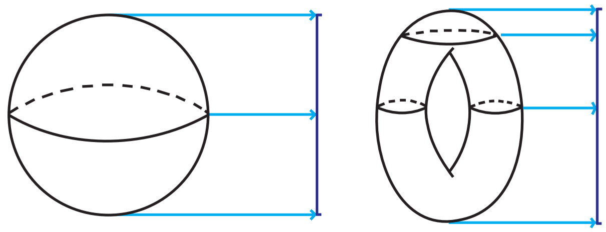

FIGURE 2. Height functions on $S^2$ and $T^2$

It turns out that differentiable manifolds locally look like the Euclidean space $\mathbb{R}^n$. Therefore, Analysis is a natural tool to use in studying these functions. More specifically, since we can diffeomorphically map a neighborhood of each point on a manifold to a subset of $\mathbb{R}^n$, it is not difficult to apply results from Analysis to our study of differentiable manifolds.

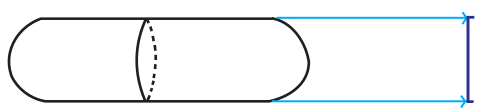

FIGURE 3. Height function without isolated critical points

Next, we look at functions and observe that the height functions are nice candidates for our study (see Figure 2). Height functions allow us to easily look at slices of manifolds since each slice is the pre-image of a single point in $\mathbb{R}$. However, not all height functions will be of interest to us. Consider the height function in Figure 3. We have infinitely many points mapping to a critical value. It turns out that this is a highly undesirable condition because it does not allow us to *isolate* critical points. However, we see in Figure 4 that any slight global perturbation of the manifold, such as tilting, yields a height function with isolated critical points.

In view of the discussion above, it becomes natural to consider the *Hessian* of a map. Using Analysis, we see that if the Hessian is *non-degenerate*, the critical points of the function are isolated. Hence, we

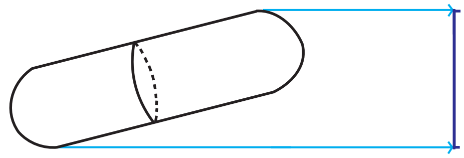

FIGURE 4. Height function with isolated critical points

will study maps $f : M \to \mathbb{R}$ with a non-degenerate Hessian where $M$ is a compact differentiable manifold. We observe that compactness allows some degree of finiteness in our results, since continuous functions on compact sets attain their maximums and minimums, and so have finitely many critical points.

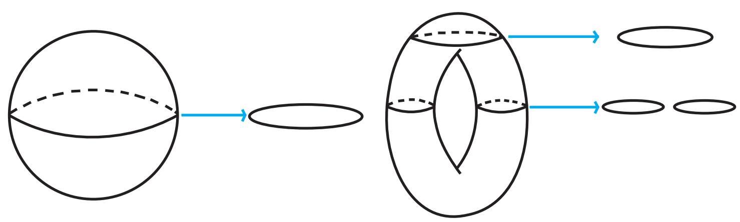

FIGURE 5. Horizontal cross-sections (slices) of $S^2$ and $T^2$

Once we consider the real-valued maps, we re-evaluate our discussion of height functions on the torus and sphere. From Figure 5, we can geometrically observe that a topological change occurs in the slices at critical points. For instance, the number of connected components changes. Hence, we say that the level sets only change at critical points. An equivalent way to say this is that the slices do not change between critical points. These two statements give us two very important results. The first statement leads us to a result called Morse Lemma, which is proven using almost entirely Analytical techniques. We note that Morse Lemma gives us a local description of the manifold at critical points. In contrast, the second statement leads us to a result called the Flow Lemma, which shows that between critical points the level sets are homeomorphic to one another via the gradient flow. This is a topological result that uses ideas from Dynamical Systems and allows us to give a global characterization of the manifold. We provide the details of both of these proofs in sections 2 and 3 of the paper.

From the discussion above, we are now interested in seeing exactly happens to the structure of the manifolds at these critical points. We know that the slices change, but we would like to know how exactly the structure of the manifold changes. An application of the Morse Lemma and the Flow Lemma yields that at a critical point of index $\lambda$ we change the topology by attaching a $\lambda$-handle to the level set. This result is formalized in the Fundamental Structure Theorem, which we prove in section 3.2.

We conclude the paper with some applications of Morse theory to the study of differentiable manifolds. Among the most important of these applications is Reeb Theorem, which states that any manifold that admits a Morse function with exactly two critical points is homeomorphic to a sphere. Finally, we return to our example comparing the sphere and the torus to consider the Poincaré polynomial associated with each manifold. Using this polynomial, we show that there is a lower bound on the number of index-1 critical points for $T^2$, showing that $T^2$ and $S^2$ cannot be *homotopic*. In addition, we can use the Poincaré polynomial to show that the torus does not admit a Morse function with two critical points, thus showing by way of Reeb Theorem that $T^2$ and $S^2$ cannot be *homeomorphic*.

1.1. **Differentiable Manifolds.** We begin with the definition of a differentiable manifold. For a more in-depth discussion of this definition, see [7] and [11].

**Definition 1.** *An n-*dimensional differentiable manifold *M is a Hausdorff topological space that has a covering of countably many open sets* $U_1, U_2, \ldots$ *satisfying the following conditions:*

*(i) For each* $U_i$ *there is a homeomorphism* $\psi_i : U_i \to \mathbb{R}^n$.

*(ii) If* $U_i \cap U_j \neq \emptyset$, *the homeomorphisms* $\psi_i$ *and* $\psi_j$ *combine to give a diffeomorphism* $\psi_{ji} = \psi_j \psi_i^{-1}$ *of* $\psi_i(U_i \cap U_j)$ *onto* $\psi_j(U_i \cap U_j)$.

*The pair* $(U_i, \psi_i)$ *is called a* coordinate chart, *and the set of all coordinate charts, called the* atlas, *is denoted* $\{U_i, \psi_i\}$.

So, for each $i$, $\psi_i$ maps $U_i$ to an open subset of $\mathbb{R}^n$. Thus, each point $p \in M$ is contained in a neighborhood, say $U_p$, which is diffeomorphic to $\mathbb{R}^n$. In simpler terms, an $n$-dimensional differentiable manifold locally looks like $\mathbb{R}^n$. We see an example of a 2-dimensional differentiable manifold in Figure 6. The neighborhood $U_p$ is mapped to a subset of $R^2$ by $\varphi_p$.

