Asymptotic preconditioning of linear homogeneous systems of differential equations

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Linear Algebra and Its Applications

Abstract

We consider the asymptotic behavior of solutions of a linear differential system $x' = A(t)x$, where $A$ is continuous on an interval $[a, \infty)$. We are interested in the situation where the system may not have a desirable asymptotic property such as stability, strict stability, uniform stability, or linear asymptotic equilibrium, but its solutions can be written as $x = Pu$, where $P$ is continuously differentiable on $[a, \infty)$ and $u$ is a solution of a system $u' = B(t)u$ that has the property in question. In this case we say that $P$ preconditions the given system for the property in question.

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1 Introduction

In this paper $J = [a, \infty)$ and $\mathbb{C}^n$, $\mathbb{C}^{n \times n}$, $\mathbb{C}_0^n(J)$, $\mathbb{C}_0^{n \times n}(J)$, $\mathbb{C}_1^n(J)$, and $\mathbb{C}_1^{n \times n}(J)$ are respectively the sets of $n$-vectors with complex entries, $n \times n$ matrices with complex entries, continuous complex $n$-vector functions on $J$, continuous complex $n \times n$ matrix functions on $J$, continuously differentiable $n$-vector functions on $J$, and continuously differentiable $n \times n$ complex matrix functions on $J$. ("Complex" and "$\mathbb{C}$" can

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just as well be replaced by “real” and “$\mathbb{R}$.” If $\xi \in \mathbb{C}^n$ and $C \in \mathbb{C}^{n \times n}$ then $\|\xi\|$ is a vector norm and $\|C\|$ is the corresponding induced matrix norm; i.e., $\|C\| = \max\{\|C\xi\| : \|\xi\| = 1\}$. Throughout the paper $A \in \mathbb{C}^{n \times n}(J)$, $\mathcal{S}_A$ is the set of solutions of

$$x' = A(t)x, \quad t \in J, \quad (1)$$

$\mathcal{J} = \{(t, \tau) \mid a \leq \tau \leq t\}$, and $\mathcal{R} = \{R \in \mathbb{C}^{n \times n}(J) \mid R^{-1} \in \mathbb{C}^{n \times n}(J)\}$.

We recall that if $X \in \mathbb{C}^{n \times n}(J)$ satisfies $X' = A(t)X$, $t \in J$, then either $X(t)$ is invertible for all $t \in J$ or $X(t)$ is noninvertible for all $t \in J$. In the first case $X$ is said to be a fundamental matrix for (1), and $x \in \mathcal{S}_A$ if and only if $x = X(t)\xi$ for some $\xi$ in $\mathbb{C}^n$ or, equivalently,

$$x(t) = X(t)X^{-1}(\tau)x(\tau) \quad \text{for all } t, \tau \in J.$$  

We begin with some standard definitions.

**Definition 1**

(a) Eq. (1) is *stable* if for each $\tau \in J$ there is a constant $M_\tau$ such that $\|x(t)\| \leq M_\tau \|x(\tau)\|$ for all $t \in J$ and $x \in \mathcal{S}_A$.

(b) Eq. (1) is *strictly stable* if there is a constant $M$ such that $\|x(t)\| \leq M \|x(\tau)\|$ for all $t, \tau \in J$ and $x \in \mathcal{S}_A$.

(c) Eq. (1) is *uniformly stable* if there is a constant $M$ such that $\|x(t)\| \leq M \|x(\tau)\|$ for all $(t, \tau) \in \mathcal{J}$ and $x \in \mathcal{S}_A$.

(d) Eq. (1) is *uniformly asymptotically stable* if there are constants $M$ and $\nu > 0$ such that $\|x(t)\| \leq M \|x(\tau)\|e^{-\nu(t-\tau)}$ for all $(t, \tau) \in \mathcal{J}$ and $x \in \mathcal{S}_A$.

(e) Eq. (1) has linear asymptotic equilibrium if every nontrivial solution of (1) approaches a nonzero constant vector as $t \to \infty$.

