Asymptotically Constant Solutions of Functional Difference Systems

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Dedicated to Allan Peterson on the Occasion of His 60th Birthday.

Abstract

We consider the functional difference system (A) $\Delta x_i(n) = f_i(n; X)$, $1 \le i \le k$, where $X = (x_1, \dots, x_k)$ and $f_1(\cdot; X), \dots, f_k(\cdot; X)$ are real-valued functionals of X, which may depend quite arbitrarily on values of $X(\ell)$ for multiple values of $\ell \in \mathcal{Z}$. We give sufficient conditions for (A) to have solutions that approach specified constant vectors as $n \to \infty$. Some of the results guarantee only that the solutions are defined for n sufficiently large, while others are global. The proof of the main theorem is based on the Schauder-Tychonoff theorem. Applications to specific quasi-linear systems are included.

Keywords: Functional difference system; Nonsingular; Quasi-linear; Schauder-Tychonoff theorem; Singular

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1 Introduction

Throughout this paper $\mathcal{Z}$ is the set of all integers. If m is an integer, then $\mathcal{Z}_m = \{ n \in \mathcal{Z} \mid n \ge m \}$.

We consider the functional difference system

$$\Delta x_i(n) = f_i(n; X), \quad 1 \le i \le k,$$

where $X = (x_1, \dots, x_k) : \mathcal{Z} \to \mathcal{R}^k$ and $f_1(\cdot; X), \dots, f_k(\cdot; X)$ are real-valued functionals of X. We view $X = \{X(\ell)\}_{\ell \in \mathcal{Z}}$ as a two-way infinite sequence; for a given n, $f_i(n; X)$ may depend quite arbitrarily on values of $X(\ell)$ for multiple values of $\ell \in \mathcal{Z}$. We also write the system as

$$\Delta X(n) = F(n; X). \tag{1}$$

Definition 1 If n_0 is an integer, then $\mathcal{C}_{n_0}$ is the space of bounded sequences $X = \{X(n)\}_{n\in\mathcal{Z}}$ that are constant for $n \le n_0$, with norm $\|X\| = \sup_{n\in\mathcal{Z}} |X(n)|$, where $|U| = \max\{|u_1|, \dots, |u_k|\}$ if $U = (u_1, \dots, u_k)$. If $\{X_\nu\}$ is an infinite sequence of elements in $\mathcal{C}_{n_0}$, we say that $X_\nu \to X$ if $\lim_{\nu\to\infty} \|X_\nu - X\| = 0$. We say that $X \in \mathcal{C}_{n_0}$ is a solution of (1) if $\Delta X(n) = F(n; X)$ for $n \ge n_0$.

Note that $\mathcal{C}_{n_0}$ is a Banach space. We make the following standing assumption.

Assumption 1 *Let m and r be integers, with $0 \le r \le k$, and let $\rho_1, \dots, \rho_k$ be arbitrary positive numbers. If $X \in \mathcal{C}_m$ and*

$$|x_i(n)| \le \rho_i, \quad n \in \mathcal{Z}, \quad 1 \le i \le r, \tag{2}$$

and

$$|x_i(n)| \ge \rho_i, \quad n \in \mathcal{Z}, \quad r+1 \le i \le k, \tag{3}$$

then

$$|f_i(n; X)| \le w_i(n, \rho_1, \dots, \rho_k), \quad n \in \mathcal{Z}_m, \quad 1 \le i \le k, \tag{4}$$

where $w_i : \mathcal{Z}_m \times (0,\infty)^k \to (0,\infty)$ and

$$\sum_{n=m}^{\infty} w_i(n, \rho_1, \dots, \rho_k) < \infty, \quad 1 \le i \le k,$$

for all $\rho_1, \dots, \rho_k > 0$. Finally, if $\{X_\nu\} \subset \mathcal{C}_m$ with

$$|x_{i\nu}(n)| \le \rho_i, \quad n \in \mathcal{Z}, \quad 1 \le i \le r,$$

and

$$|x_{i\nu}(n)| \ge \rho_i, \quad n \in \mathcal{Z}, \quad r+1 \le i \le k,$$

for all ν, and $X_\nu \to X$, then

$$\lim_{\nu\to\infty} F(n; X_\nu) = F(n; X), \quad n \in \mathcal{Z}_m. \tag{5}$$

We say that the system (1) is *nonsingular in* $x_1, \dots, x_r$ and *singular in* $x_{r+1}, \dots, x_k$. We also say that (1) is *purely singular* if $r = 0$, *purely nonsingular* if $r = k$, or *mixed* if $0 < r < k$.

