LINEAR PERTURBATIONS OF A NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATION II

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Abstract. Let \( y_1 \) and \( y_2 \) be principal and nonprincipal solutions of the nonoscillatory differential equation \((r(t)y')' + f(t)y = 0\). In an earlier paper we showed that if \( \int_{\infty}^\infty (f - g)y_1 y_2 \, dt \) converges (perhaps conditionally), and a related improper integral converges absolutely and sufficiently rapidly, then the differential equation \((r(t)x')' + g(t)x = 0\) has solutions \( x_1 \) and \( x_2 \) that behave asymptotically like \( y_1 \) and \( y_2 \). Here we consider the case where \( \int_{\infty}^\infty (f - g)y_2^2 \, dt \) converges (perhaps conditionally), without any additional assumption requiring absolute convergence.


1. Introduction

We consider the differential equation

\[(r(t)x')' + g(t)x = 0\]  

as a perturbation of

\[(r(t)y')' + f(t)y = 0,\]  

under the following standing assumption.

Assumption A. Let \( r \) and \( f \) be real-valued and continuous, with \( r > 0 \), on \([a, \infty)\). Suppose that (2) is nonoscillatory at infinity. Let \( g \) be continuous and possibly complex-valued on \([a, \infty)\).

It is known [4, p. 355] that since (2) is nonoscillatory at infinity, it has solutions \( y_1 \) and \( y_2 \) which are positive on \([b, \infty)\) for some \( b \geq a \) and satisfy the following conditions:

\[r(y_1'y_2 - y_1'y_2) = 1, \quad t \geq a,\]

\[\lim_{t \to \infty} \frac{y_2(t)}{y_1(t)} = \infty.\]

Without loss of generality we let \( b = a \). Henceforth \( t \geq a \). It is convenient to define

\[\rho = \frac{y_2}{y_1}.\]
From (3) and (4),

$$\rho' = 1 / ry_1^2 \geq 0 \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \infty.$$  

We use the Landau symbols “$$o$$” and “$$O$$” in the standard way to denote behavior as $$t \to \infty$$. In [6] we proved the following theorem.

**Theorem 1.** Suppose that \( \int_0^\infty (f - g)y_1y_2 \, dt \) converges (perhaps conditionally) and

\[
\sup_{\tau \geq t} \left| \int_\tau^\infty (f - g)y_1y_2 \, ds \right| \leq \phi(t),
\]

where \( \phi(t) \to 0 \) monotonically as \( t \to \infty \). Define

\[
G(t) = \int_t^\infty (f - g)y_1^2 \, ds,
\]

and suppose that

\[
\int_0^\infty |G|\rho' \, dt < \infty
\]

and

\[
\limsup_{t \to \infty} (\phi(t))^{-1} \int_t^\infty |G| \rho' \, ds = A < 1/3.
\]

Then (1) has a solution \( x_1 \) such that

\[
x_1 = y_1(1 + O(\phi))
\]

and

\[
(x_1/y_1)' = O(\phi' / \rho),
\]

and a solution \( x_2 \) such that

\[
x_2 = y_2(1 + O(\phi_m))
\]

and

\[
(x_2/y_2)' = O(\phi_m \rho' / \rho),
\]

where

\[
\phi_m = \max\{\phi, \dot{\phi}\},
\]

with

\[
\dot{\phi}(t) = \frac{1}{\rho(t)} \int_t^\infty \rho' \phi \, ds.
\]

This result was an improvement on a theorem of Hartman and Wintner [4, p. 379], and it was subsequently improved by Chen [1] and Šimša [5]. (For more on the Hartman-Wintner problem, see [2] and [3].) In this continuation of [6] we consider the case where \( \int_0^\infty (f - g)y_2^2 \, dt \) converges, perhaps conditionally. To motivate the present work, we first apply Theorem 1 under this assumption.

Let

\[
H(t) = \int_t^\infty (f - g)y_1y_2 \, ds,
\]

and recall from (7) that

\[
\sup_{\tau \geq t} \{|H(\tau)|\} \leq \phi(t).
\]

