THE METHOD OF FROBENIUS WITHOUT THE MESS AND THE MYSTERY

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1 Introduction

In this note we try to simplify a messy topic usually covered in an elementary differential equations course. The method of Frobenius is a way to solve equations of the form

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0, \tag{1}$$

where α_i, β_i, and γ_i are constants and $\alpha_0 \neq 0$. Important examples are

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{and} \quad xy'' + (1-x)y' + \lambda y = 0,$$

where ν and λ are constants, and multiplying the second equation through by x puts it in the form (1). The first is Bessel's equation, named after the German mathematician and astronomer Frederick Wilhelm Bessel (1784-1846), and the second is Laguerre's equation, named after the French mathematician Edward Nicholas Laguerre (1834-1886), who contributed extensively to the theory of infinite series.

The method is named after its inventor, Ferdinand George Frobenius (1849–1917), a professor at the University of Berlin. Frobenius developed it in his doctoral thesis, written under the direction of Karl Weierstrass (1815-1897), one of the founders of the theory of functions of a complex variable. Frobenius studied more general problems (for example, see [1]–[4]), but most equations treated by his method in elementary courses can be written as in (1).

If $\alpha_1 = \beta_1 = \gamma_1 = 0$ then (1) reduces to

$$\alpha_0 x^2 y'' + \beta_0 xy + \gamma_0 y = 0,$$

which is called Euler's equation in honor of Leonhard Euler (1707-1783), a Swiss mathematician who made important contributions to many areas of mathematics and is considered to be one of the greatest mathematicians of all time. We don't need the method of Frobenius to solve Euler's equation. In any elementary differential equations textbook you'll see that its general solution is determined by the zeros of

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 = \alpha_0(r - r_1)(r - r_2),$$

which is called the *indicial polynomial* of (1). Specifically, the general solution of Euler's equation is

$$y = \begin{cases} c_1x^{r_1} + c_2x^{r_2} & \text{if } r_1 \neq r_2, \\ x^{r_1}(c_1 + c_2 \ln x) & \text{if } r_1 = r_2. \end{cases}$$

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Therefore we assume that at least one of α_1, β_1, and γ_1 is nonzero, and we define

$$p_1(r) = \alpha_1 r(r-1) + \beta_1 r + \gamma_1.$$

In most textbooks it is assumed that r_1, r_2, and the constants in (1) are real numbers, but this doesn't simplify matters, so we'll allow them to be complex.

If $r_1 - r_2$ is not an integer then the method of Frobenius yields solutions of the form

$$\text{(a)} \quad y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad \text{(b)} \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

where the series converge in some interval containing the origin. If $r_1 = r_2$ then (a) and (b) are the same, and finding a second solution that isn't just a multiple of the first is somewhat complicated and treated incompletely in most textbooks. Worse, if $r_1 = r_2 + k$ where k is a nonnegative integer then (1) has a solution of the form (a), but not (b), and methods usually presented for finding a second solution are a mystery to most students. We will try to clarify all three cases.

For brevity, when we say that a y_2 is a second solution of (1), we mean that y_2 is not a constant multiple of y_1. Then every solution of (1) can be written as $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants.

2 Preliminaries

For convenience, let

$$Ly = x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y,$$

so we want find y such that $Ly = 0$. A series

$$y(x, r) = \sum_{n=0}^{\infty} a_n(r) x^{n+r}$$

where r and $a_0(r)$, $a_1(r)$, …, are independent of x can be differentiated term by term with respect to x, so

$$y'(x, r) = \sum_{n=0}^{\infty} (n+r) a_n(r) x^{n+r-1} \quad \text{and} \quad y''(x, r) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n(r) x^{n+r-2}.$$

Therefore

$$\alpha_i x^2 y''(x, r) + \beta_i x y'(x, r) + \gamma_i y(x, r) = \sum_{n=0}^{\infty} p_i(n+r) a_n(r) x^{n+r}, \quad i = 1, 2,$$

so

$$
\begin{aligned}
Ly(x,r) &= \sum_{n=0}^{\infty} p_0(n+r)a_n(r)x^{n+r} + \sum_{n=0}^{\infty} p_1(n+r)a_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} p_0(n+r)a_n(r)x^{n+r} + \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1}(r)x^{n+r} \\
&= p_0(r)a_0(r)x^r + \sum_{n=1}^{\infty}[p_0(n+r)a_n + p_1(n+r-1)a_{n-1}]x^{n+r}. \quad (2)
\end{aligned}
$$