2 The Main Theorem

The following theorem is our main result.

Theorem 1 *Suppose that $n_0 \ge m$ and $\rho_1, \dots, \rho_k$, and $\alpha_1, \dots, \alpha_k$ are positive numbers such that $\alpha_i < 1$ if $1 \le i \le r$, and*

$$\sum_{n=n_0}^{\infty} w_i(n, \rho_1, \dots, \rho_k) \le \alpha_i \rho_i, \quad 1 \le i \le k. \tag{6}$$

Let $c_1, \ldots, c_k$ be constants such that

$$|c_i| \le (1-\alpha_i)\rho_i, \quad 1 \le i \le r, \tag{7}$$

and

$$|c_i| \ge (1+\alpha_i)\rho_i, \quad r+1 \le i \le k. \tag{8}$$

Then there is an $\hat{X}$ in $\mathcal{C}_{n_0}$ such that

$$\Delta\hat{X}(n) = F(n;\hat{X}), \quad n \in \mathcal{Z}_{n_0}, \tag{9}$$

$$|\hat{x}_i(n) - c_i| \le \alpha_i\rho_i, \quad n \in \mathcal{Z}, \quad 1 \le i \le k, \tag{10}$$

and

$$\lim_{n\to\infty} \hat{x}_i(n) = c_i, \quad 1 \le i \le k. \tag{11}$$

Proof. We obtain $\hat{X}$ as a fixed point of the transformation $Y = \mathcal{T}X$ defined by

$$y_i(n) = \begin{cases} c_i - \displaystyle\sum_{\ell=n}^{\infty} f_i(\ell;X), & n \ge n_0, \\ c_i - \displaystyle\sum_{\ell=n_0}^{\infty} f_i(\ell;X), & n < n_0, \end{cases} \qquad 1 \le i \le k, \tag{12}$$

acting on the subset $\mathcal{S}_{n_0}$ of $\mathcal{C}_{n_0}$ such that

$$|x_i(n) - c_i| \le \alpha_i\rho_i, \quad n \in \mathcal{Z}, \quad 1 \le i \le k. \tag{13}$$

If $\hat{X} = \mathcal{T}\hat{X}$ for some $\hat{X} \in \mathcal{S}_{n_0}$, then $\hat{X}$ satisfies (9), (10), and (11).

Since $\mathcal{S}_{n_0}$ is a closed convex subset of a Banach space, the Schauder-Tychonoff theorem [1, p. 405] asserts that $\hat{X} = \mathcal{T}\hat{X}$ for some $\hat{X}$ in $\mathcal{S}_{n_0}$ if

(a) $\mathcal{T}$ is defined on $\mathcal{S}_{n_0}$;

(b) $\mathcal{T}(\mathcal{S}_{n_0}) \subset \mathcal{S}_{n_0}$;

(c) $\mathcal{T}X_\nu \to \mathcal{T}X$ if $\{X_\nu\} \subset \mathcal{S}_{n_0}$ and $X_\nu \to X$;

(d) $\mathcal{T}(\mathcal{S}_{n_0})$ has compact closure.

For the rest of the proof we assume that $X \in \mathcal{S}_{n_0}$. Then (7) and (13) imply (2), while (8) and (13) imply (3). By Assumption 1, (2) and (3) imply (4); hence, (6) and (12) imply that $Y = \mathcal{T}X$ is defined, and that

$$|y_i(n) - c_i| \le \alpha_i\rho_i, \quad n \in \mathcal{Z}, \quad 1 \le i \le k.$$

This establishes hypotheses (a) and (b) of the Schauder-Tychonoff theorem.

Now suppose $\{X_\nu\} \subset \mathcal{S}_{n_0}$ and $X_\nu \to X$. Let $Y_\nu = \mathcal{T}X_\nu = (y_{1\nu}, \ldots, y_{k\nu})$ and $Y = \mathcal{T}X = (y_1, \ldots, y_k)$. From (12),

$$|y_{i\nu}(n) - y_i(n)| \le \sum_{\ell=n_0}^{\infty} |f_i(\ell;X_\nu) - f_i(\ell;X)|, \quad n \in \mathcal{Z}, \quad 1 \le i \le k. \tag{14}$$