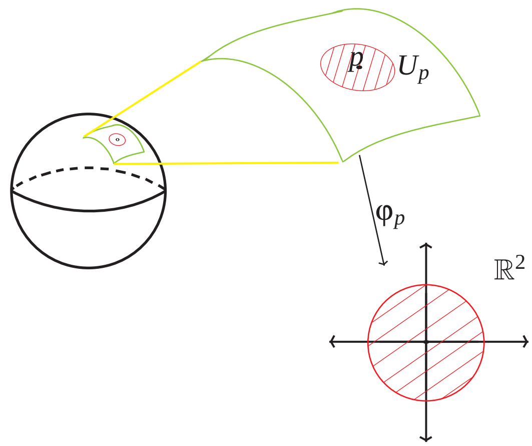


FIGURE 6. The coordinate function for a point $p \in S^2$

We now define the *tangent space* at a point $p$, an important concept needed later in the paper. For a more detailed discussion, see [7] and [11].

**Definition 2.** *Let $M$ be an $n$-dimensional differentiable manifold contained as a submanifold of $\mathbb{R}^m$ for some $m > n$, and let $p \in M$. Then the* tangent space *of $M$ at $p$, denoted $T_pM$, is the set of all velocity vectors of curves contained in $M$ which pass through $p$.*

**Remark 1.** *Notice that for any point $p \in \mathbb{R}^n$, $T_p\mathbb{R}^n$ is $\mathbb{R}^n$ itself.*

Given a differentiable manifold, there is a natural structure that we can put on it which allows us to consider distance and angles; that is, we have a Riemannian metric. The fact that $M$ has a countable cover and is indeed Hausdorff ensures that a metric always exists (see [7]). Once we have a metric, we can consider an inner product and gradient as follows:

**Definition 3.** *Let $M$ be a differentiable manifold. Choose a Riemmanian metric on $M$ and let $\langle X, Y \rangle$ denote the inner product of two tangent vectors. Then, given $f : M \to \mathbb{R}$, the* gradient *of $f$ is the vector field $\nabla f$ on $M$ which is characterized by the identity $\langle X, \nabla f \rangle_g = X(f)$.*

A standard concept is the idea of a critical point, which we now define.

**Definition 4.** *Let $f$ be a smooth real-valued function on a manifold $M$. A point $p \in M$ is a* critical point *of $f$ if the induced map $df : T_pM \to T_{f(p)}\mathbb{R}$ is zero.*

Note that this vector field vanishes at the critical points of $f$.

1.2. **Morse Functions.** Before we discuss what Morse functions are, let us consider some preliminary Analytical concepts. We begin by defining the Hessian, which plays a very important role. It lets us characterize the functions we want to consider when studying differentiable manifolds, and also lets us define the *index* of a critical point. Note that we define the Hessian only for maps in Euclidean space. The nature of differentiable manifolds allows us to extend this definition to fit our purposes. More specifically, because a differentiable manifold is locally diffeomorphic to $R^n$ with diffeomorphic transition functions and we assume smoothness in the atlas, we can apply the Analytic notion of the Hessian to manifolds. For more details see [1], [3], and [4].

**Definition 5.** *Given a smooth function* $f : \mathbb{R}^n \to \mathbb{R}$ *the* Hessian *of* $f$ *is the* $n \times n$ *matrix*

$$H_f(x) = \left[ \frac{\partial^2 f}{\partial x_i x_j} \right].$$

We can now define exactly the functions which we wish to study.

**Definition 6.** *Given a compact differentiable manifold* $M$*, a function* $f : M \to \mathbb{R}$ *is a* Morse function *if* $\det H_f(p) \neq 0$ *for every critical point* $p$ *of* $f$.

As mentioned before, the Hessian also allows us to define the very important concept of the *index* of a critical point.

**Definition 7.** *Given a differentiable manifold and a Morse function* $f : M \to \mathbb{R}$ *with critical point* $p$ *the* index *of* $p$ *is the dimension of the largest negative definite subspace of* $H_f(p)$.

1.3. **Analytical Results.** In this section, we revisit several results from Analysis which prove to be very important. We begin with the Inverse Function Theorem, a famous analytical result, adapted to differentiable manifolds by Lee in [7].

**Inverse Function Theorem for Manifolds.** *Suppose* $M$ *and* $N$ *are smooth manifolds,* $p \in M$*, and* $dF : T_pM \to T_{F(p)}N$ *is bijective. Then there exist connected neighborhoods* $U$ *of* $p$ *and* $V$ *of* $F(p)$ *such that* $F|_U : U \to V$ *is a diffeomorphism.*

We skip the proof of this important result as we choose instead to focus on its pertinent application in the proof of Morse Lemma. For a detailed proof, see [7].

Our first result shows that non-degenerate critical points of Morse functions are isolated. First, we prove a technical lemma on the submultiplicity of norms of invertible matrices.

**Lemma 1.** *If $H : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then there exists $c > 0$ such that $|Hx| \geq c|x|$.*

*Proof.* Let $H$ be as above and take $c = \dfrac{1}{|H^{-1}|}$. Then

$$|x| = |H^{-1}(Hx)| \leq |H^{-1}||Hx| = \frac{1}{c}|Hx|.$$

□

**Theorem 2.** *Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable for all $1 \leq i \leq n$. If $A = \left[\dfrac{\partial f_i}{\partial x_j}(a)\right]$ has a non-zero determinant, then there exists $\delta > 0$ such that $F(x) \neq F(a)$ with $0 < |x - a| < \delta$.*

*Proof.* Since $f_i \in C^1$ we have that Taylor's Formula holds. So, for all $i$

$$f_i(x) = f_i(a) + \sum_{j=1}^{\infty} \frac{\partial f_i}{\partial x_j}(a) \cdot (x_j - a) + \rho_i(x)|x - a|$$

where $\rho_i(x)|x-a|$ is the error term with the property that $\lim_{x \to a} \rho_i(x) \to 0$. Then,

$$F(x) = F(a) + A \cdot (x - a) + R(x)|x - a|$$

where $R(x) = (\rho_1(x), \ldots, \rho_n(x))$, and thus $\lim_{x \to a} R(x) = (0, \ldots, 0)$. Now using Lemma 1, we can pick $c$ so that $|Ax| \leq c|x|$. Next, there exists $\delta > 0$ such that for all $x \in \mathbb{R}, 0 < |x - a| < \delta$ and $|R(x)| < \dfrac{c}{2}$. Hence,

$$\begin{aligned} |F(x) - F(a)| &\geq |A(x - a)| - |R(x)||x - a| \\ &\geq c|x - a| - \tfrac{c}{2}|x - a| \\ &\geq \tfrac{c}{2}|x - a| \\ &> 0. \end{aligned}$$

□

For a more detailed treatment of such Analytical results, see [1] and [10]. Using these results, we can prove a corollary which confirms our initial observation about non-degenerate critical points.