It is convenient to include (c) and (d) in the following definition, which may be new. Let $\rho$ be continuous and positive on $\mathcal{J}$ and suppose that

$$\rho(t, t) = 1 \quad \text{and} \quad \rho(t, \tau) \leq \rho(t, s)\rho(s, \tau), \quad a \leq \tau \leq s \leq t. \quad (2)$$

We say that (1) is *$\rho$-stable* if there is a constant $M$ such that

$$\|x(t)\| \leq M \|x(\tau)\|/\rho(t, \tau) \quad \text{for all } (t, \tau) \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.$$  

We consider the following problem: given a system that does not have one of the properties defined above, is it possible to analyze (1) in terms of a related system that has the property?

Henceforth $P$ is a given member of $\mathcal{R}$. We offer the following definition.

**Definition 2**

(a) Eq. (1) is *stable relative to $P$* if for each $\tau \in J$ there is a constant $M_\tau$ such that

$$\|P^{-1}(t)x(t)\| \leq M_\tau \|P^{-1}(\tau)x(\tau)\| \quad \text{for all } t, \tau \in J \text{ and } x \in \mathcal{S}_A.$$  

We recall that if $X \in \mathbb{C}^{n \times n}(J)$ satisfies $X' = A(t)X$, $t \in J$, then either $X(t)$ is invertible for all $t \in J$ or $X(t)$ is noninvertible for all $t \in J$. In the first case $X$ is said to be a fundamental matrix for (1), and $x \in \mathcal{S}_A$ if and only if $x = X(t)\xi$ for some $\xi$ in $\mathbb{C}^n$ or, equivalently,
Eq. (1) is strictly stable relative to \( P \) if there is a constant \( M \) such that
\[
\| P^{-1}(t) x(t) \| \leq M \| P^{-1}(\tau) x(\tau) \| \quad \text{for all } t, \tau \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.
\]

Eq. (1) is \( \rho \)-stable relative to \( P \) if there is a constant \( M \) such that
\[
\| P^{-1}(t) x(t) \| \leq M \| P^{-1}(\tau) x(\tau) \| / \rho(t, \tau) \quad \text{for all } (t, \tau) \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.
\]

Eq. (1) has linear asymptotic equilibrium relative to \( P \) if \( \lim_{t \to \infty} P^{-1}(t) x(t) \) exists and is nonzero for every nontrivial \( x \in \mathcal{S}_A \).

Lemma 1 If \( x \in \mathbb{C}^n_{\mathcal{J}}(\mathcal{J}) \) and \( u = P^{-1} x \), then \( x' = Ax, t \in \mathcal{J} \), if and only if
\[
\begin{align*}
  \quad u' &= P^{-1}(AP - P')u, & t \in \mathcal{J},
\end{align*}
\]

or, equivalently, if and only if \( x = PU \xi \) where \( U \) is a fundamental matrix for (3) and \( \xi \in \mathbb{C} \).

Proof. Since \( x = Pu, x' = Pu' + P'u \) and \( Ax = APu \), so \( x' = Ax \) if and only if \( Pu' + P'u = APu \), which is equivalent to (3).

To illustrate the problem that we study here, we cite a theorem attributed by Wintner [8] to Bôcher, which says that (1) has linear asymptotic equilibrium if \( \int_{\mathcal{J}} \| A(t) \| \, dt < \infty \). This theorem does not apply to (1) if \( \int_{\mathcal{J}} \| A(t) \| \, dt = \infty \), but, by Lemma 1 it does imply that (1) has linear asymptotic equilibrium relative to \( P \) if
\[
\int_{\mathcal{J}} \| P^{-1}(AP - P') \| \, dt < \infty.
\]

Adapting terminology commonly used in computational linear algebra, we will in this case refer to the transformation \( u = P^{-1} x \) as asymptotic preconditioning, and we say that \( P \) preconditions (1) for asymptotic equilibrium. More generally, if \( \mathcal{P} \) is a given property of linear differential systems (for example, one of the properties mentioned earlier), we say that \( P \) preconditions (1) for property \( \mathcal{P} \) if (3) has property \( \mathcal{P} \) or, equivalently, if (1) has property \( \mathcal{P} \) relative to \( P \).

