SPECTRAL DECOMPOSITION OF KAC-MURD0CK-SZEGÖ MATRICES

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December 2012

1 Introduction

Kac, Murdock, and Szegö [1] showed that the eigenvalues of

$$K = \left(\rho^{|r-s|}\right)_{r,s=1}^{n}, \quad 0 < \rho < 1,$$

are

$$\lambda_j = \frac{1-\rho^2}{1-2\rho\cos\theta_j+\rho^2}, \quad 1 \le j \le n, \tag{1}$$

where θ_1, θ_2 $\dots$, θ_n are the zeros of

$$G(\theta) = \sin(n+1)\theta - 2\rho\sin n\theta + \rho^2\sin(n-1)\theta \tag{2}$$

in $(0,\pi)$. They also showed that

$$\frac{(j-1)\pi}{n+1} < \theta_j < \frac{j\pi}{n+1}, \quad 1 \le j \le n. \tag{3}$$

They did not find the associated eigenvectors. We address this and other elementary properties of K simply for the sake of exposition; there is nothing new here.

Since K is a symmetric Toeplitz matrix, it has $\lceil n/2 \rceil$ linearly independent eigenvectors

$$x = \left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}\right]^T \tag{4}$$

such that **(a)** $x_r = x_{n-r+1}$, $1 \le r \le n$, and $\lfloor n/2 \rfloor$ linearly independent eigenvectors such that that **(b)** $x_r = -x_{n-r+1}$, $1 \le r \le n$. Vectors that satisfy **(a)** are *symmetric* and those that satisfy **(b)** are *skew-symmetric*. In the terminology we introduced in [3], an eigenvalue associated with a symmetric eigenvector is *even* and an eigenvector associated with a skew-symmetric eigenvector is *odd*. In [4] we showed that if

$$t_r = \frac{1}{\pi}\int_0^\pi f(\theta)\cos r\theta\, d\theta, \quad 0 \le r \le n-1,$$

and f is nonincreasing, then $T = [t_{r-s}]_{r,s=1}^n$ has distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, where λ_{2k-1} is even, $1 \le k \le \lceil n/2 \rceil$, and λ_{2k} is odd, $1 \le k \le \lfloor n/2 \rfloor$. Based on numerical experiments, Laurie [2] conjectured that the components of a λ_j-eigenvector have exactly $j-1$ changes in sign. A matrix with this property is *oscillatory*. Here we show explicitly that K has these properties.

2 Results

Obviously, λ is an eigenvalue of K if and only if $\lambda = 1/\mu$, where μ is an eigenvalue of

$$K^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}.$$

Denote

$$a = \frac{1+\rho^2}{1-\rho^2} \quad \text{and} \quad b = \frac{\rho}{1-\rho^2}.$$

Then (4) is a μ-eigenvector of K^{-1} – and therefore a $1/\mu$-eigenvector of K – if and only if

$$-bx_{r-1} + (a-\mu)x_r - bx_{r+1} = 0, \quad 1 \le r \le n, \tag{5}$$

where

$$x_0 = \rho x_1 \quad \text{and} \quad x_{n+1} = \rho x_n. \tag{6}$$

The general solution of (5) is

$$x_r = c(\alpha\zeta^r + \beta\zeta^{-r}), \quad 1 \le r \le n, \tag{7}$$

where ζ is a zero of

$$p(z) = -bz^2 + (a-\mu)z - b = -b(z-\zeta)(z-1/\zeta),$$

and the associated eigenvalue of K^{-1} is

$$\mu = a - b(\zeta + 1/\zeta).$$

Since μ is real, $\zeta = e^{i\theta}$; hence $\mu = a - 2b\cos\theta$ is an eigenvalue of K^{-1} and

$$\frac{1}{\mu} = \frac{1}{a - 2b\cos\theta} = \frac{1-\rho^2}{1 - 2\rho\cos\theta + \rho^2}$$

is an eigenvalue of K.

From (6) and (7),

$$\begin{bmatrix} 1-\rho\zeta & 1-\rho/\zeta \\ \zeta^{n+1} - \rho\zeta^n & \zeta^{-n-1} - \rho\zeta^{-n} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}$$

The determinant of this matrix is

$$D(\zeta) = \zeta^{-n-1}(1-\rho\zeta)^2 - \zeta^{n+1}(1-\rho/\zeta)^2 = G(\theta).$$

(Recall (2).) If $D(\zeta)=0$, then

$$\alpha = 1-\rho/\zeta, \quad \beta = -(1-\rho\zeta)$$

is a solution of (8), and (7) becomes

$$x_r = c(1-\rho/\zeta)\zeta^r - (1-\rho\zeta)\zeta^{-r}, \quad 1 \le r \le n. \tag{9}$$