**Corollary 1.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ such that $F \in C^2$. Then every non-degenerate critical point of $F$ is isolated.*

*Proof.* Let $F$ be defined as above and let $p$ be a critical point of $F$. Let $f = (f_1, \ldots, f_n)$ where $f_i = \dfrac{\partial F}{\partial x_i}$. Then $f(p) = 0$ and $f'(p) = \dfrac{\partial F}{\partial x_i \partial x_j}(p) \neq 0$ since $p$ is non-degenerate. Then $\left[\dfrac{\partial F}{\partial x_i \partial x_j}(p)\right]$ has a non-zero determinant. Thus by Theorem 1, there exists $\delta > 0$ such that for all $x \in \mathbb{R}^n$ such that $0 < |x - p| < \delta$, we have that $f(x) \neq f(p)$. Thus, $p$ is isolated. □

## 2. Morse Lemma

In this section, we provide a detailed proof of Morse Lemma, which gives us a local description of a manifold in a neighborhood of its critical points. In simple terms, it states that regardless of the coordinate map on a manifold $M$ there is a change of coordinates around a critical point which gives us a quadratic expression of $f$ depending on its index. Before we can state and prove it, we need a technical lemma.

**Lemma 2.** *Let $f : V \to \mathbb{R}$ be a smooth function, where $V$ is a convex neighborhood of $0$ in $\mathbb{R}^n$, and let $f(0) = 0$. Then there exist smooth functions, $g_i$, defined on $V$ such that*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$$

*for some suitable smooth functions $g_i$ defined in $V$.*

*Proof.* Let $f$ be defined as above, where $f(0) = 0$. Then, by the Fundamental Theorem of Calculus, we have that

$$\begin{aligned} f(x_1, \ldots, x_n) &= f(x_1, \ldots, x_n) - f(0) \\ &= \int_0^1 \frac{df}{dt}(tx_1, \ldots, tx_n)\, dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_1}(tx_1, \ldots, tx_n)\, x_i dt. \end{aligned}$$

Since the sum is finite, we have that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt.$$

Let us define $g_i(x_1, \ldots, x_n) = \displaystyle\int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt$. Then $f(x_1, \ldots, x_n) = \displaystyle\sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$. In particular, observe that $g_i(0) = \dfrac{\partial f}{\partial x_i}(0)$.

□

**Morse Lemma.** *Let $M$ be a differentiable manifold and let $f : M \to \mathbb{R}$ be a smooth function where $p$ is critical point of $f$. Then there is a local coordinate system $(y_1, \dots, y_n)$ in a neighborhood, $U$ of $p$ with $y_i(p) = 0$ for all $i$ and such that the identity*

$$f(y_1, \dots, y_n) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$$

*holds where $\lambda$ is the index of $f$ at $p$.*

The general idea of the proof is that we can use Lemma 2.1 to obtain a symmetric expression of $f$ at a critical point. Then, following Milnor's proof in [8], we can use a diagonalization technique for symmetric forms to obtain our desired coordinate change. Thus, we can show that regardless of the given expression of $f$, we can always change coordinates to express it in the desired form. We conclude the proof by showing that $\lambda$ is indeed the index of $f$.

*Proof.* Let $f$ be a smooth, real-valued function on a manifold $M$, and $p$ be a critical point of $f$. By translations, we can assume without loss of generality that $f(p) = f(0) = 0$. Let $(U, \psi)$ be a coordinate chart for $p \in M$. Apply Lemma 2.1 to $f$, so that for $j = 1, 2, \dots, n$, there exist smooth functions $g_j(x_1, \dots, x_n)$ such that

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n} x_j g_j(x_1, \dots, x_n).$$

Observing that the proof of Lemma was constructive, we see that $g_i(0) = \dfrac{\partial f}{\partial x_j}(0) = 0$ as it is a critical point. Therefore, we can apply Lemma 2.1 to $g_j$ to get

$$g_j(x_1, \dots, x_n) = \sum_{i=1}^{n} x_j h_{ij}(x_1, \dots, x_n), \text{ for each } j = 1, \dots, n. \tag{1}$$

Substituting (1) into our original expression of $f$, we have

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \dots, x_n)$$

where $h_{ij}(0) = \dfrac{\partial g_j}{\partial x_i}(0)$. We claim that $h_{ij}$ may be considered as a symmetric function with respect to $i$ and $j$.

Indeed, let $\overline{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$. Then

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \dots, x_n) = \sum_{i,j=1}^{n} x_i x_j \overline{h}_{ij}(x_1, \dots, x_n).$$

First we note that $\overline{h}_{ij}$ is symmetric, since

$$\begin{aligned}\overline{h}_{ij} &= \tfrac{1}{2}\left(h_{ij}+h_{ji}\right)\\ &= \tfrac{1}{2}\left(h_{ji}+h_{ij}\right)\\ &= \overline{h}_{ji}.\end{aligned}$$

Next, for any $1 \leq i, j \leq n$, we see that

$$\begin{aligned}x_i x_j \overline{h}_{ij} + x_j x_i \overline{h}_{ji} &= x_i x_j \left(\tfrac{1}{2}\left(h_{ij}+h_{ji}\right)\right) + x_j x_i \left(\tfrac{1}{2}\left(h_{ji}+h_{ij}\right)\right)\\ &= x_i x_j \left(2 \cdot \tfrac{1}{2}\left(h_{ij}+h_{ji}\right)\right)\\ &= x_i x_j \left(h_{ij}+h_{ji}\right)\\ &= x_i x_j h_{ij} + x_j x_i h_{ji}.\end{aligned}$$

Hence, it is now clear that

$$f\left(x_1,\ldots,x_n\right) = \sum_{i,j=1}^{n} x_i x_j h_{ij}\left(x_1,\ldots,x_n\right) = \sum_{i,j=1}^{n} x_i x_j \overline{h}_{ij}\left(x_1,\ldots,x_n\right).$$

For the rest of the proof, we write the function $f\left(x_1,\ldots,x_n\right)$ as $f$; $h_{ij}\left(x_1,\ldots,x_n\right)$ as $h_{ij}$; and $\overline{h}_{ij}\left(x_1,\ldots,x_n\right)$ as $\overline{h}_{ij}$ for the sake of brevity.

We show that there exists an inductive transformation which gives us our desired expression of $f$ as quadratic forms. We begin by finding a change of coordinates where $f$ is quadratic with respect to the first variable. Finally, we iterate this process $n$ times total to obtain our desired expression of $f$. Let us now begin.

First, we may assume that $|h_{11}| \neq 0$. Notice, $h_{ij}(0) = \dfrac{\partial g_j}{\partial x_i}(0) = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)$. Since the Hessian, $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)$, is non-degenerate, there exists some $x_k$ such that $h_{k1} \neq 0$, where $1 < k \leq n$. Thus, there exists some linear transformation, say $L$, such that $L\left(x_1,\ldots,x_{k-1},x_k,\ldots,x_{n-1},x_n\right) = \left(x_k,x_2,\ldots,x_{k-1},x_1,\ldots,x_{n-1},x_n\right)$. So, $|h_{11}| \neq 0$.