This paper is strongly influenced by Conti’s work [2, 3, 4] on \( t_\infty \)-similarity of systems of differential equations and our extensions [5, 6] of his results. However, we believe that our reformulation of these results in the context of asymptotic preconditioning is new and useful. We offer the paper not as a breakthrough in the asymptotic theory of linear differential systems, but as an expository approach to what we believe is a new application of standard results on this subject.

## 2 Preliminary considerations

The proof of most of the following lemma can be pieced together from applying various results in our references to the system (3); however, in keeping with our expository goal, we present a self-contained proof here.
Lemma 2 Let $U$ be a fundamental matrix for (3). Then:

(a) Eq. (1) is stable relative to $P$ if and only if $U$ is bounded on $\mathcal{J}$.

(b) Eq. (1) is $\rho$-stable relative to $P$ if and only if there is a constant $M$ such that

$$
\|U(t)U^{-1}(\tau)\| \leq M/\rho(t, \tau), \quad (t, \tau) \in \mathcal{J}.
$$

(c) Eq. (1) is strictly stable relative to $P$ if and only if $\|U\|$ and $\|U^{-1}\|$ are bounded on $\mathcal{J}$ or, equivalently, if and only if there is a constant $M$ such that

$$
\|U(t)U^{-1}(\tau)\| \leq M, \quad t, \tau \in \mathcal{J}.
$$

(d) Eq. (1) has linear asymptotic equilibrium relative to $P$ if and only if $\lim_{t \to \infty} U(t)$ exists and is invertible.

Proof. From Lemma 1, it suffices to show that the assumptions (a)–(d) are respectively equivalent to stability, $\rho$-stability, strict stability, and linear asymptotic equilibrium of (3). Since every solution of (3) can be written as $u(t) = U(t)\xi$ with $\xi \in \mathbb{C}^n$, (d) is obvious. For the rest of the proof, let $\mathcal{U}$ denote the set of all solutions of (3). Then $u \in \mathcal{U}$ if and only if

$$
u(t) = U(t)U^{-1}(\tau)u(\tau) \text{ for all } t, \tau \in \mathcal{J}.
$$

If $\tau$ is arbitrary but fixed and $K_\tau = \|U^{-1}(\tau)\|$, then (6) implies that

$$
\|u(t)\| \leq K_\tau \|U(t)\|\|u(\tau)\| \text{ for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}.
$$

This implies sufficiency for (a). Also from (6),

$$
\|u(t)\| \leq \|U(t)U^{-1}(\tau)\|\|u(\tau)\| \text{ for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}.
$$

Therefore (4) implies that

$$
\|u(t)\| \leq M\|u(\tau)\|/\rho(t, \tau) \text{ for all } (t, \tau) \in \mathcal{J} \text{ and } u \in \mathcal{U},
$$

which implies sufficiency for (b). Moreover, (5) implies that

$$
\|u(t)\| \leq M\|u(\tau)\| \text{ for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}
$$

which implies sufficiency for (c).

We use contrapositive arguments to establish necessity in (a), (b), and (c). In all three cases let $M$ be an arbitrary positive constant. For (a), if $U$ is unbounded and $\tau$ is fixed in $\mathcal{J}$, then $U(t)U^{-1}(\tau)$ is also unbounded as a function of $t$ (since $U(t) = U(t)U^{-1}(\tau)U(\tau)$). Therefore there is a $t_0 \in \mathcal{J}$ and a $\xi \in \mathbb{C}^n$ such that $\|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\|$. Hence, if $u_0(t) = U(t)U^{-1}(\tau)\xi$ then $u_0 \in \mathcal{U}$ and

$$
\|u(t_0)\| = \|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\| = M\|u(\tau)\|;
$$

hence (3) is not stable.
For (b), if there is a \((t_0, \tau_0) \in \mathcal{J}\) such that

\[ \|U(t_0, \tau_0)\| > M/\rho(t_0, \tau_0). \]

then

\[ \|U(t_0, \tau_0)\xi\| > M\|\xi\|/\rho(t_0, \tau_0) \]

for some \(\xi \in \mathbb{C}^n\). If \(u(t) = U(t)U^{-1}(t_0)\xi\) then

\[ \|u(t_0)\| = \|U(t_0)U^{-1}(t_0)\xi\| > M\|\xi\|/\rho(t_0, \tau_0) = M\|u(t_0)\|/\rho(\rho_0(t_0)). \]

so (3) is not \(\rho\)-stable. A similar argument shows that if (3) is strictly stable, then (5) holds for some \(M\).

