THE RELATIONSHIP BETWEEN TRENCH'S TOEPLITZ INVERSION ALGORITHM AND THE GOHBERG-SEMENCUL FORMULA

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UNPUBLISHED NOTE

Abstract. It is shown that a formula of Gohberg and Semencul for the inverse of a Toeplitz matrix is equivalent to an earlier formula of the author, and that a similar formula of Heinig follows from a formula of the author for the inverse of a Hankel matrix.

This manuscript was submitted for publication in Linear Algebra and Its Applications in 1989 or 1990 with the title "A note on Toeplitz Inversion" Formulas." Two referee reports essentially said that the connection between my Toeplitz inversion algorithm (1964) and the Gohberg-Semencul (1972) formula for the inverse of a Toeplitz matrix was well known and I should therefore shorten the paper to focus on the connection between my Hankel inversion algorithm and Heinigs formula for the inverse of a Toeplitz matrix. This struck me as strange, since most papers citing the Gohberg-Semencul formula up until that time either avoided mentioning my inversion algorithm altogether or were vague on its main point, which is central to the Gohberg-Semencul formula: If T_n is invertible and $det(T_{n-1}) \neq 0$ then T_n^{-1} is completely determined by its first row and column by means of a formula stated explicitly in the last four equations of my paper, which are computationally equivalent to the Gohberg-Semencul formula. However, I yielded to the referees and the result was “A note on a Toeplitz inversion formula,” Linear Algebra Appl. 129 (1990), 55-62.

TEXT OF AN EMAIL MESSAGE SENT TO LEONID LEHRER, VADIM OLSHEVSKY, AND ILYA SPITKOVSKY ON MARCH 31, 2011

I write concerning two paragraph that I read in the Amazon.com preview of Convolution Equations and Singular Integral Operators (Operator Theory: Advances and Applications, 2010, Volume 206), concerning the connection between the Gohberg-Semencul formula (1972) and my Toeplitz inversion algorithm (1964):

“At the time of the publication of [GS72] the authors were unaware of the recursive inversion algorithm that was derived earlier in [Tre64] for the case of positive definite Toeplitz matrices. The paper [Tre64] also presents [without a proof] a generalization to non-Hermitian matrices, but it is stated that all principal minors have to be nonzero. Although the Gohberg-Semencul formula is absent in [Tre64], it can be derived from the recursion in [Tre64], at least for the special cases considered there.”

As for “[without a proof]:”

In [Tre64, Section 3] I stated that ”The derivation of the algorithm follows the same lines as in the Hermitian case and will be omitted.” Surely you must have made similar statements where appropriate in your own published papers. This remark is clearly intended to create the impression that the non-Hermitian algorithm was in some way not on solid ground until [GS72] appeared. Sorry, that doesn’t work: S. Zohar presented a detailed derivation of the general algorithm in “Toeplitz matrix inversion: the algorithm of W. F. Trench”, (Journal of the Association for Computing Machinery 16 (1969) 592–601), three years before the publication of [GS72].

As for “it is stated that all principal minors have to be nonzero” and “at least for the special cases considered there:"

With your collective expertise in this field you certainly know that the assumption concerning nonzero principal minors has nothing to do with the Gohberg-Semencul formula. I imposed it to support the $O(n^2)$ Levinson-type recursion that I used to compute the first and last columns of T_n^{-1}. Having accomplished this, the rest of the inverse is computed from the last four equations in [Tre64], which are computationally equivalent to the Gohberg-Semencul. This method works if the upper left corner entry of T_n is nonzero, which is exactly what Gohberg and Semencul assumed. There are no "special cases" here.

The computational equivalence of my work and the Gohberg-Semencul formula has been noted by many authors. I will supply specific sources on request. Moreover, you can read about the equivalence in my technical note "The Relationship between Trench's Toeplitz Inversion Algorithm and the Gohberg–Semencul Formula," which you can access by simply entering 'Gohberg-Semencul formula' in any major US search engine and looking at the first page of returns, or by logging onto

http://ramanujan.math.trinity.edu/wtrench/research/index.shtml

Some months ago I sent this document – with a different title – to Vadim Olshevsky. I will soon post an update dealing with your comments on my work and your response to this message.