Then, we have that the following holds for $u = \left(x_1,\ldots,x_n\right)$ throughout a neighborhood, $U_1$ of $f(p)$:

$$\begin{aligned}f &= \sum_{i,j=1}^{n} x_i x_j h_{ij}\\ &= x_1^2 h_{11} + x_1 x_2 h_{12} + \ldots + x_1 x_n h_{1n} + \sum_{i,j\geq 2}^{n} x_i x_j h_{ij}\\ &= \pm\left(\sqrt{|h_{11}|}x_1 + \frac{x_2 h_{12} + x_3 h_{13} + \ldots + x_n h_{1n}}{\sqrt{|h_{11}|}}\right)^2 + \sum_{i,j\geq 2}^{n} x_i x_j \overline{h}_{ij} - \left(\frac{x_2 h_{12} + x_3 h_{13} + \ldots + x_n h_{1n}}{\sqrt{|h_{11}|}}\right)^2.\end{aligned}$$

This provides a quadratic term using $x_{1j}$ for all $j = 1,2,\ldots,n$ and a remainder term with $x_{ij}$ where $i, j \geq 2$. So we may define a change of coordinate function $\varphi_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi_1(u) = (v_1,\ldots,v_n)$ where

$$v_1 = \sqrt{|h_{11}|}\left(x_1 + \frac{x_2h_{12} + \ldots + x_nh_{1n}}{|h_{11}|}\right) \text{ and } v_i = x_i \text{ for all } i \neq 1.$$

Thus, $f(u) = \pm v_1^2 + \sum_{i,j\geq 2}^{n} x_ix_jH_{ij}$.

We claim that $H_{ij} = \frac{h_{ij} - h_{1i}h_{1j}}{|h_{11}|}$ and is in fact symmetric. First, notice that

$$\begin{aligned} H_{ij} &= \overline{h}_{ij} - \frac{h_{1i}h_{1j}}{|h_{11}|} \\ &= \overline{h}_{ji} - \frac{h_{1j}h_{1i}}{|h_{11}|} \\ &= H_{ji}. \end{aligned}$$

Also, we see that

$$\begin{aligned} &\sum_{i,j\geq 2}^{n}\left(x_ix_j\overline{h}_{ij} - \left(\frac{x_2h_{12} + \ldots + x_nh_{1n}}{|h_{11}|}\right)^2\right) = \\ &= \left(x_2^2\overline{h}_{22} - \frac{x_2^2h_{12}^2}{|h_{11}|}\right) + \left(x_2x_3\overline{h}_{23} - \frac{x_2x_3h_{12}h_{13}}{|h_{11}|}\right) + \ldots + \left(x_n^2\overline{h}_{nn} - \frac{x_n^2h_{1n}^2}{|h_{11}|}\right) \\ &= x_2^2\left(\overline{h}_{22} - \frac{h_{12}^2}{|h_{11}|}\right) + x_2x_3\left(\overline{h}_{23} - \frac{h_{12}h_{13}}{|h_{11}|}\right) + \ldots + x_n^2\left(\overline{h}_{nn} - \frac{h_{1n}^2}{|h_{11}|}\right) \\ &= \sum_{i,j\geq 2}^{n} x_ix_j\left(\overline{h}_{ij} - \frac{h_{1i}h_{1j}}{|h_{11}|}\right) \\ &= \sum_{i,j\geq 2}^{n} x_ix_jH_{ij}. \end{aligned}$$

Thus,

$$f = \pm\left(\sqrt{|h_{11}|}x_1 + \frac{x_2h_{12} + x_3h_{13} + \ldots + x_nh_{1n}}{\sqrt{|h_{11}|}}\right)^2 + \sum_{i,j\geq 2}^{n} x_ix_jH_{ij},$$

where $H_{ij}$ is a symmetric form.

Observe that $\varphi_1$ is a local diffeomorphism. Hence from the Inverse Function Theorem, we can compose $\varphi_1$ with our original map in the atlas, $\psi$, perhaps in a smaller neighborhood contained in $U$, to obtain an expression which is quadratic on $x_1$.

We now provide the details of our observation. Indeed, note that

$$\begin{aligned} d\varphi_1 &= \frac{\partial \varphi_1}{\partial v_i} dv_i \\ &= \frac{\partial \varphi_1}{\partial v_1} dv_1 + \frac{\partial \varphi_1}{\partial v_2} dv_2 + \ldots + \frac{\partial \varphi_1}{\partial v_n} dv_n \\ &= \sqrt{|h_{11}|} dv_1 + 0 \cdot dv_2 + \ldots + 0 \cdot dv_n \\ &= \sqrt{|h_{11}|} dv_1 \neq 0. \end{aligned}$$

Then by the Inverse Function Theorem, there exist $U_2 \subset U_1$ and $V_2 \subset V_1$, neighborhoods of $\psi(p)$ and $\varphi_1(\psi(p))$ respectively, such that $\varphi_1|_{U_2} : U_2 \to V_2$ is a diffeomorphism. We now iterate this process $n-1$ more times. In the $r$th step, we obtain a function $\varphi_r : \mathbb{R}^n \to \mathbb{R}^n$ such that $(x_1, \ldots, x_n) \mapsto (v_1, \ldots, v_n)$ given by

$$v_i = x_i \text{ for all } i \neq r \text{ and } v_r = \sqrt{|h_{ii}|} \left( x_r + \sum_{i>r} \frac{x_i H_{ir}}{|H_{rr}|} \right).$$

Note that $\varphi_r$ is a local diffeomorphism. Next, by the Inverse Function Theorem, we may restrict $\varphi_r$ to a smaller neighborhood of $p$, say $U_r$. Since we iterate finitely many times, after the $n$th step, we find a neighborhood, namely $\cap_{i=1}^n U_i$, where $f = \pm v_1^2 + \ldots + \pm v_n^2$.

Finally, we show that this expression of $f$ is unique up to the number of $+$'s and $-$'s; that is, we may write $f = f(p) - v_1^2 - \ldots - v_\lambda^2 + v_{\lambda+1}^2 + \ldots + v_n^2$, for some $1 \leq \lambda \leq n$ and where $\lambda$ is unique. We do this by showing that $\lambda$ is the index of $f$ at the critical point $p$. Notice that

$$\frac{\partial^2 f}{\partial v_i \partial v_j}(p) = \begin{cases} -2 \text{ if } i = j \leq \lambda, \\ 2 \text{ if } i = j > \lambda, \\ 0 \text{ otherwise}. \end{cases}$$

So we have that

$$H_f(p) = \begin{bmatrix} -2 & 0 & \ldots & & & 0 \\ 0 & \ddots & & & & \\ \vdots & & -2 & & & \vdots \\ & & & 2 & & \\ & & & & \ddots & 0 \\ 0 & & \ldots & & 0 & 2 \end{bmatrix}.$$