Eq. (5) obviously holds for some \(M\) if \(U\) and \(U^{-1}\) are bounded on \(\mathcal{J}\). It remains to show that (5) implies that \(U\) and \(U^{-1}\) are bounded on \(\mathcal{J}\). If \(t \in \mathcal{J}\) is fixed and \(t\) is arbitrary, then (5) implies that

\[ \|U(t)\| = \|U(t)U^{-1}(t)U(t)\| \leq \|U(t)U^{-1}(t)\|\|U(t)\| \leq M\|U(t)\|. \]

so \(U\) is bounded on \(I\). To complete the proof, we must show that if \(U^{-1}\) is unbounded then (5) is false for every \(M\). Let \(t_0 \in \mathcal{J}\) be fixed and let \(\sigma = \min \{\|U(t_0)\eta\| : \|\eta\| = 1\}\), which is positive, since \(U(t_0)\) is invertible. If \(U^{-1}\) is unbounded on \(\mathcal{J}\) there is a \(t \in \mathcal{J}\) and \(\xi \in \mathbb{C}^n\) such that \(\|\xi\| = 1\) and \(\|U^{-1}(t)\xi\| > M/\sigma\). Then

\[ \|U(t_0)U^{-1}(t)\xi\| > \sigma\|U^{-1}(t)\xi\| > M\|\xi\|. \]

so \(\|U(t_0)U^{-1}(t)\| > M\).

**Lemma 3** Suppose that \(R, Q \in \mathcal{R}\) and let

\[ F = R' - Q'Q^{-1}R + RP^{-1}(P' - AP). \]  

Then \(X = PU \in \mathbb{C}^{n \times n}(\mathcal{J})\) satisfies \(X' = AX, t \in \mathcal{J}\), if and only if

\[ (Q^{-1}RU)' = Q^{-1}FU, \quad t \in \mathcal{J}. \]  

**Proof.** From (7),

\[ (Q^{-1}RU)' = Q^{-1}(R'U - Q'Q^{-1}RU + RU') = Q^{-1}FU + Q^{-1}R(U' - P^{-1}(P' - AP)U), \]

so Lemma 1 implies the conclusion.

This lemma provides an infinite family of linear differential systems, all with the same solutions; namely, \(u\) is a solution of (3) (and consequently \(x = Pu\) is a solution of (1)) if and only if \(u\) is a solution of every system of the form (8). Therefore, if (8) has a given property \(\mathcal{P}\) for some suitably chosen \(R\) and \(Q\) in \(\mathcal{R}\), then \(P\) preconditions (1) for \(\mathcal{P}\).
3 Main results

Theorem 1 Suppose that there are $R, Q \in \mathbb{R}$ such that $R$ and $R^{-1}$ are bounded on $J$ and
\[ \int_{-\infty}^{\infty} \|F(s)\| \, ds < \infty. \quad (9) \]

Then:
(a) $P$ preconditions Eq. (1) for $\rho$-stability if there is a constant $M$ such that
\[ \|Q(t)Q^{-1}(\tau)\| \leq M/\rho(t, \tau), \quad a \leq \tau \leq t. \quad (10) \]
(b) $P$ preconditions Eq. (1) for strict stability if $Q$ and $Q^{-1}$ are bounded on $J$.