We want to find r and $a_0(r)$, $a_1(r)$, $\ldots$, so that $Ly(x,r) = 0$ for all x. For a big step in the right direction, let r be a complex number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$
a_0(r) = 1 \quad \text{and} \quad a_n(r) = -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \quad n \geq 1. \tag{3}
$$

Then (2) and (3) imply that

$$
Ly(x,r) = p_0(r)x^r. \tag{4}
$$

Applying the second equality in (3) repeatedly yields

$$
a_1(r) = -\frac{p_1(r)}{p_0(r+1)}, \quad a_2(r) = \frac{p_1(r+1)p_1(r)}{p_0(r+2)p_0(r+1)},
$$

and, for $n \geq 2$,

$$
a_n(r) = (-1)^n \frac{p_1(n+r-1)\cdots p_1(r+1)p_1(r)}{p_0(n+r)\cdots p_0(r+2)p_0(r+1)},
$$

which we write more compactly as

$$
a_n(r) = (-1)^n \frac{\prod_{j=1}^{n} p_1(j+r-1)}{\prod_{j=1}^{n} p_0(j+r)}, \quad n \geq 0, \tag{5}
$$

where

$$
\prod_{j=1}^{0} p_1(j+r-1) = \prod_{j=1}^{0} p_0(j+r) = 1.
$$

We'll see that it is useful to define

$$
z(x,r) = \frac{\partial y}{\partial r}(x,r) = y(x,r)\ln x + \sum_{n=1}^{\infty} a_n'(r)x^{n+r}.
$$

(Since $a_0(r) = 1$ for all r, $a_0'(r) = 0$.) Then (4) implies that

$$
Lz(x,r) = L\frac{\partial y}{\partial r}(x,r) = \frac{\partial}{\partial r}Ly(x,r) = p_0'(r)x^r + p_0(r)x^r \ln x. \tag{6}
$$

3 Case 1: $r_1 - r_2 \neq$ an integer

If $r_1 - r_2$ is not an integer then $p_0(n+r_1)$ and $p_0(n+r_2)$ are both nonzero for all $n \geq 1$. Therefore (4) implies that $Ly(x, r_1) = Ly(x, r_2) = 0$ (since $p(r_1) = p(r_2) = 0$), so

$$y_1 = y(x, r_1) = \sum_{n=0}^{\infty} a_n(r_1) x^{n+r_1} \tag{7}$$

and

$$y_2 = y(x, r_2) = \sum_{n=0}^{\infty} a_n(r_2) x^{n+r_2} \tag{8}$$

are linearly independent solutions of (1).

Example 1 For the equation

$$3x^2 y'' + x(1+x)y' - (1+3x)y = 0,$$

$$p_0(r) = (r-1)(3r+1), \quad p_1(r) = r - 3,$$

$r_1 = 1$, and $r_2 = -1/3$. From (3),

$$a_n(r) = -\frac{n+r-4}{(n+r-1)(3n+3r+1)} a_{n-1}(r).$$

Hence

$$a_n(1) = -\frac{n-3}{n(3n+4)} a_{n-1}(1) \quad \text{and} \quad a_n(-1/3) = -\frac{3n-13}{3n(3n-4)} a_{n-1}(-1/3), \quad n \geq 1,$$

so

$$y_1 = x + \frac{2}{7}x^2 + \frac{1}{70}x^3 \quad \text{and} \quad y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!} \left(\prod_{j=1}^{n} \frac{3j-13}{3j-4} \right) x^{n-1/3}.$$

4 Case 2: $r_1 = r_2$

If $r_1 = r_2$ then (7) and (8) define the same solution, which we will call y_1. However, in this case

$$p(r) = \alpha_0 (r - r_1)^2 \quad \text{and} \quad p'(r) = 2\alpha_0 (r - r_1), \quad \text{so} \quad p(r_1) = p'(r_1) = 0.$$

Therefore (6) implies that

$$y_2 = z(x, r_1) = y_1 \ln x + \sum_{n=1}^{\infty} a_n'(r_1) x^{n+r_1}$$

is a second solution if (1).