So, there is a subspace $U$ of $T_pM$ of dimension $\lambda$ where $H_f(p)$ is negative definite and a subspace $V$ of dimension $n-\lambda$ where $H_f(p)$ is positive definite. We claim that $\lambda$ is the maximal dimension of a negative definite subspace. Indeed, suppose there is a subspace $U'$ of $T_pM$ of dimension $k \geq \lambda + 1$ where

$H_f(p)$ is negative definite. Since $U'$ and $V$ are both subspaces of $T_pM$ which is dimension $n$, and $k+n-\lambda \geq \lambda+1+n-\lambda = n+1$, we have that $U'$ and $V$ must intersect. This is a contradiction since $U'$ is negative definite and $V$ is positive definite. Thus, $U$ is the maximal negative definite subspace of $T_pM$ and therefore $\lambda$ is the index of $f$ at $p$. This concludes the proof. □

## 3. Reconstructing Manifolds Using Morse Theory

3.1. **The Flow Lemma.** In this section, we confirm our intuitive idea that the topology of the level sets does not change between critical points, as well as proving a theorem which allows us to completely describe the changes in topology at a critical level set using the index of the critical point. The first result is the Flow Lemma and the second result is the Fundamental Structure Theorem. Before we discuss these results, we need a definition about flows and 1-parameter diffeomorphisms.

**Definition 8.** *A* 1-parameter group of diffeomorphisms *of a manifold, $M$, is a smooth map $\varphi : \mathbb{R} \times M \to M$ such that*

(1) *For all $t \in \mathbb{R}$, $\varphi_t : M \to M$ defined by $\varphi_t(q) = \varphi(t,q)$ is a diffeomorphism of $M$ onto itself.*
(2) *For all $t,s \in \mathbb{R}$, we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$.*

Throughout this section, for a real-valued function $f$ on a compact differentiable manifold $M$, i.e. $f : M \to \mathbb{R}$, we let $M^a = f^{-1}(-\infty,a] = \{p \in M | f(p) \leq a\}$ for $a \in \mathbb{R}$.

**Flow Lemma.** *Let $f$ be a smooth real-valued function on a manifold $M$. Let $a < b$ and suppose that the set $f^{-1}[a,b]$, consisting of all $p \in M$ with $a \leq f(p) \leq b$, is compact and contains no critical points of $f$. Then $M^a$ is diffeomorphic to $M^b$. Furthermore, $M^a$ is a deformation retract of $M^b$, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence.*

The idea of this result is to use the gradient flow to carry each point of $M^a$ diffeomorphically up to $M^b$. Since the manifold is compact, we are guaranteed that the diameter of the neighborhoods we are using to flow does not vanish; hence we can always flow up to $M^b$. This establishes surjectivity. Injectivity will follow from the uniqueness of solutions in Ordinary Differential Equations. Finally, the flow will induce a family of diffeomorphisms by the transversality of the gradient. Before we continue with the proof, recall that a continuous map $H : M \times [0,1] \to M$ is a *deformation retraction* of $M$ onto $A \subseteq M$ if $H(a,t) = a$ for all $a \in M, t \in [0,1]$ and $H(x,0) = x$ for all $x \in M$ and $H(x,1) \in A$ for all $x \in M$.

*Proof.* Let $c:\mathbb{R}\to M$ be a curve and $\frac{dc}{dt}$ be its velocity vector. Note that $\left\langle \frac{dc}{dt},\nabla f\right\rangle = \frac{d}{dt}(f\circ c)$. Let $\rho : M\to\mathbb{R}$ be a smooth function defined by $\rho(x)=\frac{1}{\langle\nabla f(x),\nabla f(x)\rangle}$ throughout the compact set $f^{-1}[a,b]$ and which vanishes outside a compact neighborhood of this set.

Now consider the vector field defined by $X_p=\frac{1}{\langle\nabla f(p),\nabla f(p)\rangle}(\nabla f)(p)$ which is smooth and vanishes outside the compact set $f^{-1}[a,b]$. We consider the 1-parameter group of diffeomorphisms, $\varphi_t : M\to M$, generated by $X_p$ (see Lemma 2.4 in [8]). Our idea here is to use this parameterization of the gradient flow to push $M^b$ diffeomorphically onto $M^a$.

Then,

$$\begin{aligned}
\frac{df}{dt}(\varphi_t(p)) &= \frac{df(\varphi_t(p))}{dt}\\
&= \left\langle \frac{d\varphi_t(p)}{dt},\nabla f\right\rangle\\
&= \langle X_p(\varphi_t(p)),\nabla f\rangle\\
&= \left\langle \frac{\nabla f(\varphi_t(p))}{\langle\nabla f(\varphi_t(p)),\nabla f(\varphi_t(p))\rangle},\nabla f\right\rangle\\
&= 1.
\end{aligned}$$

So, the fucntion $F$ is linear and has derivative 1; thus $\varphi_t(p)\in f^{-1}[a,b]$.

Now, consider the diffeomorphism $\varphi_{b-a} : M\to M$. We will show that $\varphi_{b-a}$ carries $M^a$ diffeomorphically onto $M^b$. The idea of this process is that $\varphi_{b-a}$ takes a point in $M^a$ and carries it to $M^b$ along the flow of integral curves of the vector field given by the gradient. As shown by the computation above, this flow has the right *speed*. By the uniqueness of solutions of ordinary differential equations, we have that $\varphi_{b-a}$ is injective. We see that it is surjective since we could map $M^b$ to $M^a$ by carrying it backwards along the flow of integral curves. In addition, we have that $\varphi_{b-a}$ is continuous with a continuous inverse by the orthogonality of the gradient at the level sets of $M$.

Now, we show that $M^a$ is a deformation retract of $M^b$. To do so, let us define a 1-parameter family of maps $r_t : M^b\to M^b$ by

$$r_t(q)=\begin{cases} q, & \text{if } f(q)\le a;\\ \varphi_{t(a-f(q))}(q), & \text{if } a\le f(q)\le b.\end{cases}$$

Then $r_0$ is the identity, and $r_1$ is a retraction from $M^b$ to $M^a$. Hence $M^a$ is a deformation retract of $M^b$. For further details see [8].

□

3.2. **Fundamental Structure Theorem.** In this section, we will consider a compact manifold $M$, and a Morse function $f : M \to \mathbb{R}$. From our previous observation that the topology of the level sets does not change in regions without critical points, we can show that, $\lambda$, the index of a critical point which is an analytic property, allows us to completely describe the change in the topology of the level sets that occurs at the critical point. Namely, we *attach* a $\lambda$-cell to the level set. Now, we define these terms and provide a precise statement

**Definition 9.** *Let $\lambda \in \mathbb{N}$ and $D^\lambda$ be a $\lambda$-disk. Let $M$ be a manifold. We attach a $\lambda$-*handle *to $M$ by $f : \partial\left(D^\lambda\right) = S^{\lambda-1} \to M$. So, $M$ with a $\lambda$-handle attached is*

$$M \dot{\cup} D^\lambda \big/_{x \sim f(x), x \in \partial\left(D^\lambda\right)} = M \cup_f D^\lambda.$$

We now state the main result. Although our main concern is compact manifolds, we give the result in its full generality.