PROOF. Integrating (8) yields
\[ U(t) = R^{-1}(t)Q(t) \left( Q^{-1}(\tau)R(\tau)U(\tau) + \int_{\tau}^{t} Q^{-1}(s)F(s)U(s) \, ds \right), \quad (11) \]
for $t, \tau \in J$. Therefore
\[ U(t)U^{-1}(\tau) = R^{-1}(t)Q(t) \left( Q^{-1}(\tau)R(\tau) + \int_{\tau}^{t} Q^{-1}(s)F(s)U(s)U^{-1}(\tau) \, ds \right). \quad (12) \]
To prove (a), let
\[ g(t, s) = \|Q(t)Q^{-1}(s)\|\rho(t, s) \quad \text{and} \quad h(s, \tau) = \|U(s)U^{-1}(\tau)\|\rho(s, \tau). \quad (13) \]
By Lemma 2(b), we must show that $h(t, \tau)$ is bounded for $(t, \tau) \in J$. If $\tau \leq s \leq t$ then (2) implies that
\[ \rho(t, \tau)\|Q(t)Q^{-1}(s)F(s)U(s)U^{-1}(\tau)\| \leq g(t, s)\|F(s)\|h(s, \tau). \]
Since $R$ and $R^{-1}$ are bounded, multiplying both sides of (12) by $\rho(t, \tau)$ yields the inequality
\[ h(t, \tau) \leq c_1g(t, \tau) + c_2\int_{t}^{\tau} g(t, s)\|F(s)\|h(s, \tau) \, ds, \quad a \leq \tau \leq t, \]
for suitable constants $c_1$ and $c_2$. Now (10) and (13) imply that
\[ h(t, \tau) \leq M \left[ c_1 + c_2\int_{t}^{\tau} \|F(s)\|h(s, \tau) \, ds \right], \quad a \leq \tau \leq t. \quad (14) \]
Therefore
\[ \frac{c_2h(t, \tau)\|F(t)\|}{c_1 + c_2\int_{t}^{\tau} \|F(s)\|h(s, \tau) \, ds} \leq M c_2\|F(r)\| \quad a \leq \tau \leq r. \quad (15) \]

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Integrating this with respect to $t$ yields
\[ \log \left( c_1 + c_2 \int_{\tau}^{t} \| F(s) \| h(s, \tau) \, ds \right) - \log c_1 \leq Mc_2 \int_{\tau}^{t} \| F(s) \| \, ds. \] (16)

This and (14) imply that
\[ \sup \{ \| h(t, \tau) \| \mid (t, \tau) \in J \} \leq Mc_1 \exp \left( M \int_{a}^{\infty} \| F(s) \| \, ds \right) < \infty, \] (17)
from (9). This completes the proof of (a).

To prove (b), replace (13) by
\[ g(t, s) = \| Q(t) Q^{-1}(s) \| \quad \text{and} \quad h(s, \tau) = \| U(s) U^{-1}(\tau) \|. \]

Now we must show that $h(t, \tau)$ is bounded for all $t, \tau \in J$. If (1) is strictly stable then there is a constant $M$ such that $g(t, s) \leq M$ for $s, t \geq a$. This, (12) and the boundedness of $R$ and $R^{-1}$ imply that
\[ h(t, \tau) \leq M \left[ c_1 + c_2 \int_{\tau}^{t} \| F(s) \| \| h(s, \tau) \| \, ds \right], \quad t, \tau \geq a, \]
for suitable positive constants $c_1$ and $c_2$. Now the argument used in the proof of (a) again implies (17). If $a \leq t \leq \tau$ then (14)–(17) all hold with $t$ and $\tau$ interchanged, which completes the proof of (b). \( \square \)

**Remark 1** The use of logarithmic integration that produced (16) was motivated by the proof of Gronwall’s inequality [1, p. 35], a standard tool for studying the asymptotic behavior of solutions of differential equations.

**Theorem 2** In addition to the assumptions of Theorem 1(b), suppose that
\[ \lim_{t \to \infty} R^{-1}(t)Q(t) = J \] is invertible. (18)

Then $P$ preconditions (1) for linear asymptotic equilibrium.