Let

\[
I(t) = \int_t^\infty (f - g)y_2^2 \, ds,
\]
and suppose that
\begin{equation}
\sup_{\tau \geq t} |I(\tau)| \leq \sigma(t),
\end{equation}
where \( \sigma(t) \to 0 \) monotonically as \( t \to \infty \). From (8), (10), and (11),
\begin{equation}
H(t) = - \int_{t}^{\infty} \frac{I'(\tau)}{\rho} \, d\tau = \frac{I(t)}{\rho(t)} + \int_{t}^{\infty} I\left(\frac{1}{\rho}\right)' \, d\tau
\end{equation}
and
\begin{equation}
G(t) = - \int_{t}^{\infty} \frac{I'(\tau)}{\rho^2} \, d\tau = \frac{I(t)}{\rho^2(t)} + \int_{t}^{\infty} I\left(\frac{1}{\rho^2}\right)' \, d\tau,
\end{equation}
so
\begin{equation}
|H(t)| \leq 2\sigma(t)/\rho(t) \quad \text{and} \quad |G(t)| \leq 2\sigma(t)/\rho^2(t).
\end{equation}
It is straightforward to verify that (9) holds with \( \phi = \sigma/\rho \) and \( A = 0 \). Therefore Theorem 1 implies that (1) has solutions \( x_1 \) and \( x_2 \) such that
\begin{equation}
x_1 = y_1(1 + O(\sigma/\rho)),
\end{equation}
\begin{equation}
(x_1/y_1)' = O(\sigma \rho'/\rho^2),
\end{equation}
\begin{equation}
x_2 = y_2(1 + O(\hat{\phi})),
\end{equation}
and
\begin{equation}
(x_2/y_2)' = O(\hat{\phi} \rho'/\rho),
\end{equation}
with
\begin{equation}
\hat{\phi}(t) = \frac{1}{\rho(t)} \int_{a}^{t} \frac{\sigma \rho'}{\rho} \, ds.
\end{equation}
At best, (17) and (18) imply that
\begin{equation}
x_2 = y_2(1 + O(1/\rho))
\end{equation}
and
\begin{equation}
(x_2/y_2)' = O(\rho'/\rho^2)
\end{equation}
if \( \int_{a}^{\infty} \sigma \rho'/\rho \, ds < \infty \), which may be false. Among other things, we will show that (17) and (18) can be replaced by
\begin{equation}
x_2 = y_2(1 + O(\sigma/\rho))
\end{equation}
and
\begin{equation}
(x_2/y_2)' = O(\sigma \rho'/\rho^2).
\end{equation}
These two equations are improvements over (17) and (18), since \( \lim_{t \to \infty} \rho(t) \hat{\phi}(t)/\sigma(t) = \infty \) in any case. In fact, it can be seen from (15), (16), (19), and (20) that \( (x_i/y_i)' - 1, \ i = 1, 2 \), approach zero at the same rate as \( t \to \infty \), as do \( (x_i/y_i)'', \ i = 1, 2 \). We also note that the results in these four equations can be written as
\begin{equation}
x_i/y_i = 1 + O(\sigma y_i/\rho^3) \quad \text{and} \quad (x_i/y_i)' = O(\rho y_i/\rho^3), \ i = 1, 2.
\end{equation}
2. Main results

Theorem 2. Suppose that \( \int_{0}^{\infty} (f - g) y_2^2 \, dt \) converges. Let \( I \) and \( \sigma \) be as in (11) and (12). Then (1) has a solution \( x_1 \) that satisfies (15) and (16), and a solution \( x_2 \) such that

\[
(21) \quad \frac{x_2 - y_2}{y_1} = O(\sigma)
\]

and

\[
(22) \quad \left( \frac{x_2 - y_2}{y_1} \right)' = O \left( \frac{\sigma \rho'}{\rho} \right).
\]

Proof. We have already proved the assertion concerning \( x_1 \). For the assertion concerning \( x_2 \), we use the contraction mapping theorem. If

\[
(23) \quad x_2(t) = y_2(t) + \int_{t}^{\infty} (y_2(s)y_1(t) - y_1(s)y_2(t))(f(s) - g(s))x_2(s) \, ds,
\]

then \( x_2 \) satisfies (1). Although this suggests a transformation to work with, it is better to use a transformation with the fixed point \( \zeta \), where

\[
\zeta = \frac{x_2 - y_2}{y_1}.
\]

Rewriting (23) in terms of \( \zeta \) yields

\[
\zeta(t) = \int_{t}^{\infty} (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s) \, ds + \int_{t}^{\infty} (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)\zeta(s) \, ds.
\]

We use the transformation \( Tz = Q + Lz \), where

\[
Q(t) = \int_{t}^{\infty} (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s) \, ds
\]

and

\[
(Lz)(t) = \int_{t}^{\infty} (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s) \, ds.
\]

From (10), (11), and (13),

\[
Q(t) = I(t) - \rho(t)H(t) = -\rho(t) \int_{t}^{\infty} I(1/\rho)' \, ds,
\]

so \( |Q(t)| \leq \sigma(t) \), from (12). Moreover,

\[
Q' = I' - \rho H' - H\rho' = -H\rho',
\]

so

\[
|Q'(t)| \leq 2\sigma(t)\rho'(t)/\rho(t),
\]

from (14). Therefore we let \( T \) act on the Banach space \( \mathcal{B} \) of functions \( z \) on \([t_0, \infty)\) such that

\[
z = O(\sigma) \quad \text{and} \quad z' = O(\sigma\rho'/\rho),
\]

with norm

\[
(24) \quad \|z\| = \sup_{t \geq t_0} \left\{ \max \left\{ \frac{|z|}{\sigma}, \frac{\rho|z'|}{\sigma\rho'} \right\} \right\}.
\]
We will show that $T$ maps $B$ into $B$, and is a contraction if $t_0$ is sufficiently large. Since $Q \in B$, it suffices to show that $L$ is a contraction of $B$ if $t_0$ is sufficiently large.