The fact is this: the Gohberg-Semencul formula is an elegant and important extension of my algorithm. Perhaps this was not immediately clear, since Gohberg and Semencul apparently did not know of my paper (published eight years earlier) or Zohar's (published three years earlier). However, since you have taken the role of contemporary chroniclers of a subject with a history going back to 1964, you have a responsibility for unbiased historical accuracy. You have not fulfilled that responsibility. This obviously crafted belittling of my work cheapens your tribute to a great mathematician.

"Moreover, the second important fact is that it was the form of the Gohberg-Semencul formula that triggered the development of the inversion of structured matrices (see the previous section)."

Without claiming credit for "triggering" anything, I submit that my Toeplitz inversion algorithm has not gone unnoticed, as I hope you can see from the attached bibliography of over 850 citations and references. I also remind you that my paper "An algorithm for the inversion of finite Hankel matrices" (SIAM J. Math 1965, 1102-1107) presented an $O(n^2)$ Hankel inversion algorithm. In a "Note on a Toeplitz inversion formula" (Lin. Alg. Appl. 129, 1990, 55-62, posted on my web page), I showed that Heinig's Toeplitz inversion formula (Beitrage zur spektraltheorie von Operatorbuscheln und zur alge-braischen Theorei von Toeplitzmatizen, Diss. B, TH KarlMarxStadt 1979) follows from my Hankel inversion algorithm, published fourteen years earlier.

A NOTE ON TOEPLITZ INVERSION FORMULAS

William F. Trench

The explicit formulas of Gohberg and Semencul [5] and Heinig [7] for the inverse of a Toeplitz matrix have important applications and provide theoretical insight into the properties of Toeplitz matrices. It is well known (see, e.g., [1]-[4], [6],[8],[9]) that the Gohberg–Semencul formula is closely related to a a recursive formula in [10]. (In fact, the two formulas are equivalent.) Although it does not seem to have been noted explicitly in the literature, the Heinig formula follows from a formula in [11] for the inverse of a Hankel matrix. Here we present explicit derivations of these two connections.

We first consider the Gohberg–Semencul formula.

Theorem 1 [Gohberg–Semencul, 1972]. *Suppose that the Toeplitz matrix*

$$T_n = (\phi_{r-s})_{r,s=0}^{n}$$

is invertible, and let $x_0, \dots, x_n$ and $y_0, \dots, y_n$ be the solutions of the systems

$$\sum_{s=0}^{n} \phi_{r-s} x_s = \delta_{r0},\ 0 \le r \le n, \tag{1}$$

and

$$\sum_{s=0}^{n} \phi_{r-s} y_s = \delta_{rn},\ 0 \le r \le n. \tag{2}$$

Suppose also that $x_0 \neq 0$, and define

$$x_r = y_r = 0 \textit{ if } r < 0 \textit{ or } r > n. \tag{3}$$

Let A_n and B_n be the lower triangular Toeplitz matrices

$$A_n = (x_{r-s})_{r,s=0}^n, \ B_n = (y_{r-s-1})_{r,s=0}^n, \tag{4}$$

and let C_n and D_n be the upper triangular Toeplitz matrices

$$C_n = (y_{n+r-s})_{r,s=0}^n, \ D_n = (x_{n+r-s+1})_{r,s=0}^n. \tag{5}$$

Then

$$T_n^{-1} = x_0^{-1}(A_nC_n - B_nD_n). \tag{6}$$

Gohberg and Semencul actually showed that if (1) and (2) have solutions with $x_0 \neq 0$, then T_n is invertible, with inverse as in (6). Since our interest is in the inversion formula itself, we simplify the presentation by assuming from the outset that T_n is invertible. A similar comment applies to the Heinig formula, given below.