From (6), if $\epsilon > 0$, there is an $N > n_0$ such that

$$\sum_{\ell=N+1}^{\infty} w_i(\ell, \rho_1, \dots, \rho_k) < \epsilon, \quad 1 \le i \le k.$$

Then (4) and (14) imply that

$$|y_{i\nu}(n) - y_i(n)| \le \sum_{\ell=n_0}^{N} |f_i(\ell; X_\nu) - f_i(\ell; X)| + 2\epsilon, \quad n \in \mathcal{Z}, \quad 1 \le i \le k. \quad (15)$$

Since

$$\lim_{\nu\to\infty} |f_i(\ell; X_\nu) - f_i(\ell; X)| = 0, \quad \ell \ge n_0,$$

from (5), (15) implies that $\limsup_{\nu\to\infty} \|Y_\nu - Y\| \le 2\epsilon$. Since $\epsilon > 0$, this implies that $\lim_{\nu\to\infty} \|Y_\nu - Y\| = 0$. This establishes hypothesis (c) of the Schauder-Tychonoff theorem.

We will now show that $\overline{\mathcal{T}(\mathcal{S}_{n_0})}$ is compact. Let $C = (c_1, \dots, c_k)$ and $\Gamma = (\gamma_1, \dots, \gamma_k)$, with

$$\gamma_i(n) = \sum_{\ell=n}^{\infty} w_i(\ell, \rho_1, \dots, \rho_k), \quad 1 \le i \le k, \quad n \ge n_0.$$

From (4) and (12),

$$\overline{\mathcal{T}(\mathcal{S}_{n_0})} \subset \mathcal{A} = \left\{ V \in \mathcal{C}_{n_0} \,\middle|\, |V(n) - C| \le |\Gamma(n)| \right\},$$

so it suffices to show that $\mathcal{A}$ is compact. From [2, pp. 51-53], this is true if $\mathcal{A}$ is totally bounded; that is, for every $\epsilon > 0$ there is a finite subset $\mathcal{A}_\epsilon$ of $\mathcal{C}_{n_0}$ such that for each $V \in \mathcal{A}$ there is a $\tilde{V} \in \mathcal{A}_\epsilon$ that satisfies the inequality $\|V - \tilde{V}\| < \epsilon$. To establish the existence of $\mathcal{A}_\epsilon$, choose an integer $n_1 \ge n_0$ such that $|\Gamma(n_1)| < \epsilon$. Now let

$$M = \max\left\{ |\Gamma(n)| \,\middle|\, n_0 \le n \le n_1 - 1 \right\},$$

let p be an integer such that $p\epsilon > M$, and let $Q = \left\{ r\epsilon \,\middle|\, r = \text{ integer }, -p \le r \le p \right\}$. Let $\mathcal{A}_\epsilon$ be the finite set of k-vector functions A on $\mathcal{Z}$ defined as follows:

(i) If $n \ge n_1$, then $A(n) = C$.

(ii) If $n \le n_0$, then $A(n) = A(n_0)$.

(iii) If $n_0 \le n \le n_1 - 1$, then $A(n) = (c_1 + q_1(n), \dots, c_k + q_k(n))$, where $q_1(n)$, $\dots$, $q_k(n)$ are in Q.

Then, since $|V(n) - C| \le M$ for $n_0 \le n \le n_1 - 1$ if $V \in \mathcal{A}$, the set $\mathcal{A}_\epsilon$ has the desired property. Therefore the Schauder-Tychonoff theorem implies that $\mathcal{T}\hat{X} = \hat{X}$ for some $\hat{X}$ in $\mathcal{S}_{n_0}$. ∎

3 Applications of Theorem 1

Since all our results follow from Theorem 1, we will simply verify (6), (7), and (8) in each case, without specifically citing Theorem 1. We say that the problem $P_r(n_0; c_1, \dots, c_k)$ has a solution if there is a sequence $\hat{X}$ in $\mathcal{C}_{n_0}$ such that $\Delta\hat{X}(n) = F(n; X)$, $n \geq n_0$, and $\lim_{n\to\infty} \hat{x}_i(n) = c_i$, $1 \leq i \leq k$. Some of our results are local at ∞, in that a solution is shown to exist only if n_0 is sufficiently large. Others are global, in that a solution is shown to exist for all $n \geq m$.