**Fundamental Structure Theorem.** *For $M$ a differentiable manifold, let $f : M \to \mathbb{R}$ be a smooth function and let $p$ be a non-degenerate critical point with index $\lambda$. Setting $f(p) = c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact, and contains no critical point of $f$ other than $p$ for some $\varepsilon > 0$. Then, for all sufficiently small $\varepsilon$, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} = \{x \in M : f(x) \leq c - \varepsilon\}$ with a $\lambda$-cell attached.*

The main idea of the proof is to define a new function $F$, which is the same as $f$ outside an $\varepsilon$-neighborhood of $f(p) = c$, but has no critical values between $c-\varepsilon$ and $c+\varepsilon$. Then, by an application of the Flow Lemma we will diffeomorphically shrink the manifold so it becomes a handle above $M^{c-\varepsilon}$.

*Proof.* From Morse Lemma, we have that there exists a neighborhood $N$ of $p$ such that the equality $f = c - \left(x_1^2 + \cdots + x_\lambda^2\right) + \left(x_{\lambda+1}^2 + \cdots + x_n^2\right)$ holds throughout $N$. For convenience, let us define $\xi(x_1, \ldots, x_n) = x_1^2 + \cdots + x_\lambda^2$ and $\eta(x_1, \ldots, x_n) = x_{\lambda+1}^2 + \cdots + x_n^2$. Then $f = c - \xi(x_1, \ldots, x_n) + \eta(x_1, \ldots, x_n)$. We let $U = \{x \in M : \xi(x_1, \ldots, x_n) + \eta(x_1, \ldots, x_n) < 2\varepsilon\}$. For convenience, we abbreviate $\xi(x_1, \ldots, x_n)$ and $\eta(x_1, \ldots, x_n)$ to $\xi$ and $\eta$, respectively, for the remainder of the proof.

We now build a function $F : M \to \mathbb{R}$ in terms of $f$ and a suitable function, $\mu$ as follows. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

(*i*) $\mu(0) > \varepsilon$

(*ii*) $\mu(r) = 0$ for $r \geq 2\varepsilon$

(*iii*) $-1 < \mu'(r) \leq 0$ for all $r \in \mathbb{R}$.

Note the graph of $\mu$ in Figure 7.

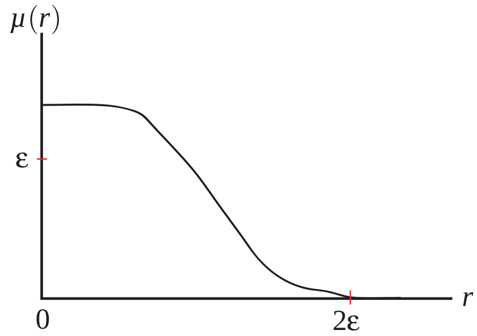


FIGURE 7. Graph of $\mu : \mathbb{R} \to \mathbb{R}$

Therefore consider $F : M \to \mathbb{R}$ such that $F(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) - \mu(\xi(x_1,\ldots,x_n) + 2\eta(x_1,\ldots,x_n))$ or simply, $F = f - \mu(\xi + 2\eta) = c - \xi + \eta - \mu(\xi + 2\eta)$.

Observe that outside the neighborhood $U$, $F$ and $f$ coincide by condition (*ii*). Thus, outside of $U$, the possible critical points of $F$ are trivially the same as those of $f$. However $f$ has no critical points outside of $U$; thus $F$ has no critical points outside $U$.

Now let us consider $F$ inside $U$ and in particular, investigate its possible critical points. First, let us show that $F^{-1}(-\infty, c+\varepsilon) = M^{c+\varepsilon}$. We have two cases: $\xi + 2\eta \geq 2\varepsilon$ or $\xi + 2\eta < 2\varepsilon$. We consider the first case. Suppose $\xi + 2\eta \geq 2\varepsilon$. Then $\xi + \eta \geq 2\varepsilon$ and $F = f - \mu(\xi + 2\eta) = f$. So, if $\xi + \eta \geq 2\varepsilon$, we have $F^{-1}(-\infty, c+\varepsilon) = M^{c+\varepsilon}$.

Now, we consider the second case and show that if $\xi + 2\eta < 2\varepsilon$, we have $F^{-1}(-\infty, c+\varepsilon) = M^{c+\varepsilon}$. Since $0 \leq \mu(\xi + 2\eta)$, we have that $F = f - \mu(\xi + 2\eta) \leq f$. Let $y \in M^{c+\varepsilon}$. Then, $F(y) \leq f(y) < c + \varepsilon$ and $y \in F^{-1}(-\infty, c+\varepsilon)$. So, $M^{c+\varepsilon} \subseteq F^{-1}(-\infty, c+\varepsilon)$.

What remains to be shown is that $F^{-1}(-\infty, c+\varepsilon) \subseteq M^{c+\varepsilon}$. Let $x \in F^{-1}(-\infty, c+\varepsilon)$. Then, $F(x) \leq f(x)$ by argument above.

Now, back to our analysis of the critical points of $F$. We have already determined that $F$ has no critical points outside $U$, we must only consider the possibility of a critical point inside $U$. Notice

$$\frac{\partial F}{\partial \xi} = 1 - 2\mu'(\xi + 2\eta) \geq 1 \text{ and}$$
$$\frac{\partial F}{\partial \eta} = -1 - \mu'(\xi + 2\eta) < 0$$

since $-1 < \mu'(r) \leq 0$ for all $r \in \mathbb{R}$. At a critical point, we have $dF = 0$. Then, since $dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$, we have that $d\xi = d\eta = 0$ at a critical point. This implies

$$(2x_1, \ldots, 2x_\lambda) = (2x_{\lambda+1}, \ldots, 2x_n) = 0,$$

which can only happen if $x = (0, \ldots, 0)$. The only point at which this is true is $p$. Thus, $p$ is the only possible critical point for $F$.

However, we see that

$$\begin{aligned} F(p) &= c - \xi + \eta - \mu(\xi + 2\eta) \\ &= c - \mu(0) \\ &< c - \varepsilon \end{aligned}$$

since $\mu(0) > \varepsilon$. Hence $F^{-1}[c-\varepsilon, c+\varepsilon]$ contains no critical points. So, we can apply the Flow Lemma.