With the appropriate choice of c,

$$x_r = \sin r\theta - \rho\sin(r-1)\theta = (1-\rho\cos\theta)\sin r\theta + \rho\sin\theta\cos r\theta, \quad 1 \le r \le n,$$

which we rewrite as

$$x_r = A\sin(r\theta+\alpha), \quad 1 \le r \le n, \tag{10}$$

where

$$A = \sqrt{1-2\rho\cos\theta+\rho^2}, \quad \cos\alpha = \frac{1-\rho\cos\theta}{A}, \quad \text{and} \quad \sin\alpha = \frac{\rho\sin\theta}{A}.$$

To separate the even and odd spectra of K, we note that

$$\begin{aligned} D(\zeta) &= \left[\zeta^{-(n+1)/2}(1-\rho\zeta) + \zeta^{(n+1)/2}(1-\rho/\zeta)\right] \\ &\times \left[\zeta^{-(n+1)/2}(1-\rho\zeta) - \zeta^{(n+1)/2}(1-\rho/\zeta)\right]. \end{aligned}$$

If ζ is a zero of the first factor, (9) becomes

$$\begin{aligned} x_r &= c(1-\rho/\zeta)(\zeta^r + \zeta^{n-r+1}) \\ &= c(1-\rho/\zeta)\zeta^{(n+1)/2}\left(\zeta^{-(n-2r+1)/2} + \zeta^{(n-2r+1)/2}\right), \quad 1 \le r \le n, \end{aligned}$$

so the associated eigenvector is symmetric. With $\zeta = e^{i\theta}$, the first factor is multiple of

$$\begin{aligned} C(\theta) &= \cos\frac{(n+1)\theta}{2} - \rho\cos\frac{(n-1)\theta}{2} \\ &= (1-\rho)\cos\frac{\theta}{2}\cos\frac{n\theta}{2} - (1+\rho)\sin\frac{\theta}{2}\sin\frac{n\theta}{2}, \end{aligned}$$

and a suitable choice of c yields

$$x_r = \cos\frac{(n-2r+1)\theta}{2}, \quad 1 \le r \le n.$$

If ζ is a zero of the second factor, (9) becomes

$$\begin{aligned} x_r &= c(1-\rho/\zeta)(\zeta^r + \zeta^{n-r+1}) \\ &= c(1-\rho/\zeta)\zeta^{(n+1)/2}\left(\zeta^{-(n-2r+1)/2} - \zeta^{(n-2r+1)/2}\right), \quad 1 \le r \le n, \end{aligned}$$

so the eigenvector is skew-symmetric. With $\zeta = e^{i\theta}$, the second factor is a multiple of

$$\begin{aligned} S(\theta) &= \sin\frac{(n+1)\theta}{2} - \rho\sin\frac{(n-1)\theta}{2} \\ &= (1-\rho)\cos\frac{\theta}{2}\sin\frac{n\theta}{2} + (1+\rho)\sin\frac{\theta}{2}\cos\frac{n\theta}{2}, \end{aligned}$$

and an appropriate choice of c yields

$$x_r = \sin\frac{(n-2r+1)\theta}{2}, \quad 1 \le r \le n.$$

It is easy to verify that C changes sign in

$$\left(\frac{2(j-1)\pi}{n}, \frac{(2j-1)\pi}{n}\right), \quad 1 \le j \le \lceil n/2 \rceil,$$

and S changes sign in

$$\left(\frac{(2j-1)\pi}{n}, \frac{2j\pi}{n}\right), \quad 1 \le j \le \lfloor n/2 \rfloor.$$

Since the right side of (1) is a decreasing function of j, it follows that the even and odd eigenvalues of K are interlaced and the largest is even. Moreover,

$$\frac{(j-1)\pi}{n} < \theta_j < \frac{j\pi}{n}, \quad 1 \le j \le n.$$

This and (3) yield the slightly improved estimates

$$\frac{(j-1)\pi}{n} < \theta_j < \frac{j\pi}{n+1}, \quad 1 \le j \le n.$$

Either of the last two inequalities implies that $r\theta_j$ increases by at least $(j-1)\pi$ but less than $j\pi$ as $r = 1, 2, \ldots, n$. This and (10) imply that K is oscillatory.

References

[1] U. Grenander, G. Szegö, Toeplitz Forms and Their Applications, Univ. of California Press, Berkeley and Los Angeles, 1958.

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[3] W.F. Trench, Spectral evolution of a one-parameter extension of a real symmetric Toeplitz matrix, SIAM J. Matrix Anal. Appl. 11 (1990), 601-611.

[4] Interlacement of the even and odd spectra of real symmetric Toeplitz matrices, Linear Algebra Appl., 195 (1993), 59-68