Denote

$$T_n^{-1} = (b_{rs})_{r,s=0}^n.$$

It is clear from (1) and (2) that $(x_0, \ldots, x_n)^t$ and $(y_0, \ldots, y_n)^t$ are the first and last columns of T_n^{-1}. Thus, (6) shows that T_n^{-1} is completely specified by its first and last columns, provided that $x_0 \neq 0$. To facilitate the comparison of Theorem 1 with Theorem 2 (below), we point out that since the inverse of a Toeplitz matrix is symmetric about its secondary diagonal, an equivalent statement is that T_n^{-1} is in this case determined by its first row and first column. This is easily made explicit by replacing r and s by $n-r$ and $n-s$ in (2), to obtain

$$\sum_{s=0}^{n} \phi_{s-r} y_{n-s} = \delta_{r0}, \ 0 \leq r \leq n. \tag{7}$$

This makes it obvious that $(y_n, \ldots, y_0)$ is the first row of T_n^{-1}. It is also now obvious that

$$y_n = x_0 = b_{00} = \det(T_{n-1})/\det(T_n); \tag{8}$$

hence, assuming that $x_0 \neq 0$ is equivalent to assuming that T_{n-1} is also invertible.

The following theorem is clearly implicit in results stated in Section 3 of [10]. These results, although stated for general (not necessarily Hermitian) matrices, are proved in [10] only for the Hermitian case; however, a general proof was later supplied by Zohar [12].

THEOREM 2 [TRENCH, 1964]. *Suppose that T_n and T_{n-1} are both invertible, and let $\psi_0, \ldots, \psi_{n-1}$ and $\eta_0, \ldots, \eta_{n-1}$ be the solutions of the systems*

$$\sum_{s=0}^{n-1} \phi_{r-s}\psi_s = \phi_{r+1}, \ 0 \leq r \leq n-1, \tag{9}$$

and

$$\sum_{s=0}^{n-1} \phi_{s-r}\eta_s = \phi_{-r-1}, \ 0 \leq r \leq n-1. \tag{10}$$

For convenience, define

$$\psi_{-1} = \eta_{-1} = -1 \tag{11}$$

and $\psi_r = \eta_r = 0$ if $r = n$. Then the elements of T_n^{-1} are given by

$$b_{rs} = b_{r-1,s-1} + b_{00}(\psi_{r-1}\eta_{s-1} - \eta_{n-r}\psi_{n-s}), \ 0 \leq r, \ s \leq n, \tag{12}$$

if we define

$$b_{-1,\ell} = b_{\ell,-1} = 0, \ 0 \leq \ell \leq n. \tag{13}$$

To deduce Theorem 1 from this, we first observe that the definition (11) and manipulation of indices enables us to rewrite (9) and (10) as

$$\sum_{s=0}^{n} \phi_{r-s}\psi_{s-1} = 0, \ 1 \leq r \leq n, \tag{14}$$

and

$$\sum_{s=0}^{n} \phi_{s-r}\eta_{s-1} = 0, \ 1 \le r \le n. \tag{15}$$

With $r = 0$, the left sides of (14) and (15) both equal $-1/x_0$. To see this for (14), note that subtracting ψ_{s-1} times column $s+1$ of T_n from the first column of T_n $(1 \le s \le n)$ and invoking (9) yields

$$\det(T_n) = \begin{vmatrix} d & \phi_{-1} & \phi_{-2} & \dots & \phi_{-n} \\ 0 & & & & \\ \vdots & & T_{n-1} & & \\ 0 & & & & \end{vmatrix},$$

where

$$d = \phi_0 - \sum_{s=1}^{n} \phi_{-s}\psi_{s-1},$$

so that

$$\det(T_n) = (\phi_0 - \sum_{s=1}^{n} \phi_{-s}\psi_{s-1}) \det(T_{n-1});$$

therefore, from (8) and (11),

$$\sum_{s=0}^{n} \phi_{-s}\psi_{s-1} = -1/b_{00} = -1/x_0. \tag{16}$$

A similar argument applied to T_n^t shows that

$$\det(T_n) = (\phi_0 - \sum_{s=1}^{n} \phi_s\eta_{s-1}) \det(T_{n-1});$$

therefore

$$\sum_{s=0}^{n} \phi_s \eta_{s-1} = -1/x_0. \tag{17}$$

Now we combine (14) and (16) as

$$\sum_{s=0}^{n} \phi_{r-s} \psi_{s-1} = -\delta_{0r}/x_0,\ \ 0 \le r \le n, \tag{18}$$

and (15) and (17) as

$$\sum_{s=0}^{n} \phi_{s-r} \eta_{s-1} = -\delta_{0r}/x_0,\ \ 0 \le r \le n. \tag{19}$$