To this end, suppose $z \in B$ and $t_0 \leq t < T$, and consider the finite integral

$$w_T(t; z) = \int_t^T (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s)\, ds.$$  

From (5) and (8),

$$w_T(t; z) = -\int_t^T (\rho(s) - \rho(t))z(s)G'(s)\, ds$$

$$= -(\rho(T) - \rho(t))z(T)G(T)$$

$$+ \int_t^T (\rho(s) - \rho(t))G(s)z'(s)\, ds$$

$$+ \int_t^T z(s)G(s)\rho'(s)\, ds.$$  

From (14) and (24),

$$|\rho(T) - \rho(t)|z(T)G(T)| < 2\|z\|\sigma^2(T)/\rho(T) \to 0 \text{ as } T \to \infty,$$

$$|(\rho(s) - \rho(t))G(s)z'(s)| \leq 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s), \quad s \geq t,$$

and

$$|z(s)G(s)\rho'(s)| \leq 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s).$$

Therefore we can let $T \to \infty$ in (25) and conclude that

$$(Lz)(t) = -\int_t^\infty (\rho(s) - \rho(t))z(s)G'(s)\, ds$$

exists and satisfies the inequality

$$(Lz)(t) < 4\|z\|\int_t^\infty \frac{\sigma^2/\rho^2}{\rho^3} \, ds < 4\|z\|\frac{\sigma^2(t)/\rho(t)}{\rho(t)}.$$  

From (26),

$$(Lz)'(t) = \rho'(t)\int_t^\infty zG'\, ds = -\rho'(t) \left( z(t)G(t) + \int_t^\infty Gz'\, ds \right).$$

From (14) and (24), the last integral converges absolutely and

$$|(Lz)'(t)| \leq 2\|z\|\rho'(t) \left( \frac{\sigma^2(t)}{\rho^2(t)} + \int_t^\infty \frac{\sigma^2/\rho'}{\rho^3} \, ds \right) < 4\|z\|\frac{\sigma^2(t)\rho'(t)}{\rho^2(t)}.$$  

From this and (27),

$$\|(Lz)\| < 4\|z\|\sigma(t)/\rho(t).$$

Hence $L$ (and consequently $T$) is a contraction of $B$ if $\sigma(t_0)/\rho(t_0) < 1/4$. Therefore there is a (unique) $\zeta \in B$ such that $T\zeta = \zeta$, and the function $x_2$ defined by $x_2 = y_2 + y_1\zeta$ ($t \geq t_0$) is a solution of (1) that satisfies (21) and (22). We can extend the definition of $x_2$ back to $t = a$. 

\textbf{Corollary 1.} Under the assumptions of Theorem 2, $x_2$ satisfies (19) and (20).
Proof. Since $y_2/y_1 = \rho$, (21) implies that $y_2$ satisfies (19) and
\[ x_2/y_1 = \rho + O(\sigma). \]
From (22),
\[ (x_2/y_1)' = \rho' (1 + O(1/\rho)). \]
Therefore
\[
\left( \frac{x_2}{y_2} \right)' = \left( \frac{x_2}{y_1} \right)' \frac{1}{\rho} - \frac{x_2 \rho'}{y_1 \rho^2} = \frac{\rho'}{\rho} (1 + O(\sigma/\rho)) - \frac{\rho'}{\rho^2} (\rho + O(\sigma)) = O\left( \frac{\sigma \rho'}{\rho^3} \right).
\]
\[ \square \]

It is natural to ask whether the convergence of \( \int_1^\infty (f-g)y_2^2 \, dt \) is necessary for the existence of a solution \( x_2 \) of (1) such that
\[ x_2 = y_2 (1 + o(1/\rho)) \quad \text{and} \quad (x_2/y_2)' = o(\rho'/\rho^2). \]
Although we do not know the answer to this question, we offer the following related theorem.