In fact, we use it to show that $F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract of $M^{c+\varepsilon}$. Notice that since there are no critical values of $F$ between $c-\varepsilon$ and $c+\varepsilon$, we have by the Flow Lemma $F^{-1}(-\infty, c+\varepsilon] \cong F^{-1}(-\infty, c-\varepsilon]$. We remark that up to this point, our analysis was localizing our investigation of $M^{c+\varepsilon}$ to the region near the critical points, since away from it the Flow Lemma ensures that the topology of the level sets does not change.

In order to conclude, let $F^{-1}(-\infty, c+\varepsilon) = M^{c-\varepsilon} \cup \overline{F^{-1}(-\infty, c-\varepsilon] \setminus M^{c-\varepsilon}}$. If we can show that $H = \overline{F^{-1}(-\infty, c-\varepsilon] \setminus M^{c-\varepsilon}}$ retracts to a $\lambda$-disk, we can use the Glueing Lemma from point-set topology to continuously attach the two spaces (for details, see [2]). More precisely, let $D^\lambda$ be a $\lambda$-cell in $U$ with respect to the coordinates $x_1, \ldots, x_\lambda$. We will show that $D^\lambda \subseteq H$ and then that $D^\lambda$ is a deformation retract of $H$. This allows us to prove that $M^{c+\varepsilon}$ is diffeomorphic to $M^{c-\varepsilon}$ with a $\lambda$-handle ($D^\lambda$) attached.

Let us show that $D^\lambda \subseteq H$. Notice for all $x \in D^\lambda$ we have that $\eta = 0$. So for all such $x$, $F(x) = c - \xi - \mu(\xi) < c - \mu(0) < c - \varepsilon$. Indeed, we claim that $\xi + \mu(\xi) > \varepsilon$. Notice for all $r \in \mathbb{R}$ we have that $\mu(r) > -1$. Thus,

$$\begin{aligned} \int_0^{\xi(q)} \mu'(r)\, dr &> \int_0^{\xi(q)} -1 dr && \Rightarrow \\ \mu(\xi(q)) - \mu(0) &> 0 - \xi(q) && \Rightarrow \\ \mu(\xi(q)) - \mu(0) &> \xi(q). && \Rightarrow \\ \mu(\xi(q)) + \xi(q) &> \mu(0) > \varepsilon. \end{aligned}$$

So, $F(x) < c - \varepsilon$. Also, observe that $f(x) = c - \xi \geq c - \varepsilon$. Thus $D^\lambda \subseteq H$.

Notice, $\{x \in M : F(x) \leq c - \varepsilon\} \cong \{x \in M : f(x) \leq c - \varepsilon\} \cup D^{\lambda}$. So, $\{x \in M : F(x) \leq c - \varepsilon\} \setminus \{x \in M : f(x) \leq c - \varepsilon\} = H$. Then, $\{x \in M : F(x) \leq c - \varepsilon\} \cap \{x \in M : f(x) > c - \varepsilon\} \subseteq U$, which implies that $H \subseteq U$ since $F = f$ only inside $U$.

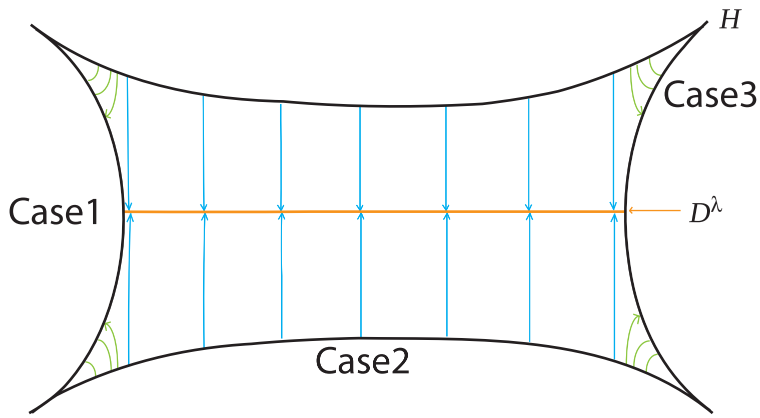


FIGURE 8. We retract $H$ to $D^{\lambda}$ according the these three cases.

Now we show that $D^{\lambda}$ is a deformation retraction of $H$, following Milnor's proof in [8]. From Figure 8, we observe that we can map the points identified in Case 1 by using the identity, and the points identified in Case 2 by using a linear retraction. Finally, for the points in case 3, we will use a nonlinear retraction. We proceed as follows. Let us define a function $r_t : M^{c-\varepsilon} \cup H \to M^{c-\varepsilon} \cup H$ where $r_t$ is the identity outside $U$, and $r_t$ is defined within $U$ according to the cases below:

**Case 1**: Within the region $\eta + \varepsilon \leq$ (that is, inside $M^{c-\varepsilon}$, let $r_t$ be the identity.

**Case 2**: Within the region $\xi \leq \varepsilon$, let $r_t(u_1, \ldots, u_n) = (u_1, \ldots, u_{\lambda}, tu_{\lambda+1}, \ldots, tu_n)$. Thus, $r_1$ is the identity, and $r_0$ maps the entire region into $D^{\lambda}$. The fact that each $r_t$ maps $F^{-1}(-\infty, c-\varepsilon]$ into itself follows from the inequality $\dfrac{\partial F}{\partial \eta} > 0$.

**Case 3**: Within the region $\varepsilon \leq \xi \leq \eta + \varepsilon$, let $r_t(u_1, \ldots, u_n) = (u_1, \ldots, u_{\lambda}, s_t u_{\lambda+1}, \ldots, s_t u_n)$ where $s_t \in [0,1]$ is defined by

$$s_t = t + (1-t)\left(\frac{\xi - \varepsilon}{\eta}\right)^{\frac{1}{2}}.$$

Then, $r_1$ is again the identity, and $r_0$ maps the entire region into $f^{-1}(c-\varepsilon)$. Note that this definition coincides with Case 2 if $\eta = \varepsilon$. Thus, $D^{\lambda}$ is a deformation retraction of $H$. $\square$

Notice that when $M$ is compact and $f$ is a Morse function, we can apply the Fundamental Structure Theorem, thus obtaining a global characterization of $M$ up to homotopy using the critical points of $f$ as a guide for the homotopic building blocks of $M$. Without loss of generality we can assume $p$ is the only critical point whose value is $c$. Indeed, if there were another such point, say $q$, where $f(q)=c$, we could diffeomorphically map a neighborhood of $q$ using a local gradient flow so that the value at $q$ was no longer at the level set $c$.

## 4. Applications of Morse Theory

In order to better see the usefulness of Morse theory, it is beneficial to see a concrete example of how the theory can be applied. Below, we introduce the theorem of Reeb, which states that all $n$-dimensional manifolds that admit a Morse function with exactly two non-generate critical points are homeomorphic to $S^n$. This fits in as a surprising result with the overall goal of Morse theory, which is to classify differentiable manifolds up to homotopy. This theorem goes even further, and gives us classification up to homeomorphism.