Comparing (1) with (18) and (7) with (19) shows that $\psi_{j-1} = -x_j/x_0$ and $\eta_{j-1} = -y_{n-j}/x_0$ $(0 \le j \le n)$; therefore, (12) can be rewritten in terms of the Gohberg–Semencul parameters as

$$b_{rs} = b_{r-1,s-1} + x_0^{-1}(x_r y_{n-s} - y_{r-1} x_{n-s+1}),\ \ 0 \le r,\ \ s \le n. \tag{20}$$

This equation and (13) are equivalent to the Gohberg–Semencul formula. To see this, simply perform the matrix operations in (6) and recall (3), (4) and (5) to obtain

$$b_{rs} = x_0^{-1} \sum_{k=0}^{\min(r,s)} (x_{r-k} y_{n+k-s} - y_{r-k-1} x_{n+k-s+1})$$

$$= x_0^{-1}(x_r y_{n-s} - y_{r-1} x_{n-s+1}) + x_0^{-1} \sum_{k=1}^{\min(r,s)} (x_{r-k} y_{n+k-s} - y_{r-k-1} x_{n+k-s+1}).$$

Taking the new summation index $k' = k - 1$ in the last sum yields (20).

We now consider the Heinig formula.

THEOREM 3 [HEINIG, 1979]. *Suppose that T_n is invertible, let $x_0, \dots, x_n$ be as in Theorem 1, and let $z_0, \dots, z_n$ be the solution of*

$$\sum_{s=0}^{n} \phi_{r-s} z_s = \phi_{r-n-1}, \ 0 \le r \le n, \tag{21}$$

where ϕ_{-n-1} (which does not appear in T_n) is arbitrary. Let $x_r = 0$ if $r < 0$ or $r > n+1$,

$$z_{n+1} = -1, \ z_r = 0 \text{ if } r < 0 \text{ or } r > n+1, \tag{22}$$

and define

$$U_n = (z_{r-s})_{r,s=0}^{n} \ \text{ and } \ V_n = (z_{n+r-s+1})_{r,s=0}^{n}.$$

Then

$$T_n^{-1} = U_n D_n - A_n V_n, \tag{23}$$

where A_n and D_n are as in Theorem 1.

We will deduce (23) from an inversion formula given in [11] for a Hankel matrix

$$H_n = (C_{r+s})_{r,s=0}^{n}. \tag{24}$$

The following lemma is clearly implicit in the first paragraph of Section 3 of [11]. For the reader's convenience, we use the notation of that paragraph here.

LEMMA 1 [TRENCH, 1965]. *Suppose that the matrix H_n in (24) and its principal submatrix*

$$H_{n-1} = (C_{r+s})_{r,s=0}^{n-1}$$

are both invertible, and let $u_{0,n-1},\dots,u_{n-1,n-1}$ *and* $u_{0n},\dots,u_{nn}$ *be the solutions of the systems*

$$\sum_{s=0}^{n-1} C_{r+s}u_{s,n-1} = -C_{n+r},\ 0\le r\le n-1, \tag{25}$$

and

$$\sum_{s=0}^{n} C_{r+s}u_{sn} = -C_{n+r+1},\ 0\le r\le n, \tag{26}$$

where C_{2n+1} *in (26) (which does not appear in* H_n*) is arbitrary. Define*

$$\lambda_n = C_{2n} + \sum_{s=0}^{n-1} C_{n+s}u_{s,n-1}, \tag{27}$$

$$u_{n,n-1} = u_{n+1,n} = 1 \tag{28}$$

and $u_{n+1,n-1}=0$*. Then*

$$H_n^{-1} = (a_{rs})_{r,s=0}^{n},$$

where

$$a_{rs} = a_{r-1,s+1} + \lambda_n^{-1}(u_{r,n-1}u_{s+1,n} - u_{rn}u_{s+1,n-1}),\ 0\le r,\ s\le n, \tag{29}$$

with

$$a_{-1,s+1} = a_{r-1,n+1} = 0,\ 0\le r,\ s\le n. \tag{30}$$

By adding $u_{s,n-1}$ times column s $(s=0,\dots,n-1)$ of H_n to its last column and invoking (25), it can be seen that

$$\det(H_n) = \lambda_n \det(H_{n-1});$$

hence, our assumptions imply that $\lambda_n \neq 0$, so that the division in (29) is legitimate. However, to deduce the Heinig formula in complete generality, it is necessary to eliminate the assumption that H_{n-1} is invertible. The following stronger version of Lemma 1 was motivated by this requirement. It had not occurred to the author earlier.