THEOREM 2 *If $c_i \neq 0$ for $r+1 \leq i \leq k$, then $P_r(n_0; c_1, \dots, c_k)$ has a solution if n_0 is sufficiently large.*

PROOF. Let $\alpha_1, \dots, \alpha_k$ be positive, with $\alpha_i < 1$ for $1 \leq i \leq r$. Choose $\rho_1, \dots, \rho_k$ to satisfy (7) and (8). Then choose n_0 to satisfy (6). ∎

THEOREM 3 *If*

$$\sum_{n=n_0}^{\infty} w_i(n, \rho_1, \dots, \rho_k) < \rho_i, \quad 1 \leq i \leq r, \tag{16}$$

then $P_r(n_0, c_1, \dots, c_k)$ has a solution if $|c_1|, \dots, |c_r|$ are sufficiently small and $|c_{r+1}|, \dots, |c_k|$ are sufficiently large.

PROOF. Choose $\alpha_1, \dots, \alpha_k$ sufficiently large to satisfy (6). (Because of (16), this can be achieved with $\alpha_i < 1$, $1 \leq i \leq r$.) Then $P(n_0, c_1, \dots, c_k)$ has a solution if (7) and (8) hold. ∎

THEOREM 4 *If $|c_1|, \dots, |c_k|$ are sufficiently large, then $P_0(m; c_1, \dots, c_k)$ has a solution.*

PROOF. Let $\rho_1, \dots, \rho_k$ be positive. Choose $\alpha_1, \dots, \alpha_k$ to satisfy (6) with $n_0 = m$. Then choose $c_1, \dots, c_k$ to satisfy (8) with $r = 0$. ∎

THEOREM 5 *Suppose that*

$$\limsup_{\rho\to 0+} \rho^{-1} \sum_{n=m}^{\infty} w_i(n, \rho, \dots, \rho) = \psi_i < 1, \quad 1 \leq i \leq k. \tag{17}$$

Then $P_k(m; c_1, \dots, c_k)$ has a solution if $|c_1|, \dots, |c_k|$ are sufficiently small.

PROOF. Let $\psi_i < \alpha_i < 1$, $1 \leq i \leq k$. From (17), we can choose ρ_0 so small that

$$\sum_{n=m}^{\infty} w_i(n, \rho_0, \dots, \rho_0) \leq \alpha_i \rho_0, \quad 1 \leq i \leq k.$$

Now choose $c_1, \dots, c_k$ so that $|c_i| \leq (1 - \alpha_i)\rho_0$, $1 \leq i \leq k$. ∎

Theorem 6 *Suppose that*

$$\limsup_{\rho\to\infty} \rho^{-1} \sum_{n=m}^{\infty} w_i(n,\rho,\dots,\rho) = \eta_i < 1, \quad 1 \le i \le k. \tag{18}$$

Let $c_1, \dots, c_k$ *be arbitrary. Then* $P_k(m; c_1, \dots, c_k)$ *has a solution.*

Proof. Let $\eta_i < \alpha_i < 1$, $1 \le i \le k$. From (18), we can choose ρ_0 so large that $\rho_0 \ge |c_i|/(1-\alpha_i)$, $1 \le i \le k$, and

$$\sum_{n=m}^{\infty} w_i(n,\rho_0,\dots,\rho_0) \le \alpha_i \rho_0, \quad 1 \le i \le k.$$

□

Assumption 2 *In addition to Assumption* 1, *assume that* (1) *is mixed* (*that is,* $0 < r < k$), *and*

$$w_i(n,\rho_1,\rho_2,\dots,\rho_k) = u_i(n,\rho_1,\dots,\rho_r) + v_i(n,\rho_{r+1},\dots,\rho_k), \quad 1 \le i \le r,$$

where u_i *and* v_i *are positive, and*

$$\lim_{\rho\to\infty} \sum_{n=m}^{\infty} v_i(n,\rho,\dots,\rho) = 0, \quad 1 \le i \le r. \tag{19}$$

Theorem 7 *If Assumption* 2 *holds, then* $P_r(n_0, c_1, \dots, c_k)$ *has a solution if* n_0 *and* $|c_{r+1}|, \dots, |c_k|$ *are sufficiently large and* $|c_1|, \dots, |c_r|$ *are sufficiently small.*

Proof. Let $\rho_1 > 0$ and $0 < \alpha_1 < 1$, and choose $n_0 \ge m$ so that

$$\sum_{n=n_0}^{\infty} u_i(n,\rho_1,\dots,\rho_1) < \alpha_1\rho_1, \quad 1 \le i \le r.$$