**Proof.** From (11) and (18), $\lim_{t \to \infty} U(t) = V$, where
\[ V = J \left( Q^{-1}(\tau)R(\tau)U(\tau) + \int_{\tau}^{\infty} Q^{-1}(s)F(s)U(s) \, ds \right) \]
and the integral converges because of (9), the boundedness of $Q^{-1}$ (assumed) and $U$ (from Theorem 1(b)). Now we must show that $V$ is invertible. Since Theorem 1(b) implies that (1) is strictly stable relative to $P$, there is a constant $K$ such that $\| U^{-1} \| < K, \ t \in J$. If $\xi \in \mathbb{C}^n$ then
\[ \| \xi \| = \| U^{-1}(t)U(t)\xi \| \leq \| U^{-1}(t)\| \| U(t)\xi \| \leq K \| U(t)\xi \|, \quad t \leq a, \]
so
\[ \| \xi \| \leq K \lim_{t \to \infty} \| U(t)\xi \| = K \| V\xi \|. \]
Therefore $V\xi = 0$ if and only if $\xi = 0$, so $V$ is invertible. \( \square \)
Theorem 3 If there are $Q$ and $R$ in $\mathcal{R}$ such that $R^{-1}Q$ is bounded and
\[ \int_{-\infty}^{\infty} \|Q^{-1}(s)F(s)\| \, ds < \infty, \]
then $P$ preconditions (1) for stability; moreover, if (18) holds then $P$ preconditions (1) for linear asymptotic equilibrium.

Proof. Our assumptions imply that if $0 < \rho < 1$ then there is a $\tau \geq a$ such that
\[ \|R^{-1}(t)Q(t)\| \int_{\tau}^{\infty} \|Q^{-1}(s)F(s)\| \, ds \leq \rho, \quad t \geq \tau. \]

Let $\mathcal{B}$ be the Banach space of bounded continuous $n \times n$ vector functions on $J = [\tau, \infty)$ with norm $\|U\|_\mathcal{B} = \sup_{t \in J} \|U(t)\|$, and define $T : \mathcal{B} \to \mathcal{B}$ by
\[ (TU)(t) = R^{-1}(t)Q(t) \left( C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s) \, ds \right) \]
where $C \in \mathbb{C}^{n \times n}$ is invertible. If $U_1, U_2 \in \mathcal{B}$ then
\[ (TU_1)(t) - (TU_2)(t) \leq \|R^{-1}(t)Q(t)\| \int_{t}^{\infty} \|Q^{-1}(s)F(s)\| \|U_1(s) - U_2(s)\| \, ds, \]
so $\|TU_1 - TU_2\|_\mathcal{B} \leq \rho \|U_1 - U_2\|_\mathcal{B}$. Therefore, by the contraction mapping principal [7, p. 545], there is a $U \in \mathcal{B}$ such that
\[ U(t) = R^{-1}(t)Q(t) \left( C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s) \, ds \right). \]

Since $U$ satisfies (8), Theorem 1 implies that $X = P \dot{U}$ satisfies (1). Therefore $P$ preconditions (1) for stability. Finally, if (18) holds then $\lim_{t \to \infty} U(t) = JC$ is invertible, so $P$ preconditions (1) for linear asymptotic equilibrium.

Remark 2 Strictly speaking, our proof of Theorem 3 defines $U$ only on the interval $[\tau, \infty)$, which has the appearance of leaving a gap if $\tau > a$. However, in this case we appeal to the elementary theory of linear differential systems, which guarantees that $U$ can extended uniquely over $J$ as an invertible solution of $U' = P^{-1}(AP - P')U$.

From (7),
\[ Q^{-1}F = (Q^{-1}R)' + (Q^{-1}R)P^{-1}(P' - AP). \]
Therefore we can reformulate Theorem 3 as follows.

Theorem 4 If there is a $T \in \mathcal{R}$ such that $T^{-1}$ is bounded and
\[ \int_{-\infty}^{\infty} \|T' + TP^{-1}(P' - AP)\| \, ds < \infty, \]
then $P$ preconditions (1) for stability; moreover, if $\lim_{t \to \infty} T(t)$ exists and is invertible then $P$ preconditions (1) for linear asymptotic equilibrium.

The assertion concerning linear asymptotic equilibrium can also be proved by applying a theorem of Conti [3] to (3).
References