To use this result we must compute $a_n'(r_1)$ for all $n \geq 0$. Since $a_0(r) = 1$ for all r, $a_0'(r) = 0$. If

$$p_0(j + r) \neq 0 \quad \text{and} \quad p_1(r + j - 1) \neq 0, \quad 1 \leq j \leq m,$$

(the first inequality for all j with $r_1 = 1$), then (5) implies that

$$\ln |a_n(r)| = \sum_{j=1}^{n} \ln \left| \frac{p_1(j + r - 1)}{p_0(j + r)} \right| = \sum_{j=1}^{n} (\ln |p_1(j+r-1)| - \ln |p_0(j+r)|), \quad 1 \leq n \leq m.$$

Differentiating this yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} b_j(r) \tag{9}$$

where

$$b_j(r) = \frac{p_0(j + r)p_1'(j + r - 1) - p_0'(j + r)p_1(j + r - 1)}{p_1(j + r - 1)p_0(j + r)}, \quad 1 \leq j \leq m. \tag{10}$$

Hence,

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} b_j(r), \quad 1 \leq j \leq m. \tag{11}$$

Therefore, if $p_0(r) = \alpha_0(r - r_0)^2$ and $p_1(j + r_1 - 1) \neq 0$ for all $j \geq 1$, we can let m be arbitrary in (10), so

$$y_2 = \sum_{n=0}^{\infty} a_n(r_1) \left(\sum_{j=1}^{n} b_j(r_1) \right) z^{n+r_1}$$

is a second solution of (1).

Example 2 For the equation

$$x^2(1 + 2x)y'' + x(3 + 5x)y' + (1 - 2x)y = 0$$

$$p_0(r) = (r + 1)^2, \quad p_1(r) = (r + 2)(2r - 1),$$

and $r_1 = r_2 = -1$. From (3) and (5),

$$a_n(r) = -\frac{2n + 2r - 3}{n + r + 1} a_{n-1}(r), \quad n \geq 0,$$

so

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{2j + 2r - 3}{j + r + 1}, \quad n \geq 0.$$

Therefore

$$a_n(-1) = \frac{(-1)^n}{n!} \prod_{j=1}^{n} (2j - 5), \quad n \geq 0,$$

and

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\prod_{j=1}^{n} (2j-5) \right) x^{n-1}.$$

If $a_n(r) \neq 0$ then

$$\ln |a_n(r)| = \sum_{j=1}^{n} \left(\ln|2j+2r-3| - \ln|j+r+1| \right),$$

so

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{2}{2j+2r-3} - \frac{1}{j+r+1} \right) = \sum_{j=1}^{n} \frac{5}{(j+r+1)(2j+2r-3)}$$

Hence

$$a_n'(-1) = a_n(-1) \sum_{j=1}^{n} \frac{5}{j(2j-5)},$$

so

$$y_2 = y_1 \ln x + 5 \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{j=1}^{n}(2j-5)}{n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j-5)} \right) x^{n-1}. \quad \blacksquare$$

The method we used to obtain (11) is called logarithmic differentiation. It doesn't work if p_1 contains a factor of the form $r - r_1 - m$ where m is a nonnegative integer, since this puts a zero in the denominator of (10) with $r = r_1$ and $j = m + 1$. If m is the least nonnegative integer for which p has this property, then either

$$p_1(r) = \gamma_1(r - r_1 - m) \quad \text{where} \quad \gamma_1 \neq 0 \tag{12}$$

or

$$p_1(r) = \alpha_1(r - r_1 - m)(r - s) \;\text{ where }\; \alpha_1 \neq 0 \;\text{ and } s \neq r_1 + \ell,\ 0 \leq \ell \leq m-1. \tag{13}$$