**Theorem 3.** If (1) has a solution \( x_2 \) that satisfies (19) and (20) for some positive monotonic function \( \sigma \) such that \( \lim_{t \to \infty} \sigma(t) = 0 \), then
\[ (28) \quad \int_t^\infty (f-g)y_1y_2 \, dt = O(\sigma/\rho). \]
Moreover, if
\[ (29) \quad \int_1^\infty \frac{\sigma \rho'}{\rho} \, dt < \infty, \]
then \( \int_1^\infty (f-g)y_2^2 \, dt \) converges.

**Proof.** From (20), \( R(t) = \int_t^\infty (x_2/y_2)' \, dt \) converges absolutely and
\[ (30) \quad R = O(\sigma/\rho). \]
If \( t > T \), define
\[ R_T(t) = \int_t^T \left( \frac{x_2}{y_2} \right)' \, ds. \]
From (5) and (6),
\[ (31) \quad \left( \frac{x_2}{y_2} \right)' = \frac{y_2 x_2' - x_2 y_2'}{y_2^2} = u \frac{\rho'}{\rho^2}, \]
where
\[ u = r(y_2 x_2' - x_2 y_2'). \]
From (1) and (2),
\[ u' = (f-g)y_2 x_2. \]
Therefore
\[ R_T(t) = \frac{u(t)}{\rho(t)} - \frac{u(T)}{\rho(T)} + \int_t^T (f-g)y_1 x_2 \, ds. \]
From (20) and (31), \( u = o(\sigma) \), so we can let \( T \to \infty \) and invoke (30) to conclude that

\[
\hat{R}(t) \overset{df}{=} \int_t^\infty (f - g)y_1x_2 \, ds = O(\sigma/\rho).
\]

Now let

\[
S_T(t) = \int_t^T (f - g)y_1y_2 \, ds = -\int_t^T \frac{y_2}{x_2} \hat{R}' \, ds
\]

\[
= \frac{y_2(t)}{x_2(t)} \hat{R}(t) - \frac{y_2(T)}{x_2(T)} \hat{R}(T) + \int_t^T \frac{y_2}{x_2} \, ds.
\]

But

\[
\left( \frac{y_2}{x_2} \right)' = \frac{y_2^2}{x_2^2} \left( \frac{x_2}{y_2} \right)' = O\left( \frac{\sigma \rho'}{\rho^2} \right)
\]

from (19) and (20). From this and (32), we can let \( T \to \infty \) in (33) to conclude that

\[
S(t) \overset{df}{=} \int_t^\infty (f - g)y_1y_2 = O(\sigma/\rho).
\]

This verifies (28). If (29) holds and \( T > a \), then

\[
\int_a^T (f - g)y_2^2 \, dt = -\int_a^T \rho S' \, dt = \rho(a)S(a) - \rho(T)S(T) + \int_a^T S' \rho' \, dt.
\]

Since (34) implies that \( \lim_{T \to \infty} \rho(T)S(T) = 0 \) and (29) and (34) together imply that \( \int_\infty S' \rho' \, dt \) converges, (35) implies that \( \int_\infty (f - g)y_2^2 \, dt \) converges. \( \square \)

3. Examples

Examples illustrating our results can be constructed by letting

\[
g(t) = f(t) + \frac{u(t)S(t)}{y_2^2(t)}, \quad t \geq a,
\]

where \( u \) and \( S \) are continuously differentiable and \( S \) has a bounded antiderivative \( C \) on \([a, \infty)\), while \( \lim_{t \to \infty} u(t) = 0 \) and \( \int_\infty |u'(t)| \, dt < \infty \). Then

\[
\int_t^\infty (f(s) - g(s))y_2^2(s) \, ds = -\int_t^\infty u(s)S(s) \, ds = -u(s)C(s) \bigg|_t^\infty + \int_t^\infty u'(s)C(s) \, ds
\]

converges, and the convergence may be conditional. Here we may take

\[
\sigma(t) = M \sup_{\tau \geq t} \left( |u(\tau)| + \int_\tau^\infty |u'(s)| \, ds \right),
\]

where \( M \) is an upper bound for \( C \) on \([a, \infty)\).

For a specific example, consider the equation

\[
x'' + \frac{\sin t}{t^2(\log t)^\alpha}x = 0, \quad t \geq a > 0 \quad (\alpha > 0),
\]

as a perturbation of \( y'' = 0 \). Our results imply that (36) has solutions \( x_1 \) and \( x_2 \) such that

\[
x_1(t) = 1 + O\left( t^{-1}(\log t)^{-\alpha} \right), \quad x_1'(t) = O\left( t^{-2}(\log t)^{-\alpha} \right)
\]

and

\[
x_2(t) = t + O((\log t)^{-\alpha}), \quad x_2'(t) = 1 + O(t^{-1}(\log t)^{-\alpha}).
\]
References


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