From (19), we can choose ρ_2 so large that

$$\sum_{n=n_0}^{\infty} (u_i(n,\rho_1,\dots,\rho_1) + v_i(n,\rho_2,\dots,\rho_2)) \le \alpha_1\rho_1, \quad 1 \le i \le r.$$

Now choose α_2 so that

$$\sum_{n=n_0}^{\infty} w_i(n,\rho_1,\dots,\rho_k) \le \alpha_2\rho_2, \quad r+1 \le i \le k,$$

if $\rho_i = \rho_1$, $1 \le i \le r$, and $\rho_i = \rho_2$, $r+1 \le i \le k$. Then choose $|c_i| \le (1-\alpha_1)\rho_1$, $1 \le i \le r$, and $|c_i| \ge (1+\alpha_2)\rho_2$, $r+1 \le i \le k$. □

THEOREM 8 *In addition to Assumption* 2, *suppose that*

$$\limsup_{\rho\to 0+} \rho^{-1} \sum_{n=m}^{\infty} u_i(n,\rho,\ldots,\rho) = \psi_i < 1, \quad 1 \le i \le r.$$

Then $P_r(m, c_1, \ldots, c_k)$ *has a solution if* $|c_1|, \ldots, |c_r|$ *are sufficiently small and* $|c_{r+1}|, \ldots, |c_k|$ *are sufficiently large.*

PROOF. Let $\psi_i < \alpha_i < 1$, $1 \le i \le r$. Choose ρ_1 so small that

$$\sum_{n=m}^{\infty} u_i(n,\rho_1,\ldots,\rho_1) < \alpha_i\rho_1, \quad 1 \le i \le r.$$

Now apply the argument used in the proof of Theorem 7, with $n_0 = m$. □

THEOREM 9 *In addition to Assumption* 2, *suppose that*

$$\limsup_{\rho\to \infty} \rho^{-1} \sum_{n=m}^{\infty} u_i(n,\rho,\ldots,\rho) = \eta_i < 1, \quad 1 \le i \le r.$$

Then $P_r(m, c_1, \ldots, c_k)$ *has a solution if* $|c_{r+1}|, \ldots, |c_k|$ *are sufficiently large.*

PROOF. Let $\eta_i < \alpha_i < 1$, $1 \le i \le r$. Choose ρ_1 so large that $\rho_1 \ge |c_i|/(1-\alpha_i)$, $1 \le i \le r$, and

$$\sum_{n=m}^{\infty} u_i(n,\rho_1,\ldots,\rho_1) < \alpha_i\rho_1, \quad 1 \le i \le r.$$

Now apply the argument used in the proof of Theorem 7, with $n_0 = m$. □

4 Quasi-linear Systems: I

Consider the system

$$\Delta x_i(n) = \sum_{j=1}^{k} a_{ij}(n) g_{ij}(x_j(\phi_{ij}(n))), \quad 1 \le i \le k, \tag{20}$$

assuming throughout that, for some integer m and $1 \le i \le j \le k$, $\phi_{ij} : \mathcal{Z}_m \to \mathcal{Z}$, $g_{ij} : \mathcal{Z} \to \mathcal{R}$, $a_{ij} : \mathcal{Z}_m \to \mathcal{R}$,

$$|g_{ij}(u)| = |u|^{\gamma_{ij}} \quad \text{and} \quad \sum_{n=m}^{\infty} |a_{ij}(n)| < \infty.$$

We assume that for some r in $\{0, 1, \ldots, k\}$, $\gamma_{ij} > 0$ if $1 \le i \le r$ and $\gamma_{ij} < 0$ if $r+1 \le i \le k$, for $1 \le j \le k$. Then Assumption 1 holds, with

$$w_i(n, \rho_1, \rho_2, \ldots, \rho_k) = \sum_{j=1}^{k} |a_{ij}(n)| \rho_j^{\gamma_{ij}}.$$

It is to be understood throughout this section that this is the definition of w_i. If $0 < r < k$, then (20) satisfies Assumption 2 with

$$u_i(n, \rho_1, \ldots, \rho_r) = \sum_{j=1}^{r} |a_{ij}(n)| \rho_j^{\gamma_{ij}}$$

and

$$v_i(n, \rho_{r+1}, \ldots, \rho_k) = \sum_{j=r+1}^{k} |a_{ij}(n)| \rho_j^{\gamma_{ij}}.$$

THEOREM 10 *If $\gamma_{ij} > 0$, $1 \le i, j \le k$, there is an $n_0 \ge m$, which depends upon c_1, …, c_k, such that $P_k(n_0; c_1, \ldots, c_k)$ has a solution.*