In either case (5) implies that

$$a_n(r_1) = \begin{cases} (-1)^n \dfrac{\prod_{j=1}^{n} p_1(j + r_1 - 1)}{\prod_{j=1}^{n} p_0(j + r_1)} & \text{if} \quad 0 \leq j \leq m, \\ 0 & \text{if} \quad n > m, \end{cases}$$

so

$$y_1 = \sum_{n=0}^{m} a_n(r_1) x^{n+r_1}.$$

To find y_2 we must compute $a_n'(r_1)$ for all $n \geq 1$. We can use logarithmic differentiation as in (9) and (10) for $1 \leq n \leq m$, since $p_1(j + r_1 - 1) \neq 0$ for $1 \leq j \leq m$. If $n > m$ then (5) and (12) or (13) imply that

$$a_n(r) = (r - r_1) b_n(r) \quad \text{if} \quad n > m,$$

with

$$b_n(r) = (-1)^n \frac{\widehat{\prod}_{j=1}^{n} p_1(j+r-1)}{\prod_{j=1}^{n} p_0(j+r)},$$

where the " $\frown$ " over the product symbol in the numerator indicates that the factor $(r-r_1)$ in $p_1(m+r)$ if $n>m$ is omitted. Therefore

$$a_n'(r) = (b_n(r) + (r-r_1)b_n'(r)), \quad n > m,$$

so

$$a_n'(r_1) = b_n(r_1), \quad n > m.$$

If (13) holds with $s = r_1 + q$ where q is an integer $\geq m$, then $b_n(r)$ has the factor $r - r_1$ if $n > q$, so $a_n'(r_1) = 0$ if $n > q$.

Example 3 For the equation

$$x^2(1-x)y'' + x(3-2x)y' + (1+2x)y = 0,$$

$$p_0(r) = (r+1)^2, \quad p_1(r) = -(r-1)(r+2),$$

and $r_1 = r_2 = -1$. From (3) and (5),

$$a_n(r) = \frac{n+r-2}{n+r+1}a_{n-1}(r), \quad \text{so} \quad a_n(r) = \prod_{j=1}^{n} \frac{j+r-2}{j+r+1}, \quad n \geq 0.$$

In particular,

$$a_1(r) = \frac{r-1}{r+2} \quad \text{and} \quad a_2(r) = \frac{r(r-1)}{(r+2)(r+3)},$$

so $a_1(-1) = -2$, $a_2(-1) = 1$, and $a_n(-1) = 0$ if $n \geq 3$. Therefore

$$y_1 = (1-x)^2/x.$$

Also,

$$a_1'(r) = \frac{3}{(r+2)^2} \quad \text{and} \quad a_2'(r) = \frac{6(r^2+2r-1)}{(r+2)^2(r+3)^2},$$

so $a_1'(-1) = a_2'(-1) = -3$. If $n \geq 3$, then

$$a_n(r) = (r+1)b_n(r) \quad \text{where} \quad b_n(r) = \frac{r(r-1)}{(n+r-1)(n+r)(n+r+1)},$$

so

$$a_n'(-1) = b_n(-1) = \frac{2}{(n-2)(n-1)n}, \quad n \geq 3.$$

Therefore

$$y_2 = y_1 \ln x + 3 - 3x + 2\sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)n} x^{n-1}.$$

5 Case 3: $r_1 - r_2$ = a positive integer

If $r_1 - r_2 = k$ is a positive integer then $p_0(n + r_1) \neq 0$ for $n \geq 1$, so y_1 in (7) is a solution of (1). However, y_2 in (8) is undefined, since $p_0(r_2 + k) = p_0(r_1) = 0$. In this case we construct a second solution of (1) as a linear combination of

$$w_1 = z(x, r_1) = y_1(x) \ln x + \sum_{n=0}^{\infty} a_n'(r_1) x^{n+r_1}$$

and

$$w_2(x) = \sum_{n=0}^{k-1} c_n x^{n+r_2}, \quad n \geq 1, \tag{14}$$

for suitably chosen $c_0, \ldots, c_{k-1}$.

From (6),

$$Lw_1 = p_0'(r_1) = \alpha_0(0)(r_1 - r_2) = k\alpha_0, \tag{15}$$

As in the proof of (2),

$$\begin{aligned} Lw_2(x) &= \sum_{n=0}^{k-1} p_0(n + r_2) c_n x^{n+r_2} + \sum_{n=0}^{k-1} p_1(n + r_2) c_n x^{n+r_2+1} \\ &= \sum_{n=0}^{k-1} p_0(n + r_2) c_n x^{n+r_2} + \sum_{n=1}^{k} p_1(n + r_2 - 1) c_{n-1} x^{n+r_2} \\ &= p_0(r_2) c_0 x^{r_2} + \sum_{n=0}^{k-1} \left[p_0(n + r_2) c_n + p_1(n + r_2 - 1) c_{n-1} \right] x^{n+r_2} \\ &\quad + p_1(k + r_2 - 1) c_{k-1} x^{k+r_2}. \end{aligned}$$