THEOREM 4. *Suppose that H_n in (24) is invertible, and let $\xi_0, \ldots, \xi_n$ and $\eta_0, \ldots, \eta_n$ be the solutions of the systems*

$$\sum_{s=0}^{n} C_{r+s}\xi_s = \delta_{rn},\ 0 \leq r \leq n, \tag{31}$$

and

$$\sum_{s=0}^{n} C_{r+s}\eta_s = -C_{n+r+1},\ 0 \leq r \leq n, \tag{32}$$

where C_{2n+1} is arbitrary. For convenience, define $\xi_{n+1} = 0$ and

$$\eta_{n+1} = 1. \tag{33}$$

Then $H_n^{-1} = (a_{rs})_{r,s=0}^{n}$, with

$$a_{rs} = a_{r-1,s+1} + (\xi_r\eta_{s+1} - \eta_r\xi_{s+1}),\ 0 \leq r,\ s \leq n, \tag{34}$$

subject to (30).

PROOF. Comparing (26) and (28) with (32) and (33) shows that

$$\eta_s = u_{sn},\ 0 \leq s \leq n+1. \tag{35}$$

From (25), (27) and (28),

$$\sum_{s=0}^{n} C_{r+s}u_{s,n-1} = \delta_{rn}\lambda_n,\ 0 \leq r \leq n.$$

From this and (31),

$$\xi_r = \lambda_n^{-1} u_{r,n-1},\ 0 \le r \le n, \tag{36}$$

if $\lambda_n \neq 0$. Now (35) and (36) imply that (29) can be rewritten as (34) if H_n and H_{n-1} are both nonsingular. However, the quantities in (31), (32) and (34) are all continuous functions of $C_0, \dots, C_{2n}$ so long as $\det(H_n) \neq 0$; hence, (34) is also valid for all such that $C_0, \dots, C_{2n}$, including those for which $\det(H_{n-1})$ may happen to vanish.

To obtain Theorem 3, we simply rewrite (1) and (21) as

$$\sum_{s=0}^{n} \phi_{n-r-s} x_s = \delta_{rn},\ 0 \le r \le n, \tag{37}$$

and

$$\sum_{s=0}^{n} \phi_{n-r-s} z_s = \phi_{-r-1},\ 0 \le r \le n, \tag{38}$$

and apply Theorem 4 to the Hankel matrix

$$H_n = (\phi_{n-r-s})_{r,s=0}^{n}; \tag{39}$$

i.e., with

$$C_\ell = \phi_{n-\ell}.$$

Making this substitution into (31) and (32) and comparing the results with (37) and (38) shows that $\xi_r = x_r$ and $\eta_r = -z_r$ $(0 \le r \le n)$. Moreover, $\eta_{n+1} = -z_{n+1}$ by definition (cf. (22) and (33)). Therefore, (34) can be rewritten as

$$a_{rs} = a_{r-1,s+1} + (x_{s+1} z_r - x_r z_{s+1}),\ 0 \le r,\ s \le n. \tag{40}$$

for the Hankel matrix (39). With $\{a_{rs}\}$ as in (40),

$$\sum_{k=0}^{n} a_{rk} \phi_{n-k-s} = \delta_{rs},\ 0 \le r,\ s \le n.$$

Replacing k by $n-k$ here yields

$$\sum_{k=0}^{n} a_{r,n-k}\phi_{k-s} = \delta_{rs},\ 0 \le r,\ s \le n;$$

hence, the elements of T_n^{-1} are

$$b_{rs} = a_{r,n-s}. \tag{41}$$

Finally, replacing s by $n-s$ in (30) and (40) and using (41) shows that

$$b_{rs} = b_{r-1,s-1} + (x_{n-s+1}z_r - x_r z_{n-s+1}),\ 0 \le r,\ s \le n, \tag{42}$$

with

$$b_{-1,s-1} = b_{r-1,-1} = 0,\ 0 \le r,\ s \le n. \tag{43}$$

An easy argument like the one used to show that (6) is equivalent to (12) and (13) confirms that (23) is equivalent to (42) and (43).

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