PROOF. If $0 < \alpha < 1$, choose ρ_1, ρ_2, …, ρ_k so that $|c_i| \le (1-\alpha)\rho_i$, $1 \le i \le k$. Then choose n_0 so that

$$\sum_{n=n_0}^{\infty} w_i(n, \rho_1, \rho_2, \ldots, \rho_k) \le \alpha \rho_i, \quad 1 \le i \le k.$$

□

THEOREM 11 *If $\gamma_{ij} = 1$, $1 \le i, j \le k$, there is an $n_0 \ge m$, independent of c_1, …, c_k, such that $P_k(n_0; c_1, \ldots, c_k)$ has a solution.*

PROOF. If $0 < \alpha < 1$, choose n_0 so that

$$\sum_{n=n_0}^{\infty} w_i(n, 1, \ldots, 1) < \alpha, \quad 1 \le i \le k.$$

Then

$$\sum_{n=n_0}^{\infty} w_i(n, \rho, \ldots, \rho) < \alpha\rho, \quad 1 \le i \le k,$$

for any $\rho > 0$. For arbitrary c_1, …, c_k choose ρ so that $|c_i| \le (1-\alpha)\rho$, $1 \le i \le k$. □

Theorems 5-9 imply the following theorems.

THEOREM 12 *If $\gamma_{ij} > 1$, $1 \le i, j \le k$, then $P_k(m; c_1, \ldots, c_k)$ has a solution if $|c_1|$, …, $|c_k|$ are sufficiently small.*

THEOREM 13 *If $0 < \gamma_{ij} < 1$, $1 \le i, j \le k$, and c_1, …, c_k are arbitrary, then $P_k(m; c_1, \ldots, c_k)$ has a solution.*

THEOREM 14 *If $0 < r < k$, then $P_r(n_0; c_1, \ldots, c_k)$ has a solution if n_0 and $|c_{r+1}|$, …, $|c_k|$ are sufficiently large and $|c_1|$, …, $|c_r|$ are sufficiently small.*

THEOREM 15 *If $0 < r < k$ and $\gamma_{ij} > 1$, $1 \le i \le r$, $1 \le j \le k$, then $P_r(m; c_1, \ldots, c_k)$ has a solution if $|c_1|$, …, $|c_r|$ are sufficiently small and $|c_{r+1}|$, …, $|c_k|$ are sufficiently large.*

THEOREM 16 *If $0 < r < k$ and $0 < \gamma_{ij} < 1$, $1 \le i \le r$, $1 \le j \le k$, then $P_r(m; c_1, \ldots, c_k)$ has a solution if $|c_{r+1}|$, …, $|c_k|$ are sufficiently large.*

5 Quasi-linear Systems: II

In this section we consider

$$\Delta x_i(n) = \sum_{j=1}^{k} \beta_{ij}^n \sum_{\ell=0}^{n} p_{ij}(n-\ell) g_{ij}(x_j(\phi_{ij}(n))), \quad 1 \le i \le k, \quad n \ge 0, \qquad (21)$$

where g_{ij} and ϕ_{ij} are as in the previous section, $|\beta_{ij}| < 1$, and

$$\sum_{n=0}^{\infty} |\beta_{ij}^n p_{ij}(n)| < \infty, \quad 1 \le i, j \le n.$$

Here we can take

$$w_i(n, \rho_1, \ldots, \rho_k) = \sum_{j=1}^{k} \rho_j^{\gamma_{ij}} \sum_{\ell=0}^{n} |\beta_{ij}^n p_{ij}(n-\ell)|.$$

Therefore Assumption 1 holds with

$$\sum_{n=0}^{\infty} w_i(n, \rho_1, \ldots, \rho_k) = \sum_{j=1}^{k} \sigma_{ij} \rho_j^{\gamma_{ij}},$$

where

$$\begin{aligned} \sigma_{ij} &= \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} |\beta_{ij}^n p_{ij}(n-\ell)| = \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} |\beta_{ij}^n p_{ij}(n-\ell)| \\ &= \frac{1}{1-|\beta_{ij}|} \sum_{n=0}^{\infty} |\beta_{ij}^n p_{ij}(n)|. \end{aligned}$$

All the arguments used in the previous section can now be used with $|a_{ij}|$ replaced by σ_{ij}; therefore, Theorems 10-16 all hold (with $m = 0$) for (21).

References

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