Therefore, if we define

$$c_0 = 1 \quad \text{and} \quad c_n = -\frac{p_1(n + r_2 - 1)}{p_0(n + r_2)} c_{n-1}, \quad 1 \leq n \leq k - 1$$

(we can't let $n = k$ because $k + r_2 = r_1$ and $p_0(r_1) = 0$), then

$$c_n = (-1)^n \frac{\prod_{j=1}^{n} p_1(j + r_2 - 1)}{\prod_{j=1}^{n} p_0(j + r_2)}, \quad 0 \leq n \leq k - 1, \tag{16}$$

and, since $p_0(r_2) = 0$,

$$Lw_2 = p_1(k + r_2 - 1) c_{k-1} x^{k+r_2} = p_1(r_1 - 1) c_{k-1} x^{r_1}.$$

This and (15) imply that if C is any constant, then

$$L(w_2 + Cw_1) = \left[p_1(r_1 - 1) c_{k-1} + Ck\alpha_0 \right] x^{r_1}.$$

Therefore, if

$$C = -\frac{p_1(r_1 - 1) c_{k-1}}{k\alpha_0} \tag{17}$$

then $y_2 = w_2 + Cw_1$ is a solution of (1).

Example 4 For the equation

$$x^2y'' + x(1-2x)y' - (4+x)y = 0,$$

$$p_0(r) = (r-2)(r+2), \quad p_1(r) = -2r-1,$$

$r_1 = 2$, $r_2 = -2$, and $k = r_1 - r_2 = 4$. From (3) and (5),

$$a_n(r) = \frac{2j+2r-1}{(j+r-2)(j+r+2)}a_{n-1}(r), \quad n \geq 1, \tag{18}$$

and

$$a_n(r) = \prod_{j=1}^{n} \frac{2j+2r-1}{(j+r-2)(j+r+2)}, \quad n \geq 0, \tag{19}$$

if $r \neq -2$. Therefore

$$a_n(2) = \frac{1}{n!}\prod_{j=1}^{n} \frac{2j+3}{j+4}, \quad n \geq 0,$$

so

$$y_1 = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right)x^{n+2}.$$

From (16),

$$c_n = \frac{1}{n!}\prod_{j=1}^{n} \frac{2j-5}{j-4}, \quad 0 \leq j \leq 3,$$

so

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = \frac{1}{4}, \quad \text{and} \quad c_3 = -\frac{1}{12}.$$

Therefore, from (14),

$$w_2 = x^{-2}\left(1 + x + \frac{1}{4}x^2 - \frac{1}{12}x^3\right).$$

From (17) and (18) with $r_1 = 2$ and $r_2 = -2$,

$$C = -\frac{p_1(1)c_3}{4} = -1/16.$$

From (19) and logarithmic differentiation,

$$a_n'(r) = -2a_n(r)\sum_{j=1}^{n} \frac{j^2 + j(2r-1) + r^2 - r + 4}{(j+r-2)(j+r+2)(2j+2r-1)}, \quad n \geq 1,$$

so

$$a_n'(2) = -2a_n(2)\sum_{j=1}^{n} \frac{j^2+3j+6}{j(j+4)(2j+3)}, \quad n \geq 1.$$

Therefore

$$
\begin{aligned}
y_2 &= x^{-2}\left(1 + x + \frac{1}{4}x^2 - \frac{1}{12}x^3\right) \\
&\quad -\frac{1}{16}y_1 \ln x + \frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\prod_{j=1}^{n}\frac{2j+3}{j+4}\right)\left(\sum_{j=1}^{n}\frac{j^2+3j+6}{j(j+4)(2j+3)}\right)x^{n+2}.
\end{aligned}
$$

References

[1] A. R. Forsyth, Theory of Differential Equations, Part III, Ordinary Linear Equations, Cambridge University Press, 2002.

[2] E. L. Ince, Ordinary Differential Equations, Courier Dover Publications, 1956.

[3] H. Jeffreys and B. S. Jeffreys, Methods of Mathematical Physics, Third Edition, Cambridge University Press, 1956.

[4] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Fourth Edition, Cambridge University Press, 1952.