### 4.1. Characterization of $S^n$.

**Theorem 3** (Reeb)**.** *If $M$ is a compact manifold and $f$ is a differentiable function on $M$ with exactly two critical points, both non-degenerate, then $M$ is homeomorphic to a sphere.*

The general idea of this proof is to partition the manifold $M$ into two pieces - a bowl and a cap. Using the Flow Lemma and Morse Lemma, we can show that each of these two pieces is homeomorphic to a disk in $\mathbb{R}^n$. Then we can identify the boundaries of the two disks, thus building $S^n$, and the remaining task of explicitly defining the homeomorphism is not too difficult.

*Proof.* Let $p$ and $q$ be the non-degenerate critical points of $f$. Since $M$ is compact, these critical points must be a minimum and a maximum. Without loss of generality, assume $f(p)=0$ is the minimum and $f(q)=1$ is the maximum.

For sufficiently small $\varepsilon>0$, we can use the Flow Lemma to obtain that $M^{\varepsilon}$ is homeomorphic to $M^{1-\varepsilon}$. Thus $M=M^{1-\varepsilon}\dot{\cup}f^{-1}[1-\varepsilon,1]$. Since $p$ and $q$ are non-degenerate, we know from Morse Lemma that $f^{-1}[0,\varepsilon]$ and $f^{-1}[1-\varepsilon,1]$ are diffeomorphic to closed disks in $\mathbb{R}^n$, say $D_1^n$ and $D_2^n$ respectively. Without loss of generality, identify $D_1^n$ and $D_2^n$ to $\{x\in\mathbb{R}^n : |x|\leq 1\}$. (For details on identification, see [2].)

Let $\psi_1 : M^{\varepsilon}\to D_1^n$ and $\psi_2 : f^{-1}[1-\varepsilon,1]\to D_2^n$ denote these diffeomorphisms. Notice that $\psi_1(p)=0$ and $\psi_2(q)=0$. We have that $\partial(D_1^n)=S_1^{n-1}$ and $\partial(D_2^n)=S_2^{n-1}$, where $S_1^{n-1}$ and $S_2^{n-1}$ are copies of $S^{n-1}$.

Let $f : S_1^{n-1} \to S_2^{n-1}$ given by $f(x) = x$. This function identifies the boundaries of $D_1^n$ and $D_2^n$. Thus, the n-dimensional sphere $S^n = D_1^n \dot{\cup} D_2^n \big/ _f$.

Using the representation above, we mimic it to extend the identification of the boundary of $f^{-1}(0,\varepsilon)$ to $f^{-1}(1-\varepsilon,1)$ as a homeomorphism. Indeed, define the homeomorphism between $M$ and $S^n$ as follows. We first map $x \in M$ to a copy of $D^n$ by either $\psi_1$ or $\psi_2$, and then identify the boundaries of the two disks. Let $\varphi : M \to S^n$ be given by the following: $\begin{cases} \varphi(x) = \psi_1(x), & \text{if } x \in M^{1-\varepsilon}; \\ \varphi(x) = \psi_2(x), & \text{if } x \in f^{-1}(1-\varepsilon,1]. \end{cases}$

Thus, we have that $M$ is homeomorphic to $S^n$ by $f \circ \varphi$. □

4.2. **The Poincaré Polynomial.** Given a Morse function on a compact manifold $M$, we can associate a polynomial called the *Poincaré polynomial* to the Morse function. It is a known result that this polynomial gives us important topological information; for instance, it gives us the Euler number of a manifold, which is a homotopic invariant. Then, by way of Reeb Theorem, we can use this property to differentiate between two manifolds up to homeomorphism. We illustrate this by examining the Poincaré polynomial associated with height functions on $S^2$ and $T^2$, but first let us state the definition of the Poincaré polynomial associated with $f : M \to \mathbb{R}$.

**Definition 10.** *Given a Morse function $f$ on a compact n-dimensional differentiable manifold $M$, the* Poincaré polynomial *associated with $f$ is*

$$P_f(t) = \sum_{\lambda=1}^{n} c_\lambda t^\lambda,$$

*where $c_\lambda$ denotes the number of critical points of index $\lambda$.*

Notice that the Poincaré polynomial depends on the function $f$ because the summation depends on the critical points of $f$. This fact makes the following theorem about the Poincaré polynomial quite remarkable. We simply state it as a known result. Recall that $\chi(M)$ is the Euler number of a manifold, $M$.

**Theorem 4.** *Given a Morse function $f$ on a compact n-dimensional differentiable manifold $M$, we have that*

$$\chi(M) = P_f(-1).$$

Using the height functions pictured in Figure 2, we can compute the Euler numbers of $S^2$ and $T^2$. Let $f_1 : S^2 \to \mathbb{R}$ and $f_2 : T^2 \to \mathbb{R}$. First, we compute the Poincaré polynomial associated with the function $f_1$. We see that $c_0 = 1, c_1 = 0$, and $c_2 = 1$. So, $P_{f_1}(t) = 1 + t^2$. Using this to find the Euler number, we have that

$$\chi\left(S^2\right) = P_{f_1}(-1) = 1 + (-1)^2 = 1 + 1 = 2.$$

Now, we apply this same procedure to the function $f_2$ on $T^2$. From Figure 2, we see that $c_0 = 1, c_1 = 2$, and $c_2 = 1$. So, $P_{f_2}(t) = 1 + 2t + t^2$ and furthermore,

$$\chi\left(T^2\right) = P_{f_2}(-1) = 1 + 2(-1) + (-1)^2 = 1 - 2 + 1 = 0.$$

Thus, the torus is not homotopic to the sphere since they do not have the same Euler number.

Next, we use the Euler number of $T^2$ to show that the torus always has a lower bound on the number of its critical points of index 1. We now state and prove this proposition, which we will use to show that $S^2$ and $T^2$ are not homeomorphic.

**Proposition 1.** *Given any Morse function $f : T^2 \to \mathbb{R}$, $f$ always has at least two critical points of index 1. That is,* $c_1 \geq 2$.

*Proof.* Since $P_{f_2}(-1) = 0$, we have that $\chi\left(T^2\right) = 0$ for all Morse functions on $T^2$. Since Morse functions are defined as continuous functions on compact manifolds, we have that $c_0 \geq 1$ and $c_2 \geq 1$ for such function on $T^2$. Thus,

$$\begin{aligned} 0 &= \chi\left(T^2\right) \\ &= c_0 + c_1(-1) + c_2(-1)^2 \\ &= c_0 - c_1 + c_2 \\ &\geq 1 - c_1 + 1 \\ &= 2 - c_1. \end{aligned}$$

So, $c_1 \geq 2$. □

So, any Morse function on $T^2$ always has at least four critical points: one of index 0, two of index 1, and one of index 2. Thus, by the Reeb Theorem, the torus cannot be homeomorphic to the sphere.

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